hdd

4/5

5

MECHANICS OF THE GIRDER:

A Treatise on Bridges and Roofs,

IN WHICH THE NECESSARY AND SUFFICIENT WEIGHT OF THE STRUCTURE IS CALCULATED, NOT ASSUMED;

AND

THE NUMBER OF PANELS AND HEIGHT OF GIRDER THAT RENDER THE BRIDGE WEIGHT LEAST, FOR A GIVEN SPAN, LIVE LOAD, AND WIND PRESSURE, ARE DETERMINED.

BY

JOHN DAVENPORT CREHORE, C.E.

"Inveniam viam aut faciam."

NEW YORK:
JOHN WILEY AND SONS.
1886.

Eng 748.86.5

Engineering a. iv.
66.76

RAND, AVERY, AND CO.,
ELECTROTYPERS AND PRINTERS,
BOSTON.

PREFACE.

THE "Mechanics of the Girder" is presented to the public in an unfinished condition, just as it was left at the author's death, in October, 1884. All that then remained to be done was to carry out an example in each of the twelve classes of girders in a manner similar to that of the Brunel Girder in Class I. (Sections 2 and 3, Chapter X.), and the Double Parabolic Bow and Post Truss in Class II. (Chapter XI.). Of all these, the Post Truss promised to yield the most prolific results; and it may be possible, before another edition is published, to complete this calculation at least, if not to introduce other examples from the later classes. However, the *a priori* method of the author is fully set forth previous to the tenth chapter; and it is believed that no one else has as yet published any so satisfactory results from this method, if, indeed, the method has been hitherto attempted with any degree of success.

The author's family feel deeply grateful to Professor John N. Stockwell for his kindness in devoting much of his valuable time to the supervision of the proof-reading, for the many suggestions he has given during the publi-

cation, and particularly for his offér to conduct the work of completing the remaining examples. At his own suggestion, however, it has been thought expedient to delay no longer the publication of the completed portion of the book, and to leave any additional matter to be inserted later.

WILLIAM W. CREHORE.

July 29, 1886.

ANALYTICAL TABLE OF CONTENTS.

CHAPTER I.

PRESSURES IN ONE PLANE.

CHAPTER II.

MOMENT OF A FORCE.

CHAPTER III.

MOMENTS OF THE EXTERNAL FORCES APPLIED TO A BEAM OR GIRDER.

SECTION 1.

The Semi-Beam, or Girder fixed at One End and free at the Other.

v

SECTION 2.

Horizontal Girder supported only at its Ends, which are not fixed.

SECTION 3.

Horizontal Girder of One Span, with Fixed Ends. Effects of End Moments.

CHAPTER IV.

STRAINS IN FRAMED OR BUILT GIRDERS, DEDUCED FROM THE MOMENTS OF THE EXTERNAL FORCES AND FROM THE SHEARING-FORCES, AND FROM THESE COMBINED.

CHAPTER V.

MOMENTS OF RESISTANCE OF THE INTERNAL FORCES OF A BEAM OR GIRDER HAVING A CONTINUOUS WEB.

SECTION I.

General Formula found and applied to Particular Cross-Sections of Beams with Continuous Web.

SECTION 2.

Moment of Inertia and Radius of Gyration of a Given Cross-Section.

CHAPTER VI.

DEFLECTION, END MOMENTS, AND POINTS OF CONTRARY FLEXURE FOUND. — CAMBER.

SECTION I.

Deflection of the Semi-Beam having a Uniform Cross-Section.

SECTION 2.

Deflection of a Beam of Uniform Cross-Section, supported at its Free Unfixed Ends.

SECTION 3.

The Influence of Fixed Ends upon the Deflection of a Beam of Uniform Cross-Section, supported at its Two Extremities, which are assumed to be Level, and One or Both of Them fixed Horizontally or Otherwise. Determination of the End Moments and Points of Contrary Flexure.

SECTION 4.

Deflection of a Girder of Variable Cross-Section in Terms of the Constant Unit Strain upon the Extreme Fibres of the Section; that is, Deflection of a Beam of Uniform Strength. End Moments for Fixed Beams.

SECTION 5.

Camber.

CHAPTER VII.

THE STRENGTH OF PILLARS, COLUMNS, POSTS, OR STRUTS.

SECTION I.

Strength of Pillars by Rational Formulæ.

SECTION 2.

Hodgkinson's Empirical Formulæ for the Strength of Cast-Iron and Timber Pillars.

SECTION 3.

Gordon's Empirical Formula, with Rankine's Modification.

SECTION 4.

Strength of Pillars computed by the Preceding Formulæ, and compared with the Strength experimentally determined.

CHAPTER VIII.

PROPORTIONS AND WEIGHTS OF ALL THE MEMBERS OF A BRIDGE EXCEPTING THE GIRDERS PROPER.

CHAPTER IX.

OPEN GIRDERS WITH EQUAL AND PARALLEL STRAIGHT CHORDS. CLASS IX.

SECTION 1.

The Pratt Truss of Single System and Uniform Live Load.—Wind Pressure.

SECTION 3.

The Brunel Girder of Double System.

CHAPTER XI.

BRIDGES OF CLASS II. — BEST NUMBER OF PANELS AND BEST HEIGHT DETERMINED FOR A GIVEN SPAN UNDER A GIVEN UNIFORM LIVE LOAD. — LEAST BRIDGE WEIGHT AND LIMITING SPAN FOUND.

SECTION 1.

The Parabolic Bowstring Girder of Double Triangular System, with the Extreme Diagonals omitted, and a Vertical Suspender at Extreme Panel Point.

SECTION 2.

The Post Truss with Parabolic Top Chord.

MECHANICS OF THE GIRDER.

CHAPTER I.

PRESSURES IN ONE PLANE.

1. FORCE is a cause which changes, or tends to change, the condition of matter as to rest or motion. Whether there is or is not, in fact, any difference between force and pressure, it is sufficient for the purposes of this volume to treat them as identical, since it is with their measurable effects alone that we are here concerned.

A force is said to be given when its point of application, its direction, its line of action, and its intensity are known. Two pressures are equal which, acting on the same point, along the same line, and in opposite directions, neutralize each other; and, if two equal pressures act at the same point in the same direction, the result of their combined action is twice that of each separate pressure.

Pressures, therefore, may be compared by means of numbers expressing their intensities. Since the intensity of any one of the pressures to be compared may be taken as the standard, it follows that the unit pressure is entirely arbitrary, and may be a finite or an infinitesimal pressure.

When pressures are expressed by symbols, such as P, Q, R, etc., it is to be understood that these letters stand for num-

bers denoting the number of times the concrete unit is taken. Otherwise, such an expression as P^2, being the square of a concrete pressure, would be unintelligible.

A force or pressure may be conveniently represented by a geometrical straight line; one end of the line denoting the point of application of the force, the direction of the line being coincident with the direction of the force, and the number of linear units in the line being equal to the number of force units to be represented.

2. When many pressures act at the same time on a material particle, the result of their combined action is generally a definite pressure in a definite direction. This definite pressure is called the ***resultant*** of the acting or impressed pressures; and these latter, with reference to the resultant, are styled ***components.*** When the resultant is zero, the pressures are said to be in ***equilibrium;*** when the resultant of the given pressures is not zero, equilibrium may evidently be produced by introducing a new force which shall neutralize this resultant.

3. **Parallelogram of Forces.** — It is shown in elementary works on mechanics, that if two forces act upon a single point, and their intensities and directions be represented by two adjacent sides of a parallelogram, then the diagonal of the parallelogram drawn to the intersection of those two sides will represent, both in magnitude and direction, the resultant of the two given forces.

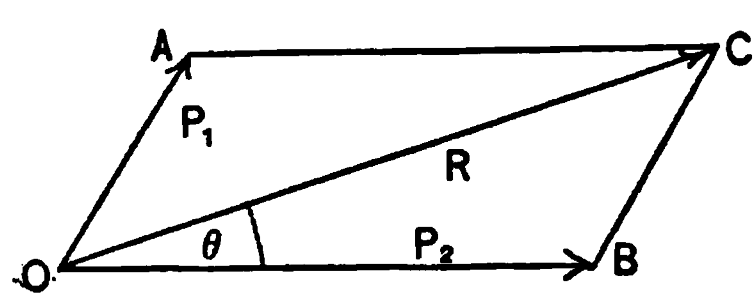


FIG. 1.

If P_1 and P_2, Fig. 1, are two forces acting at the point O, represented in magnitude and direction by the lines OA and OB, then, completing the parallelogram, $AOBC$, the resultant will be represented, in magnitude and direction, by the diagonal $OC = R$. When, therefore, a force is applied at O equal in intensity to R, and acting in the same line but in the opposite

direction, it will balance the given forces P_1, P_2, and the three forces will be in equilibrium.

4. Triangle of Forces. — Since, in Fig. 1, $AO = BC = P_1$, the three sides of the triangle BOC (or AOC) represent, in magnitude and direction, three forces, P_1, P_2, and R, which, acting in one plane on a given point, are in equilibrium; the direction of the forces being that of a point traversing the perimeter of the triangle. In this manner the value of the resultant may be constructed.

A formula for the value of R is found by solving the triangle of forces, where two sides and the angle included between them are given. Thus, if $c = AOB =$ the angle between the given lines of action of P_1 and P_2, we have, from the geometry of the figure, putting $BOC = \theta$ (*theta*),

$$R^2 = P_1^2 + P_2^2 + 2P_1P_2 \cos c. \qquad (1)$$

$$\text{Sin}\,\theta = \frac{P_1}{R} \sin c. \qquad (2)$$

Example. — Let $P_1 = 8$, $P_2 = 12$, $c = 75°$.
Then

$$R^2 = 8^2 + 12^2 + 2 \times 8 \times 12 \cos 75° = 208 + 192 \times 0.25882 = 257.6933.$$

$$R = 16.053.$$

$$\text{Sin}\,\theta = \frac{8}{16.053} \times 0.96593 = 0.48471.$$

$$\theta = 28°\ 59'\ 40''.$$

If the lines of action of the two forces, P_1, P_2, are at right angles to each other, $\cos c$ becomes zero, and equation (1) reduces to $R^2 = P_1^2 + P_2^2$, where R is the hypothenuse, and P_1 and P_2 are the other sides of a right-angled triangle.

EXAMPLE. — When $P_1 = 8$, and $P_2 = 12$,

$$R^2 = 8^2 + 12^2 = 208. \qquad R = 14.422.$$

In this case, Fig. 2, we have

$$\left.\begin{aligned} P_1 &= R \quad \sin\theta = P_2 \tan\theta. \\ P_2 &= R \quad \cos\theta = P_1 \cot\theta. \\ R &= P_1 \div \sin\theta = P_1 \operatorname{cosec}\theta. \\ R &= P_2 \div \cos\theta = P_2 \sec\theta. \end{aligned}\right\} \qquad (3)$$

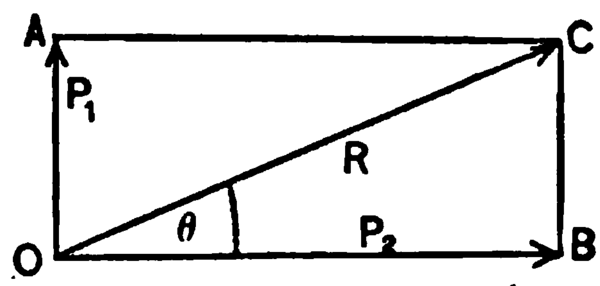


FIG. 2.

5. Resolution of a Force. — Conversely, any force, R, acting at a given point with known intensity and direction, may be resolved into two component forces acting at the same point, having definite intensities and directions. Manifestly also may each one of the two components be resolved into two components, and so on without limit.

EXAMPLE. — Resolve the force $R = 100$ tons, acting at the point O, Fig. 2, in the direction OC, into its horizontal and vertical components; θ being equal to $28° \, 59' \, 40''$.

From (3),

$$P_1 = R \sin\theta = 100 \times 0.48471 = 48.471 \text{ tons.}$$

$$P_2 = R \cos\theta = 100 \times 0.87467 = 87.467 \text{ tons.}$$

6. Resolution of Many Forces acting in One Plane at a Given Point. — Let there be any number of forces, P_1, P_2, P_3, etc., Fig. 3, acting in the plane of the axes XX', YY', at their point of intersection, O; and let α (*alpha*) symbolize the angle between the line of action of any force and the axis of x.

Resolving each force into its horizontal and vertical components, and calling the sum of the horizontal components X, and the sum of the vertical components Y, these results:

$$X = P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + P_3 \cos \alpha_3 + \ldots = \Sigma P \cos \alpha, \quad (4)$$

$$Y = P_1 \sin \alpha_1 + P_2 \sin \alpha_2 + P_3 \sin \alpha_3 + \ldots = \Sigma P \sin \alpha; \quad (5)$$

the symbol Σ (*sigma*) denoting the sum of the terms having the form $P \cos \alpha$ or $P \sin \alpha$.

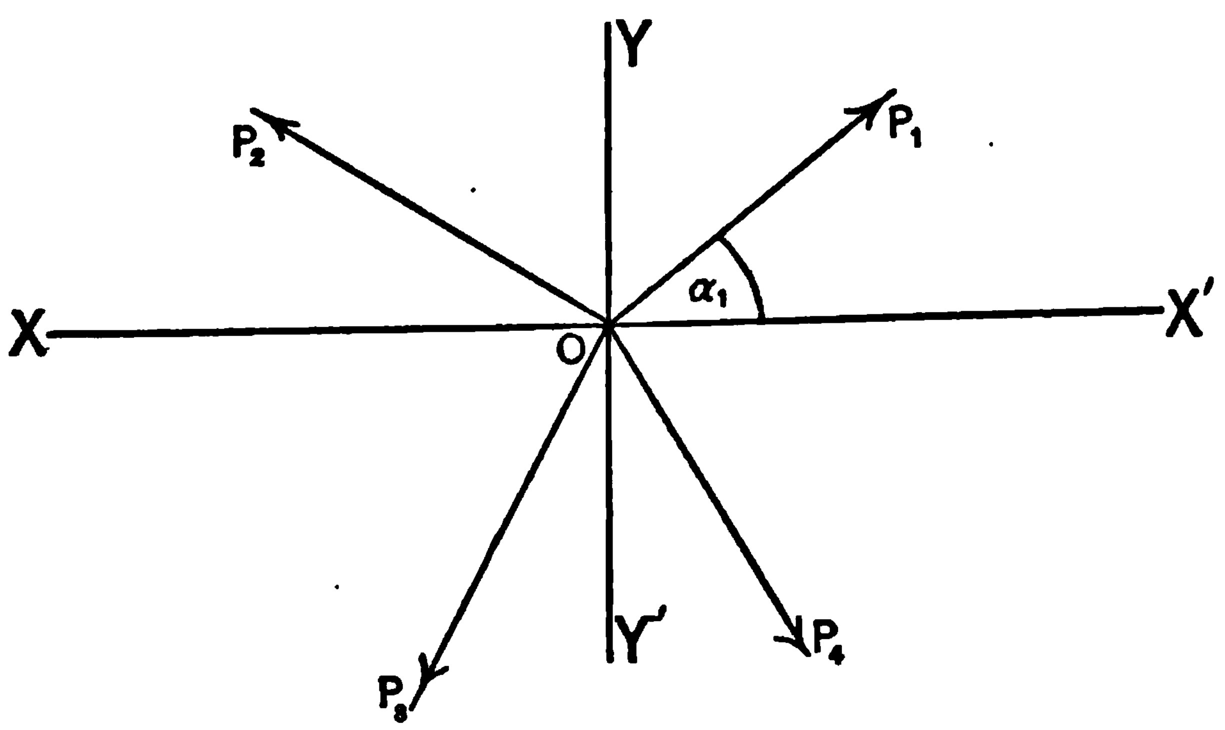


FIG. 3.

7. Thus, for all the given forces acting in their various directions on the point O have been substituted two other forces, X and Y, acting at the same point; the one horizontally, the other vertically, and in the plane of the original forces. Now, if R is the resultant of the two forces X and Y, it must also be the resultant of the forces P_1, P_2, P_3, etc.; and, θ being the angle between the resultant and the axis of x, we shall have

$$R \cos \theta = X = \Sigma P \cos \alpha, \quad (6)$$

$$R \sin \theta = Y = \Sigma P \sin \alpha, \quad (7)$$

$$\therefore R^2 = X^2 + Y^2, \quad (8)$$

$$\tan \theta = \frac{Y}{X}. \quad (9)$$

When the given forces are in equilibrium, the resultant vanishes, and

$$X = \Sigma P \cos \alpha = 0. \qquad (10)$$

$$Y = \Sigma P \sin \alpha = 0. \qquad (11)$$

EXAMPLE. — Let $P_1 = 10$ tons, $\alpha_1 = 40°$.
$P_2 = 20$ tons, $\alpha_2 = 150°$.
$P_3 = 30$ tons, $\alpha_3 = 250° = -110°$.
$P_4 = 40$ tons, $\alpha_4 = 300° = -60°$.

Required the intensity, R, and the direction, θ, of the resultant.

$$X = 10 \cos 40° + 20 \cos 150° + 30 \cos 250° + 40 \cos 300° = 10 \cos 40° - 20 \cos 30° - 30 \cos 70° + 40 \cos 60° = 10 \times 0.76604 - 20 \times 0.86603 - 30 \times 0.34202 + 40 \times 0.5 = 0.0792 \text{ tons.}$$

$$Y = 10 \sin 40° + 20 \sin 150° + 30 \sin 250° + 40 \sin 300° = 10 \sin 40° + 20 \sin 30° - 30 \sin 70° - 40 \sin 60° = 10 \times 0.64279 + 20 \times 0.5 - 30 \times 0.93969 - 40 \times 0.86603 = -46.404 \text{ tons.}$$

$$\tan \theta = \frac{-46.404}{0.0792} = -585.909, \qquad \theta = -89° \, 54' \, 8''.$$

$$R = [(0.0792)^2 + (-46.404)^2]^{\frac{1}{2}} = Y \div \sin \theta = 46.404065 \text{ tons.}$$

The resultant is therefore in the fourth quadrant, and makes an angle of $5' \, 52''$ with the axis of y.

This substitution of one force for many others is called the *composition* of forces.

8. Polygon of Forces. — Let S_1, S_2, S_3, etc., in Fig. 4, be the five sides of a closed polygon. Measure the inclination of each side to the horizon, as indicated in the figure, for c_1, c_2, etc.; then the sum of the horizontal projections of all the

sides is, in accordance with the trigonometrical signs of the cosine, found to be

$$\Sigma S \cos c = S_1 \cos c_1 + S_2 \cos c_2 + S_3 \cos c_3 + S_4 \cos c_4 + S_5 \cos c_5 = 0. \quad (12)$$

Since

$$S_1 \cos c_1 = +BC, \qquad S_2 \cos c_2 = -DE,$$
$$S_5 \cos c_5 = +AB, \qquad S_3 \cos c_3 = -FG,$$
$$S_4 \cos c_4 = -HI.$$

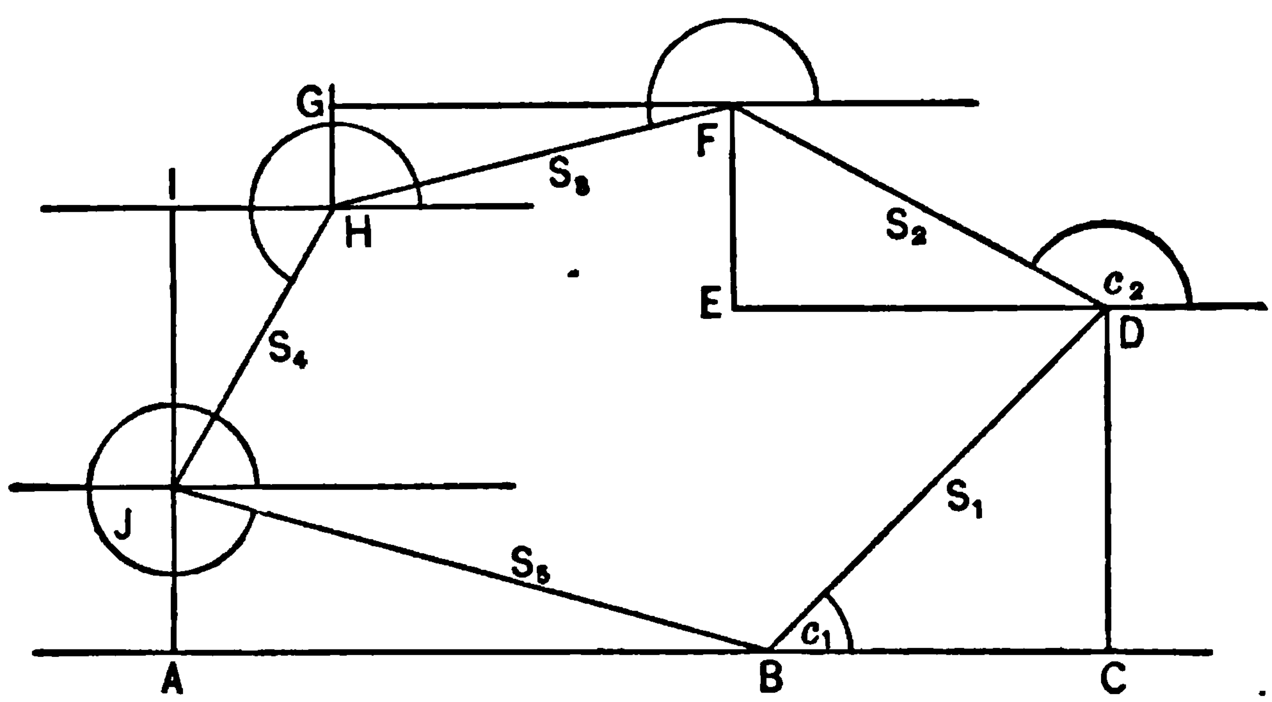


FIG. 4.

Equation (12) is true whatever be the number of sides of the polygon; and from its analogy to equation (10), viz.,

$$\Sigma P \cos \alpha = 0,$$

we may enunciate the proposition, that when any number of forces acting at the same point, with their lines of action in the same plane, are in equilibrium, then the given forces may be represented, in magnitude and direction, by the sides of a closed polygon; the direction being, for each side, that of a point traversing the perimeter.

This proposition enables us to construct the resultant of many forces acting on a point in the common plane of their

lines of action, by regarding the unknown resultant, with its direction changed, as the side required to complete or close the polygonal figure due to the given forces.

EXAMPLE. — Let us apply this proposition to the example of Art. 7.

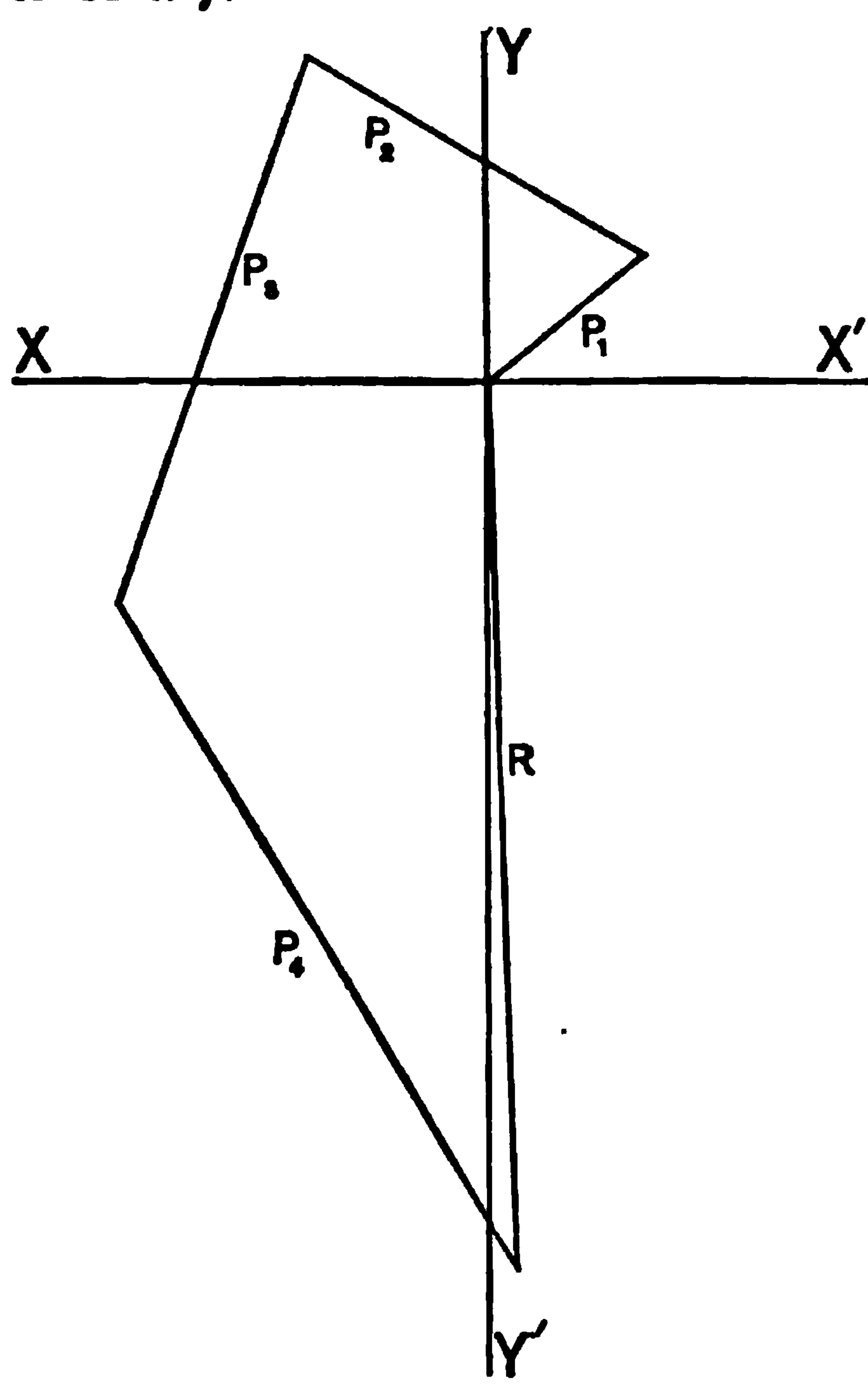


FIG. 5.

This solution consists simply in drawing (Fig. 5) a continuous figure made up of lines proportional to the given forces and respectively parallel to their given lines of action, and each force having the direction a point would take in traversing the broken line from end to end. The straight line joining the two ends of this broken line will be the resultant sought, with its direction reversed.

It will be seen that the values of Y and R in this example are very nearly equal, and that the solution by construction can show an appreciable difference in them, only when a large scale is used. In practice, however, either solution is accurate enough; and one serves to check the other.

The triangle of forces is a particular case of the closed polygon.

CHAPTER II.

MOMENT OF A FORCE.

9. The moment of a force is the effect of the force's effort to turn the body on which it acts about a given point, and is measured by the number expressing the force, multiplied by the number denoting the perpendicular distance from the given point to the line of action of the force.

Moments, therefore, may be added and subtracted, and represented by lines, like other numbers.

Since the unit of the force and the unit of the perpendicular distance are arbitrary, it is usual to express the moment as a denominate number, designating both the units. Thus, 20 foot-tons, or ton-feet, means that the moment 20 is equivalent to the effect of a force or pressure of 20 tons acting at the perpendicular distance, or lever arm, of 1 foot from the axis of rotation, or to a force of 1 ton acting at the distance of 20 feet from the same axis.

It is plain that the moment of a given force acting at a given perpendicular distance from the axis of rotation may be replaced by any one of an infinite number of equivalent moments.

10. In former articles forces have been considered as acting in straight lines in one plane and on a single point; tending in their united action, when the resultant does not vanish, to move that point or material particle in the direction of the resultant. Hence such forces are termed forces of translation.

But, in the case now under consideration, we have two forces in one plane acting at two points in a rigid body, the one force at one point tending to turn the body about the other point.

I say *two* forces, for it is manifest, that, at the point which is taken as the centre of rotation, there must be a resistance to motion equal and opposite to the rotatory or tangential force acting at the other point. Such a system of two parallel forces acting in opposite directions is called a *couple*, and the perpendicular drawn to the lines of action of the forces is called the *arm* of the couple.

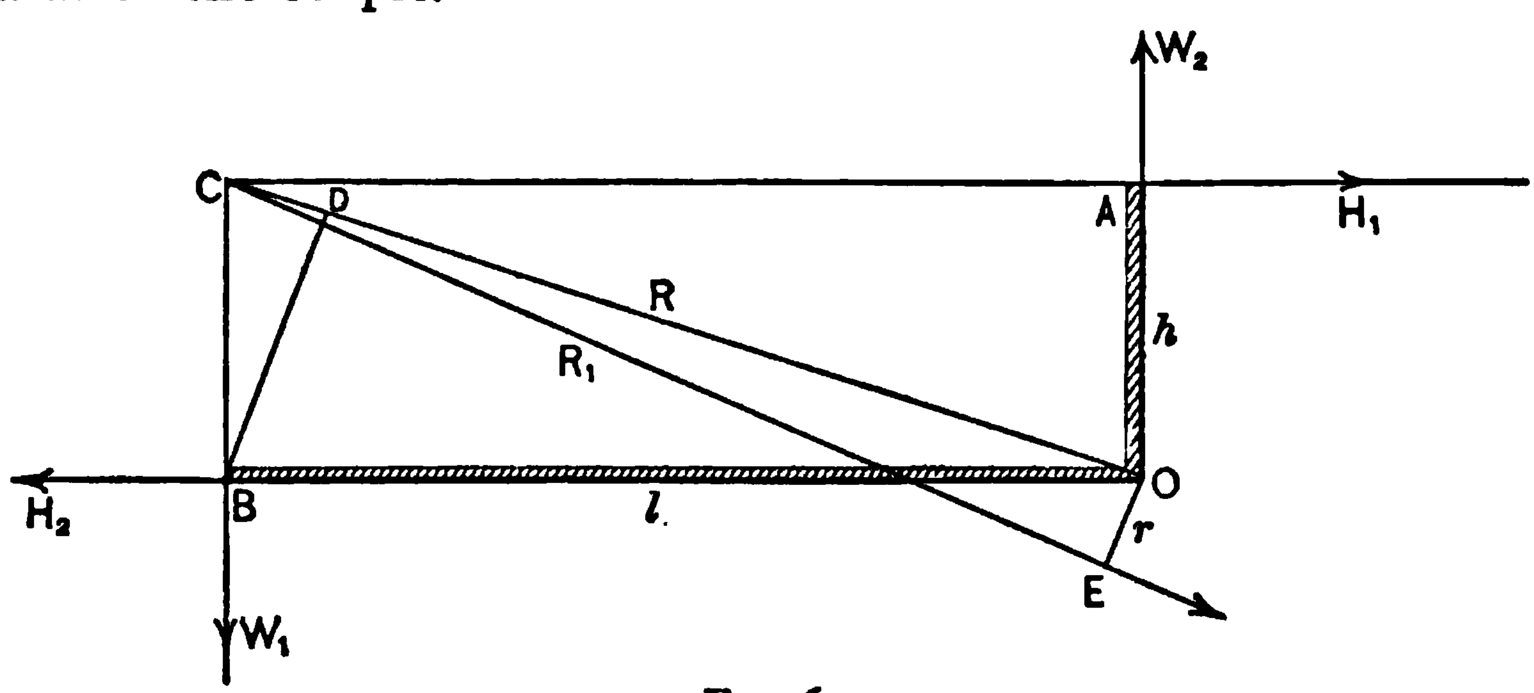


FIG. 6.

In Fig. 6, let AOB be a rigid body, beam, or bent lever, whose weight is not now to be taken into account; and let AO be perpendicular to OB, O being a fixed point about which the applied forces, W_1 and H_1, acting at right angles to their respective lever arms, tend to turn the beam. Take $OB = l$, and $OA = h$, where l and h represent each a number of linear units. Then the moment of the force W_1 is $-W_1l$, and the moment resulting from the action of H_1 is H_1h; giving them different signs because they tend to turn the beam in opposite directions about the point O.

If these two moments are equal, we have

$$H_1h - W_1l = 0, \qquad (13)$$

which shows that there is no rotation about the point O, and that the three forces W_{I}, H_{I}, and the resistance to translation offered at O, are in equilibrium.

At the fixed point O are developed two forces: the one, W_2, equal to W_{I}, but acting in the opposite direction, OA; the other, H_2, equal to H_{I}, acting at O in the direction OB Now, these two forces, acting at the same point, O, must, by Art. 3, have a resultant equal to the diagonal of the rectangle of which W_2 and H_2 are the adjacent sides. Therefore the resultant $R = \sqrt{W_2^2 + H_2^2}$, and the tangent of the angle between the line of R and the line OB is $\tan\theta = \dfrac{W_2}{H_2} = \dfrac{h}{l}$, in the case of equilibrium, or when the two forces are inversely proportional to their lever arms, as shown in equation (13).

This resultant, R, is a force of translation, and may be graphically found by producing the lines of action of W_1 and H_1 till they intersect at C; then, if AC represent the intensity of H_{I}, and BC the intensity of W_{I}, we have, from Art. 4, $R = OC$, the diagonal of the rectangle.

Otherwise, graphically, draw BD perpendicular to OC; then, if W_1 and H_1 be resolved, each into one component along OC and one at right angles to OC, we have

$$\begin{aligned} &\text{Components of } H_1 = DO \text{ and } BD, \\ &\text{Components of } W_1 = CD \text{ and } DB. \\ \therefore\ & DO + CD = R = \text{pressure at } O, \\ & BD - DB = 0 = \text{rotatory effect.} \end{aligned}$$

If we suppose the rigid body extended so as to fill the space $AOBC$, then the resultant may be conceived as acting at any point in the line OC without altering its effect of translation on the whole mass. The effect within the body will, of course, be different for every new point of application. With this we are not now concerned.

We conclude, then, that if two forces whose lines of action are in the same plane act on a rigid body, and if from any point in the line of action of their resultant, perpendiculars be drawn to the lines of action of the forces, then the resistance at the point chosen, and the two given forces, will be in equilibrium, when the intensity and direction of the resistance are respectively equal and opposite to those of the resultant.

This conclusion may also be drawn from the figure, since two lines drawn from any point in OC, respectively perpendicular to the lines of action of W_1 and H_1, must be proportional to l and h, and therefore equation (13) would be satisfied, whatever be the angle AOB.

If the two moments, W_1l and H_1h, are not equal, let us suppose that W_1l is the greater by reason of an increment given to W_1, so that l, h, and H_1 remain unaltered. Then the resultant of the forces H_1 and W_1 will not pass through the point O, but will lie somewhere between it and the line of action of the augmented force W_1.

Suppose CE to be the line of action of the new resultant R_1, and draw OE perpendicular to CE; then will $R_1 \times OE$ represent the total rotatory effect of the given pressures W_1 and H_1 with respect to the point O, and we shall have, if $r = EO$,

$$-W_1l + H_1h = -R_1r, \qquad (14)$$

where $-R_1r$ is the moment of a couple, equivalent to the difference or algebraic sum of the moments of the couples whose arms are h and l.

We see, then, that the effect of one couple may be neutralized by the moment of another couple having the same axis of rotation and an opposite direction, and that the combined effort of two couples may be balanced by the moment of a single couple having the same centre of rotation.

11. The law may clearly be extended to any number of

forces, P_1, P_2, P_3, etc., acting in one plane to turn a rigid body about a fixed point in that plane, or about a fixed axis perpendicular to that plane. Let p_1, p_2, p_3, etc., be the lengths of the perpendiculars drawn from the fixed centre of rotation to the respective lines of action of P_1, P_2, P_3, etc. Let R be the resultant of translation of all the forces, and r the length of the perpendicular drawn from the same centre to the line of action of R; then

$$Rr = P_1p_1 + P_2p_2 + P_3p_3 + \text{etc.} = \Sigma Pp. \qquad (15)$$

The algebraic signs of the terms in this equation will depend upon the directions in which the forces tend to turn the rigid body; and it will be convenient to distinguish moments as positive which tend to turn the body in the direction taken by the hands of a watch, and to call moments having the opposite tendency negative.

For equilibrium we must have

$$\Sigma Pp = Rr = O. \qquad (16)$$

Equation (16) is satisfied either when R, the resultant of the given forces, becomes zero, or when r, the arm of the resultant couple, vanishes. In the former case the given impressed forces are in equilibrium among themselves; in the latter case, if R does not also vanish, it is equal and opposite to the resistance offered at the fixed point.

As in the case of two forces, each acting tangentially at one extremity of its lever arm to cause rotation about the fixed point common to the other extremities, so in the case of many forces acting in one plane on a rigid body, and tending to turn it about a fixed point, the resistance developed at the fixed point by each of the given forces will be equal and opposite to the given force, and will have its line of action parallel to that of the given force.

Hence the intensity and direction of the resultant, R, may be found from equations (4), (5), (8), and (9), as in the case of many forces acting in one plane at a common point.

12. And, having found R, equation (15) gives

$$r = \Sigma Pp \div R. \qquad (17)$$

If, then, through the fixed point a line be drawn parallel to the line of action of R, and through the same point another line be drawn at right angles to this line of action, and if on this second line the distance, r, be laid off from the fixed point, and R, both in magnitude and direction, be applied at the outer extremity of r, we shall have a graphical representation of the resultant couple whose moment is equivalent to the combined action of all the given forces.

13. The direction of R and the sign of Rr will show on which side of the fixed point r must be laid off.

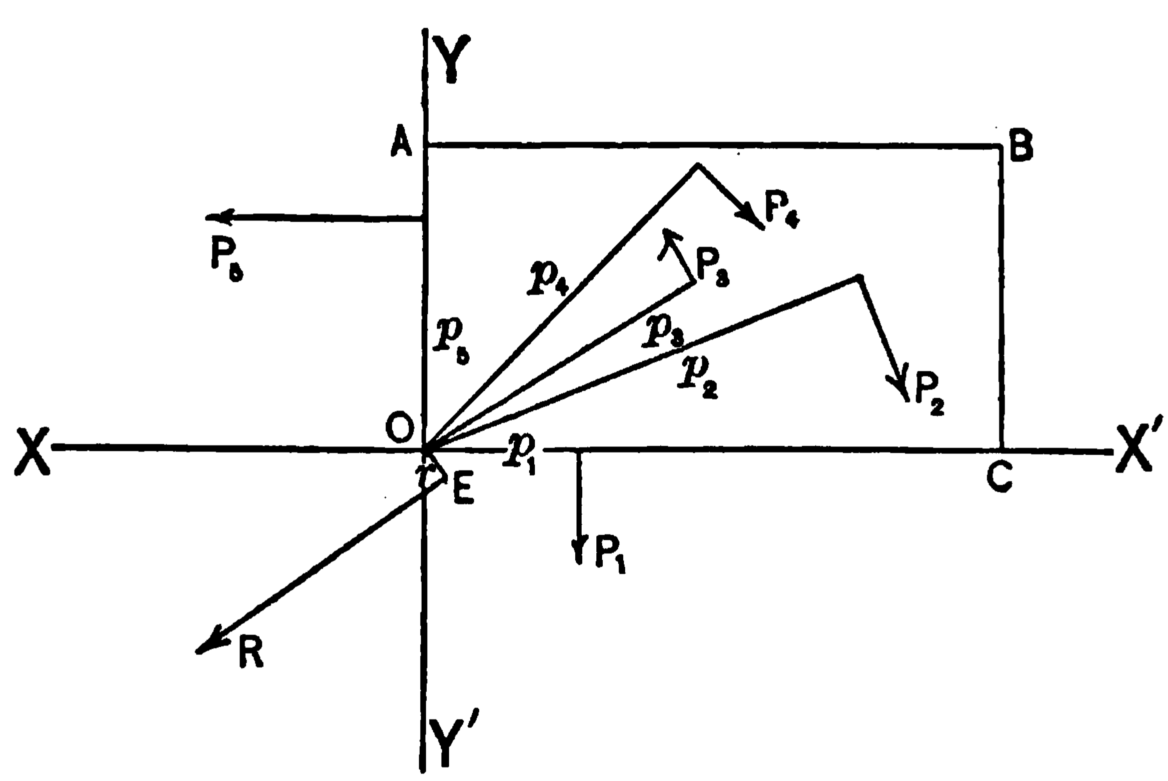


FIG. 7.

EXAMPLE. — Let $ABCO$ be a rigid body acted on by five forces, whose lines of action are in one plane, and which tend to turn the body about the fixed point O, Fig. 7.

Let the directions of the forces P_1, P_2, P_3, etc., and their points of application, be as indicated in the figure; and designate the angle between the line of action of any force and the axis of x by α, and the distance of O from any point of application by p. Take

$$\begin{array}{lll} P_1 = \ \ 10 \text{ tons}, & p_1 = \ \ 4 \text{ feet}, & \alpha_1 = 270° \text{ or } -90°. \\ P_2 = 100 \text{ tons}, & p_2 = 12 \text{ feet}, & \alpha_2 = 290° \text{ or } -70°. \\ P_3 = \ \ 20 \text{ tons}, & p_3 = \ \ 8 \text{ feet}, & \alpha_3 = 120°. \\ P_4 = \ \ 30 \text{ tons}, & p_4 = 10 \text{ feet}, & \alpha_4 = 315° \text{ or } -45°. \\ P_5 = 200 \text{ tons}, & p_5 = \ \ 6 \text{ feet}, & \alpha_5 = 180°. \end{array}$$

To find R, let these forces be considered as acting at the point O in direct opposition to the resistances there developed; then, by equations (4), (5), (8), and (9), we have

$$\begin{aligned} X = {} & 10 \cos 270° + 100 \cos 290° + 20 \cos 120° + 30 \cos 315° \\ & + 200 \cos 180° = -10 \cos 90° + 100 \cos(-70°) - 20 \cos 60° \\ & + 30 \cos(-45°) - 200 \cos 0° = 10 \times 0 + 100 \times 0.34202 \\ & - 20 \times 0.5 + 30 \times 0.70711 - 200 \times 1 = 0 + 34.202 - 10 \\ & + 21.2133 - 200 = -154.5847. \end{aligned}$$

$$\begin{aligned} Y = {} & 10 \sin 270° + 100 \sin 290° + 20 \sin 120° + 30 \sin 315° + 200 \sin 180° \\ & = 10 \times -1 + 100 \times -0.93969 + 20 \times 0.86603 + 30 \times -0.70711 \\ & + 200 \times 0 = -10 - 93.969 + 17.3206 - 21.2133 + 0 \\ & = -107.8617. \end{aligned}$$

$$R = \sqrt{(-154.5847)^2 + (-107.8617)^2} = 188.496 \text{ tons}.$$

$$\tan \theta = \frac{-107.8617}{-154.5847} = 0.697753.$$

$\theta = 34° \ 54' \ 20''$; or, since X and Y are both negative, we must have

$$\theta = 214° \ 54' \ 20'',$$

and the resultant is therefore in the third quadrant.

From equation (15),

$$Rr = \Sigma Pp = +10 \times 4 + 100 \times 12 - 20 \times 8 + 30 \times 10 - 200 \times 6$$
$$= 40 + 1200 - 160 + 300 - 1200 = +180 \text{ foot-tons},$$

$$\therefore \quad r = \frac{180}{188.496} = 0.95493 \text{ foot.}$$

Since the product Rr is positive, and the direction of the line of R, when drawn through the fixed point, is into the third quadrant, it follows that r must be laid off below the fixed point on the perpendicular to the line of R, as shown by OE in the figure; E being supposed rigidly connected with the solid $AOCB$.

CHAPTER III.

MOMENTS OF THE EXTERNAL FORCES APPLIED TO A BEAM OR GIRDER.

SECTION I.

The Semi-Beam, or Girder fixed at One End and free at the Other.

14. We can now find expressions for the moments developed in any section of a beam or girder, by the action of any forces in the plane of the beam, in whatsoever manner applied

Let us first take a beam fixed at one end and free at the other, or a *semi-beam* as it is called. Let $EOAB$, Fig. 8, represent a beam fixed to a wall along the line $AB = h$. Suppose the weight of the beam to be w pounds for every unit of its length $l = AO$. Assume, also, that the length $b = DC$ has an additional uniform load of w' pounds per linear unit, both w and w' being continuously distributed throughout their respective lengths. Also let W be a concentrated weight or pressure at the distance $d' = BJ$ from the fixed end of the beam. Let $BC = a =$ the distance from the wall to the nearer end of the uniform load $w'b$. Let P be any pressure acting at any point, G, with any inclination, α, to the arm FG; and call the horizontal distance of the point G from the wall $d'' = AK$.

Suppose the beam horizontal, and all the applied pressures, except P, vertical. Let VS be any vertical section of the beam at the distance x from the fixed end AB. It is required to find the moment of the applied forces which must be resisted by the internal forces of the beam at the section VS.

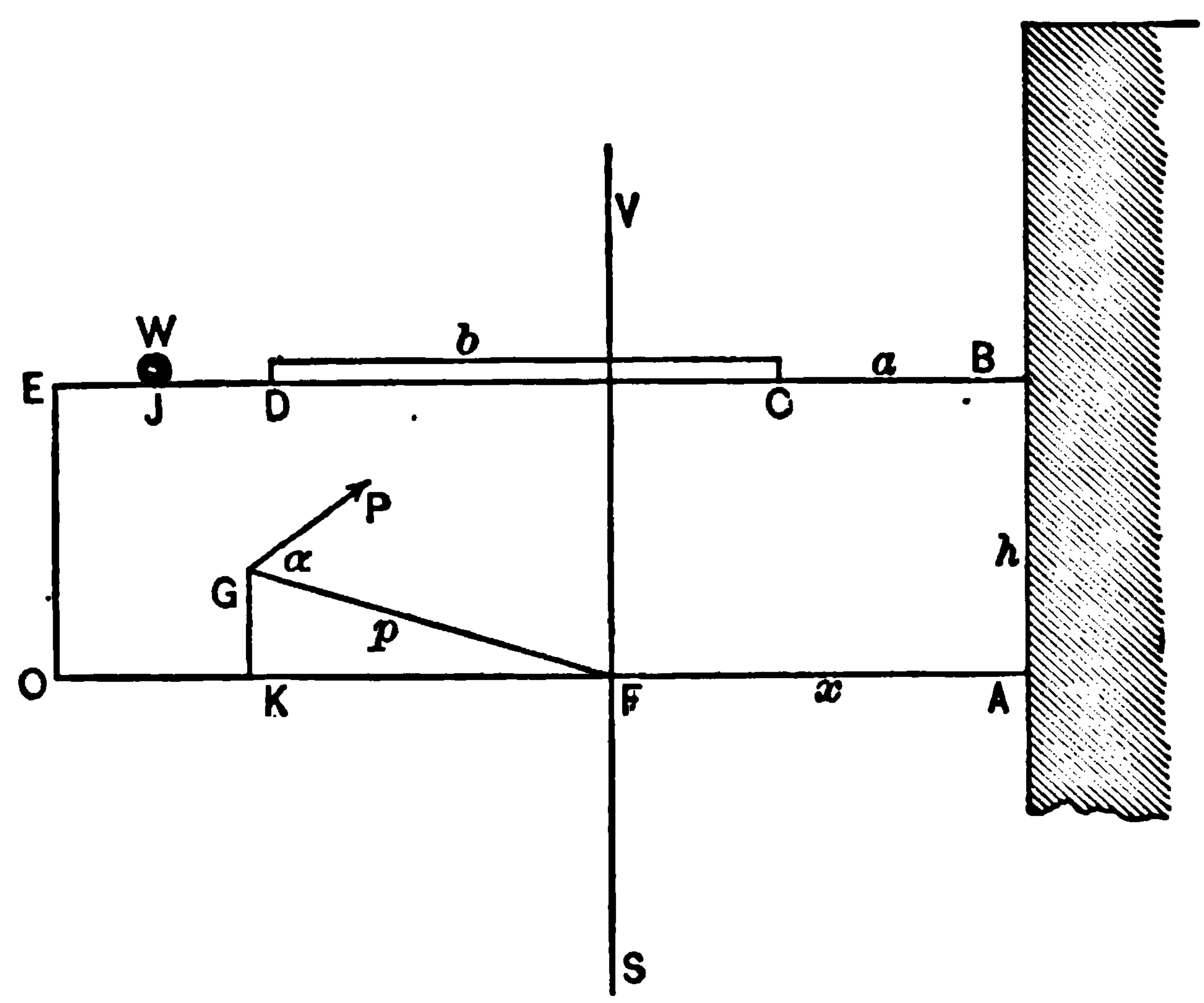


FIG. 8.

Manifestly only the pressures at the left of VS affect that section. Taking the moments of these sinister pressures about any point, F, in the vertical section VS, and remembering that downward pressures on the left of VS give negative moments, we have the following equations for the required moment M:—

SEMI-GIRDER. LENGTH = l. (See Fig. 8.)

Load.	Conditions.	Force left of VS.	Arm.	Moments about F.	
W	$x < a'$	W	$a' - x$	$M = -W(a' - x)$.	(18)
W	$x =$ or $> a'$	0		$M = 0$.	(19)
W	$x = 0$	W	a'	$M = -Wa'$.	(20)
W	$a' = l$	W	$l - x$	$M = -W(l - x)$.	(21)
W	$x = 0, a' = l$	W	l	$M = -Wl$ (max.).	(22)
wl	$x < l$	$w(l - x)$	$\frac{1}{2}(l - x)$	$M = -\frac{1}{2}w(l - x)^2$.	(23)
wl	$x = l$	0		$M = 0$.	(24)
wl	$x = 0$	wl	$\frac{1}{2}l$	$M = -\frac{1}{2}wl^2$ (max.).	(25)
$w'b$	$x > a$ and $< (a+b)$	$w'(a+b-x)$	$\frac{1}{2}(a+b-x)$	$M = -\frac{1}{2}w'(a + b - x)^2$.	(26)
$w'b$	$a = 0$	$w'(b - x)$	$\frac{1}{2}(b - x)$	$M = -\frac{1}{2}w'(b - x)^2$.	(27)
$w'b$	$a = 0, b = l$	$w'(l - x)$	$\frac{1}{2}(l - x)$	$M = -\frac{1}{2}w'(l - x)^2$.	(28)
$w'b$	$x =$ or $< a$	$w'b$	$\frac{1}{2}b + a - x$	$M = -w'b(\frac{1}{2}b + a - x)$.	(29)
$w'b$	$x = 0, a = 0, b = l$	$w'l$	$\frac{1}{2}l$	$M = -\frac{1}{2}w'l^2$ (max.).	(30)
P	$x < a''$	$P \sin a$	p	$M = P \sin ap$.	(31)

15. In applying these formulæ for W to the case of many equal weights placed at equal intervals along the beam, we may simplify the numerical computations by first summing the series resulting from assigning to a' and x their proper values.

Suppose we have n weights, each equal to W, at intervals of $\frac{l}{n}$ feet along the beam; then

The moment at the fixed end of the beam, or when $x = 0$, due to all of the equal weights is, by summing (20),

$$\left.\begin{aligned} M = -W\Sigma a' &= -Wl\left(\frac{1}{n} + \frac{2}{n} + \frac{3}{n} + \ldots \frac{n}{n}\right) \\ &= -Wl\frac{n+1}{2}. \end{aligned}\right\} \qquad (32)$$

The moment at the fixed end due to 1, 2, 3, ... r, of these equal weights, first in order is

$$\left.\begin{aligned} M = -W\Sigma d' &= -Wl\left(\frac{1}{n}+\frac{2}{n}+\frac{3}{n}+\ldots\frac{r}{n}\right) \\ &= -Wl\frac{r(r+1)}{2n}. \end{aligned}\right\} \qquad (33)$$

The moment at the fixed end due to the remaining $(n-r)$ of these equal weights is

$$\left.\begin{aligned} M = -W\Sigma d' &= -Wl\left(\frac{r+1}{n}+\frac{r+2}{n}+\frac{r+3}{n}+\ldots\frac{n}{n}\right) \\ &= -Wl\left(\frac{n+1}{2}-\frac{r(r+1)}{2n}\right). \end{aligned}\right\} \qquad (34)$$

The moment at the interval r due to these remaining $(n-r)$ equal weights is

$$\left.\begin{aligned} M = -W\Sigma d' &= -Wl\left(\frac{1}{n}+\frac{2}{n}+\frac{3}{n}+\ldots\frac{n-r}{n}\right) \\ &= -Wl\frac{(n-r+1)(n-r)}{2n}. \end{aligned}\right\} \qquad (35)$$

EXAMPLE 1. — A semi-girder projects 50 feet, and is loaded at intervals of 10 feet with a weight of 10 tons; required the moment at the fixed end due to the 5 equal weights.

From (32),

$$M = -Wl\frac{n+1}{2} = -10 \times 50 \times \frac{6}{2} = -1500 \text{ foot-tons.}$$

EXAMPLE 2. — The same conditions continuing, required the moment at the fixed end due to the first 3 of the weights.

From (33),

$$M = -Wl\frac{r(r+1)}{2n} = -10 \times 50 \times \frac{3(3+1)}{2 \times 5} = -600 \text{ foot-tons.}$$

EXAMPLE 3. — With same conditions of beam and load, required the moment due, at the fixed end, from the remaining 2 weights.

From (34),

$$M = -Wl\left(\frac{5+1}{2} - \frac{3(3+1)}{2 \times 5}\right) = -900 \text{ foot-tons.}$$

EXAMPLE 4. — At 10 feet from the fixed end of the beam, what is the moment due to the 4 weights beyond?

From (35),

$$M = -Wl\frac{(5-1+1)(5-1)}{2 \times 5} = -1000 \text{ foot-tons.}$$

EXAMPLE 5. — If the given semi-beam weighs 0.8 ton to the linear foot, what is the moment at its centre and at its fixed end?

From (23), if $x = \frac{1}{2}l$,

$$M = -\tfrac{1}{2}w(\tfrac{1}{2}l)^2 = \tfrac{1}{8} \times 0.8 \times 50^2 = -250 \text{ foot-tons.}$$

From (25),

$$M = -\tfrac{1}{2} \times 0.8 \times 50^2 = -1000 \text{ foot-tons.}$$

EXAMPLE 6. — Suppose the same beam to be covered with the uniform load 0.6 ton for the space of 15 feet, beginning 25 feet from the fixed end; required the moment due to this load at 30 feet from the fixed end.

Here

$$w' = 0.6, \qquad b = 15, \qquad a = 25, \qquad x = 30.$$

From (26),

$$M = -\tfrac{1}{2} \times 0.6(25 + 15 - 30)^2 = -30 \text{ foot-tons.}$$

EXAMPLE 7. — If the load 0.6 ton per foot covers the first 35 feet of the beam, and the moment at 10 feet is required, we have $b = 35$, $a = 0$, $x = 10$; and, from (27),

$$M = -\tfrac{1}{2} \times 0.6(35 - 10)^2 = -187.5 \text{ foot-tons.}$$

EXAMPLE 8. — If the load 0.6 ton per foot covers the entire beam, the moment at the centre is, from (28),

$$M = -\tfrac{1}{2} \times 0.6 \times (50 - 25)^2 = -187.5 \text{ foot-tons,}$$

and at the fixed end

$$M = -\tfrac{1}{2} \times 0.6 \times 50^2 = -750 \text{ foot-tons.}$$

EXAMPLE 9. — If the uniform load 0.6 ton covers 40 feet of the beam, beginning at the free end, then the moment at 5 feet from the fixed end is, from (29),

$$M = -0.6 \times 40(\tfrac{1}{2} \times 40 + 10 - 5) = -600 \text{ foot-tons.}$$

EXAMPLE 10. — If the force P, Fig. 8, is 4 tons, and its line of action makes an angle of 30° with the line $GF = P = 20$ feet, then the moment due to P at the point F is, from (31),

$$M = 4 \times 0.5 \times 20 = 40 \text{ foot-tons.}$$

16. If there are several concentrated weights, W_1, W_2, W_3, etc., or pressures, P_1, P_2, P_3, etc., or detached uniform loads, b_1w_1', b_2w_2', b_3w_3', etc., at different points on the left of the section VS, we must evidently sum the moments due to the separate pressures for the total moment.

Thus we may write

$$M_x = -\Sigma W(d' - x) - \tfrac{1}{2}w(l - x)^2 - \tfrac{1}{2}w'(a + b - x)^2 \\ - \Sigma w'(\tfrac{1}{2}b + a - x)b + \Sigma P \sin a\phi, \quad (36)$$

where M_x is the moment, with reference to any point of any vertical section of a semi-beam, due to all the forces applied to the beam between its free end and the given vertical section.

It should be observed, that, for all pressures whose lines of action are vertical, the moments will be the same, whatever point of reference is taken in the vertical section VS; for such pressures have no horizontal component.

Section 2.

17. We next take a beam or girder, horizontal, supported at its ends, and loaded in any manner whatsoever. Such a girder is also said to have its ends *free;* since they simply rest upon two level supports, and are fixed in no other manner.

Fig. 9.

Let the beam $OABE$, Fig. 9, be supported at the two points O and A, and be subjected to the following pressures:—

w = weight of beam per linear unit.
w' = uniform load per linear unit of the length $b = CD$.
W = a concentrated weight or vertical pressure at any point, K.
P = any force at distance $d'' = OF$ from O.
V_1 = vertical re-action of the left support.
V_2 = vertical re-action of the right support.
l = length of girder.
h = height of girder.
a = OD = the horizontal distance from the origin O to the nearer end of the uniform load bw'.
d' = the horizontal distance from O to the weight W.
x = the horizontal distance of any vertical section, VS, from O, the origin of co-ordinates.
y = the vertical distance of any point in the section VS from the horizontal line AO.

The vertical section VS is made by any plane cutting the beam perpendicular to the line AO.

It is required to find the moment generated at any vertical section, VS, by the action of each of the given pressures.

Since at any given cross-section, there can be but *one* moment due to the given simultaneous pressures, it follows that we may determine this moment, either by using the pressures applied upon the left side of the given section, or by using the applied pressures on the right side of the same section.

In the following table we use the pressures that act on the *left* of the section VS; and consequently downward pressures give negative moments, and upward pressures give positive moments, in accordance with our previous notation.

18. The sum of the re-actions V_1 and V_2 for the simple girder with free ends is equal to the total weight of the girder and its load.

The resistances V_1 and V_2 due to any concentrated weight, W, are, since there can be but one moment for the vertical section through W, inversely proportional to the horizontal distances of W from the points of support; and we have, from equation (13),

$$M = V_1 d' = V_2(l - d'),$$

$$\therefore \quad V_1 = V_2 \frac{l - d'}{d'} = W - V_2, \tag{37}$$

$$\therefore \quad V_2 = W \frac{d'}{l}. \tag{38}$$

$$V_1 = W \frac{l - d'}{l}. \tag{39}$$

Or, by proportion,

$$V_1 : V_2 :: l - d' : d',$$

$$\therefore \quad V_1 + V_2 : V_1 :: l : l - d', \quad \therefore \quad V_1 = W \frac{l - d'}{l}.$$

$$V_1 + V_2 : V_2 :: l : d', \qquad \therefore \quad V_2 = W \frac{d'}{l}.$$

Similarly, for the uniform load bw', the re-actions V_1 and V_2 will be inversely proportional to the distances of the centre of gravity of the uniform load from the points of support.

19. In the following table we have,—

First column, load whose moment is sought.

Second column, re-action at left support, giving $+M$.

Third column, conditions of load and plane VS.

Fourth column, part of load on left of VS, giving $-M$.

Fifth column, arm of V_1.

Sixth column, arm of load on left of VS.

Beam Supported at Both Ends. Moments at any Section.

Load.	Re-action V_1.	Conditions.	Load left of VS.	Arms. V_1.	Arms. Load.	Distance of Section from Left End = x.	
W	$W\frac{l-a'}{l}$	$x < a'$	0	x		$M = W\frac{l-a'}{l}x.$	(40)
W	$W\frac{l-a'}{l}$	$x = 0$	0	0		$M = 0.$	(41)
W	$W\frac{l-a'}{l}$	$x = a'$	0	a'		$M = W\frac{l-a'}{l}a'.$	(42)
W	$W\frac{l-a'}{l}$	$x > a'$	W	x	$x-a'$	$M = W\frac{l-a'}{l}x - W(x-a') = W\frac{l-x}{l}a'.$	(43)
W	$W\frac{l-a'}{l}$	$x = l$	W	l	$l-a'$	$M = 0.$	(44)
W	$\frac{1}{2}W$	$x > a', a' = \frac{1}{2}l$	0	x		$M = \frac{1}{2}Wx.$	(45)
W	$\frac{1}{2}W$	$x = a' = \frac{1}{2}l$	0	$\frac{1}{2}l$		$M = \frac{1}{4}Wl$ (max.).	(46)
$P \sin a$	$-P \sin a \frac{l-a''}{l}$	$x \leqq a''$	0	x	0	$M = -P \sin a \frac{l-a''}{l}x.$	(47)
$P \sin a$	$-P \sin a \frac{l-a''}{l}$	$x \geqq a''$	$P \sin a$	x	$x-a''$	$M = -P \sin a \frac{l-a''}{l}x + P \sin a(x-a'').$	(48)
wl	$\frac{1}{2}wl$		wx	x	$\frac{1}{2}x$	$M = \frac{1}{2}wlx - \frac{1}{2}wx^2 = \frac{1}{2}w(l-x)x.$	(49)
wl	$\frac{1}{2}wl$	$x = 0$	0	0		$M = 0.$	(50)
wl	$\frac{1}{2}wl$	$x = l$	wl	l	$\frac{1}{2}l$	$M = 0.$	(51)
wl	$\frac{1}{2}wl$	$x = \frac{1}{2}l$	$\frac{1}{2}wl$	$\frac{1}{2}l$	$\frac{1}{4}l$	$M = \frac{1}{4}wl^2 - \frac{1}{8}wl^2 = \frac{1}{8}wl^2$ (max.).	(52)
$w'b$	$w'b\frac{l-a-\frac{1}{2}b}{l}$	$x < a$	0	x		$M = w'b\frac{l-a-\frac{1}{2}b}{l}x.$	(53)
$w'b$	$w'b\frac{l-a-\frac{1}{2}b}{l}$	$x = a$	0	a		$M = w'b\frac{l-a-\frac{1}{2}b}{l}a.$	(54)
$w'b$	$w'b\frac{l-a-\frac{1}{2}b}{l}$	$x > a, x < (a+b)$	$w'(x-a)$	x	$\frac{1}{2}(x-a)$	$M = w'b\frac{l-a-\frac{1}{2}b}{l}x - \frac{1}{2}w'(x-a)^2.$	(55)
$w'b$	$w'b\frac{l-\frac{1}{2}b}{l}$	$a = 0, x \leqq b$	$w'x$	x	$\frac{1}{2}x$	$M = w'b\frac{l-\frac{1}{2}b}{l}x - \frac{1}{2}w'x^2.$	(56)
$w'b$	$w'b\frac{l-a-\frac{1}{2}b}{l}$	$x > (a+b)$	$w'b$	x	$x-a-\frac{1}{2}b$	$M = w'b\frac{l-a-\frac{1}{2}b}{l}x - w'b(x-a-\frac{1}{2}b) = w'b(a+\frac{1}{2}b)\frac{l-x}{l}.$	(57)
$w'b$	$\frac{1}{2}w'\frac{b^2}{l}$	$x > a, b = l-a$	$w'(x-a)$	x	$\frac{1}{2}(x-a)$	$M = \frac{1}{2}w'\frac{b^2x}{l} - \frac{1}{2}w'(x-a)^2 = \frac{1}{2}w'\left[\frac{b^2x}{l} - (x-a)^2\right].$	(58)
$w'b$	$\frac{1}{2}w'l$	$x = \frac{1}{2}l, b = l$	$\frac{1}{2}w'l$	$\frac{1}{2}l$	$\frac{1}{4}l$	$M = \frac{1}{4}w'l^2 - \frac{1}{8}w'l^2 = \frac{1}{8}w'l^2$ (max.).	(59)

20. Moments due Uniform Discontinuous Load on any Part of the Beam. — Let $r_1 - r_2$ denote the number of equal weights, W, at equal intervals, c, between any two consecutive weights on the whole or any part of the girder. We may shorten the numerical computation of moments, as in case of the semi-girder, by first summing the series that arises in the expression for M.

For this purpose let $r - r_2 =$ the number of equal weights, W, on the length x; $(r_2 + 1)c =$ the distance from the left end of the beam to the nearest weight. If this distance is less than c, that is, not a full interval, it follows that r_2 will be a negative proper fraction. Now $(r_1 - r_2) - (r - r_2) = r_1 - r =$ the number of equal weights between the point x and the right end of the beam. The three differences, $r_1 - r_2$, $r - r_2$, and $r_1 - r$, must be integers, since each denotes a number of equal weights. If one of the three quantities r, r_1, r_2, is not an integer, neither of the other two is an integer, and the decimal part of each is the same, except that, when r_2 is negative, its value is less by unity than the common decimal part of r and r_1.

Let us first find the moment due $r - r_2$ equal weights, W, at any point, x, between the last weight and right-hand end of the girder. We use equation (43), giving to d' the successive values $c(r_2 + 1)$, $c(r_2 + 2)$, $c(r_2 + 3)$, ... cr, and taking the sum; thus,

$$\Sigma d' = c[(r_2 + 1) + (r_2 + 2) + (r_2 + 3) + \dots r]$$
$$= \tfrac{1}{2}c(r - r_2)(r + r_2 + 1),$$

$$\therefore \quad M_x = \frac{Wc}{2l}(r - r_2)(r + r_2 + 1)(l - x), \qquad (60)$$

where x cannot be less than cr.

EXAMPLE I. — Length of beam $= l = 100$ feet $= 10c$, $r = 6\frac{1}{2}$, $r_2 = 2\frac{1}{2}$; what is the moment at the fourth weight, $W = 8$ tons, due the 4 weights $= 32$ tons?

Here $x = rc = 65$,

$$\therefore \quad M_r = \frac{8 \times 10}{2 \times 100} \times 4 \times 10 \times 35 = 16 \times 35 = 560 \text{ foot-tons.}$$

If $x = (r + 1)c = 75$,

$$\therefore \quad M_{r+1} = 16 \times 25 = 400 \text{ foot-tons.}$$

If $x = (r + 2)c = 85$,

$$\therefore \quad M_{r+2} = 16 \times 15 = 240 \text{ foot-tons.}$$

If $x = (r + 3)c = 95$,

$$\therefore \quad M_{r+3} = 16 \times 5 = 80 \text{ foot-tons.}$$

This shows a uniform decrease of moment for each interval beyond the given load.

Equation (40) gives for a single weight, W, applied at any point, a', the moment at any distance, x, between the weight and the left end of the beam. By giving to a' the successive values $c(r + 1)$, $c(r + 2)$, $c(r + 3)$, ... cr_1, and summing for a° and a', we find

$$\Sigma a^\circ = r_1 - r,$$
$$\Sigma a' = c[(r+1)+(r+2)+(r+3)+ \ldots r_1] = \tfrac{1}{2}c(r_1 - r)(r_1 + r + 1).$$

$$M_x = \frac{W}{2l}[2(r_1 - r)l - c(r_1 - r)(r_1 + r + 1)]x, \qquad (61)$$

which is the moment at any point, x, between the left end of the girder and the nearest weight, which is at the $(r + 1)^{\text{th}}$ point of division; the number of weights being $r_1 - r$, and x not being greater than $c(r + 1)$.

EXAMPLE 2.—Let $l = 100$ feet $= 10c$, $W = 8$ tons, $r_1 = 6\frac{1}{2}$, $r = 2\frac{1}{2}$; what is the moment at the first weight due the $r_1 - r = 4$ equal weights?

Here $x = (r + 1)c = 35$,

$$\therefore \quad M_{r+1} = \frac{8}{2 \times 100}(2 \times 4 \times 100 - 10 \times 4 \times 10)35 = 560 \text{ foot-to}$$

$$x = rc = 25,$$

$$\therefore \quad M_r \quad = 16 \times 25 = 400 \text{ foot-tons.}$$

$$x = (r - 1)c = 15,$$

$$\therefore \quad M_{r-1} = 16 \times 15 = 240 \text{ foot-tons.}$$

$$x = (r - 2)c = \ \ 5,$$

$$\therefore \quad M_{r-2} = 16 \times \ \ 5 = \ \ 80 \text{ foot-tons.}$$

This shows a uniform decrease of moment for each interval before the given load. These moments are the same as those of example 1, as they should be, since the same load is symmetrically placed on the beam in both cases.

If we add equations (60) and (61), calling the sum M_x still, we shall have

$$M_x = \frac{W}{2l}\left\{[2(r_1 - r)l - c(r_1 - r_2)(r_1 + r_2 + 1)]x \right.$$
$$\left. + cl(r - r_2)(r + r_2 + 1)\right\}, \quad (62)$$

which is the moment at any point, x, of the beam due $r_1 - r_2$ equal weights, W, placed at equal and consecutive intervals, c, over the whole or any part of its length.

Here r cannot be less than r_2 nor greater than r_1, and x lies between rc and $(r + 1)c$ for the loaded part of the beam, but may have any value between 0 and r_2c where $r = r_2$, and any value between r_1c and l where $r = r_1$.

EXAMPLE 3.—Let $l = 100$ feet $= 10\,c$, $W = 8$ tons, $r_2 = 2\frac{1}{2}$, $r_1 = 6\frac{1}{2}$; what is the moment at the fourth weight? Here $x = r_1c = 65$ feet, and (62) gives,

If $r = r_1$,

$$M_{r_1} = \frac{8}{200}\left\{(0 - 10\times4\times10)65+10\times100\times4\times10\right\} = 560 \text{ foot-tons.}$$

Or, if $r = r_1 - 1$,

$$M_{r_1} = \frac{8}{200}\left\{(2\times1\times100 - 10\times4\times10)65+10\times100\times3\times9\right\}$$
$$= 560 \text{ foot-tons.}$$

What is the moment one interval beyond the last weight?

Here $x = (r_1 + 1)c = 75$, and $r = r_1 = 6\frac{1}{2}$, in (62);

$$\therefore\quad M_{r_1+1} = \frac{8}{200}\left\{(0 - 10\times4\times10)75+10\times100\times4\times10\right\}$$
$$= 400 \text{ foot-tons.}$$

If $n =$ the whole number of intervals in the girder's length, we have $c = \frac{l}{n}$, and (62) becomes

$$M_x = \frac{Wl}{2n}\left\{[2n(r_1 - r)-(r_1 - r_2)(r_1 + r_2 + 1)]\frac{x}{l} + (r - r_2)(r + r_2 + 1)\right\}, \quad (63)$$

from which we may find the simultaneous moments at all points throughout the girder due to any uniform discontinuous partial or full load.

EXAMPLE 4.—Let a uniform load of 4 weights, each $= W = 8$ tons, spaced $c = 10$ feet from weight to weight, come upon a girder 100 feet long, and move forward to the centre; required the simultaneous moments throughout the girder as the foremost end of the load passes the points $x = 5, 15, 25, 35$, etc., $n = 10$.

Owing to the important applications of this formula which are to follow, we add the complete solution of this problem, and may remark that by giving to x the values 10, 20, 30, 40, etc., we can find the simultaneous moments at the full intervals as the foremost end of the load passes the successive points of division. Or we may give x any value we please between 0 and l and so suit the equal or unequal panel lengths of any girder. As the load now centred passes off to the right, it is evident these moments will be reversed.

SIMULTANEOUS MOMENTS DUE ADVANCING UNIFORM LOAD.

No. of Wts. $r_1 - r_2$.	r_2	r	r_1	$\frac{x}{l}$	$\frac{Wl}{2n}$	Equation (63).	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8	M_9	M_{10}
1	−0.5	0.5	0.5	0.05	40	$(0 - 1^2)0.05 + 1^2$	38	34	30	26	22	18	14	10	6	2
2	−0.5	0.5	1.5	0.05	40	$(20 \times 1 - 2^2)0.05 + 1^2$	72									
2	−0.5	1.5	1.5	0.15	40	$(0 - 2^2)0.15 + 2^2$		136	120	104	88	72	56	40	24	8
3	−0.5	0.5	2.5	0.05	40	$(20 \times 2 - 3^2)0.05 + 1^2$	102									
3	−0.5	1.5	2.5	0.15	40	$(20 \times 1 - 3^2)0.15 + 2^2$		226								
3	−0.5	2.5	2.5	0.25	40	$(0 - 3^2)0.25 + 3^2$			270	234	198	162	126	90	54	18
4	−0.5	0.5	3.5	0.05	40	$(20 \times 3 - 4^2)0.05 + 1^2$	128									
4	−0.5	1.5	3.5	0.15	40	$(20 \times 2 - 4^2)0.15 + 2^2$		304								
4	−0.5	2.5	3.5	0.25	40	$(20 \times 1 - 4^2)0.25 + 3^2$			400							
4	−0.5	3.5	3.5	0.35	40	$(0 - 4^2)0.35 + 4^2$				416	352	288	224	160	96	32
4	+0.5	0.5	4.5	0.05	40	$(20 \times 4 - 4 \times 6)0.05 + 0$	112									
4	+0.5	1.5	4.5	0.15	40	$(20 \times 3 - 24)0.15 + 1 \times 3$		336								
4	+0.5	2.5	4.5	0.25	40	$(20 \times 2 - 24)0.25 + 2 \times 4$			480							
4	+0.5	3.5	4.5	0.35	40	$(20 \times 1 - 24)0.35 + 3 \times 5$				544						
4	+0.5	4.5	4.5	0.45	40	$(0 - 24)0.45 + 4 \times 6$					528	432	336	240	144	48
4	+1.5	1.5	5.5	0.05	40	$(20 \times 4 - 4 \times 8)0.05 + 0$	96	288								
4	+1.5	2.5	5.5	0.25	40	$(20 \times 3 - 32)0.25 + 1 \times 5$			480							
4	+1.5	3.5	5.5	0.35	40	$(20 \times 2 - 32)0.35 + 2 \times 6$				592						
4	+1.5	4.5	5.5	0.45	40	$(20 \times 1 - 32)0.45 + 3 \times 7$					624					
4	+1.5	5.5	5.5	0.55	40	$(0 - 32)0.55 + 4 \times 8$						576	448	320	192	64
4	+2.5	2.5	6.5	0.05	40	$(20 \times 4 - 4 \times 10)0.05 + 0$	80	240	400							
4	+2.5	3.5	6.5	0.35	40	$(20 \times 3 - 40)0.35 + 1 \times 7$				560						
4	+2.5	4.5	6.5	0.45	40	$(20 \times 2 - 40)0.45 + 2 \times 8$					640					
4	+2.5	5.5	6.5	0.55	40	$(20 \times 1 - 40)0.55 + 3 \times 9$						640				
4	+2.5	6.5	6.5	0.65	40	$(0 - 40)0.65 + 4 \times 10$							560	400	240	80
	Maxi	ma	mo	ments	. .	. . . .	128	336	480	592	640					

In the above table all moments included by the same brace are simultaneous, and due, at their respective points, to all the weights on the girder, as indicated by the first column, $r_1 - r_9$.

Only the first moment in any horizontal line is computed by the formula in that line; the remaining moments in any line being found by the simple variation of x, using only the term containing x. In this example the constant difference to be added to the first moment in any horizontal line of moments is four times the quantity in the parenthesis for the given line.

21. Let the length, l, of the girder be divided into n equal intervals, c, so that there are $n - 1$ points of division; then, if a weight, W, be applied at each point of division beginning at the left, we may find the moment at the foremost end of this advancing load, from equation (60), by putting $r_2 = 0$, $x = rc = \frac{rl}{n}$.

Thus,

$$M_r = \frac{W}{2n} r(r + 1)(l - cr) = \frac{Wl}{2n^2} r(r + 1)(n - r), \quad (64)$$

which is the moment at the foremost end of a uniform discontinuous load when that end passes the r^{th} point of division, and r equal weights have come on.

EXAMPLE. — Let $l = 100$ feet, $n = 10$, $W = 8$ tons; what is the moment at each point of division as the foremost end of this load passes it? Using (64),

If $r = 1$, $M_1 = 1 \times 2 \times 9 \times 4 = 72$ foot-tons.
2, $M_2 = 2 \times 3 \times 8 \times 4 = 192$ foot-tons.
3, $M_3 = 3 \times 4 \times 7 \times 4 = 336$ foot-tons.
4, $M_4 = 4 \times 5 \times 6 \times 4 = 480$ foot-tons.
5, $M_5 = 5 \times 6 \times 5 \times 4 = 600$ foot-tons.
6, $M_6 = 6 \times 7 \times 4 \times 4 = 672$ foot-tons.
7, $M_7 = 7 \times 8 \times 3 \times 4 = 672$ foot-tons.
8, $M_8 = 8 \times 9 \times 2 \times 4 = 576$ foot-tons.
9, $M_9 = 9 \times 10 \times 1 \times 4 = 360$ foot-tons.

22. From equation (63), by putting $r_2 = 0$, $r_1 = n - 1$, and $x = rc = \frac{rl}{n}$, we derive

$$M_r = \frac{Wl}{2n}(n - r)r, \qquad (65)$$

which is the moment at the r^{th} weight due $n - 1$ equal weights, W, placed at equal intervals, $\frac{l}{n}$, throughout the girder.

EXAMPLE. — Uniform discontinuous load; $W = 8$ tons, $l = 100$ feet, $n = 10$.

If $r = 1$, $M_1 = 40 \times 9 \times 1 = 360$ foot-tons.
2, $M_2 = 40 \times 8 \times 2 = 640$ foot-tons.
3, $M_3 = 40 \times 7 \times 3 = 840$ foot-tons.
4, $M_4 = 40 \times 6 \times 4 = 960$ foot-tons.
5, $M_5 = 40 \times 5 \times 5 = 1000$ foot-tons.

And these moments are to be reversed for the other half-span.

23. Suppose that the first and last intervals into which the beam is divided are each $= \frac{1}{2}c = \frac{l}{2n}$, while every other is $= c$, and that we wish to find the moment at the foremost end of a uniform load of equal intervals, c, as that end passes each point of division of the beam.

For this object we employ equation (60), making $x = rc = \frac{rl}{n}$, and $r_2 = -\frac{1}{2}$, and have

$$M_r = \frac{W}{2n}(r + \tfrac{1}{2})^2(l - rc) = \frac{Wl}{2n^2}(r + \tfrac{1}{2})^2(n - r). \qquad (66)$$

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

25. Difference of Simultaneous Moments at Consecutive Points of Division. — By making $r_1 = 0$, and $x = (r + 1)c$, in equation (60), we have

$$M_{r+1} = \frac{W}{2n} r(r + 1)[l - (r + 1)c]$$
$$= \frac{Wl}{2n^2} r(r + 1)(n - r - 1), \quad (68)$$

which is the moment one interval, c, beyond the foremost end of a uniform load consisting of r equal weights, W, at the distances c, $2c$, $3c$, ... rc, respectively, from the left end of the beam.

Subtracting (64) from (68), we have

$$\Delta M = M_{r+1} - M_r = -\frac{Wc}{2n} r(r + 1), \quad (69)$$

which is the difference between the simultaneous moments at any two consecutive points of division on the unloaded end of a beam, which has r equal weights at full intervals on the other end.

For finding the difference of simultaneous moments at consecutive points of division on the loaded part of the beam, we use (62), making $r_2 = 0$, and $x = rc$, $(r + 1)c$, $(r + 2)c$, etc., in succession. Thus,

$$M_r = \frac{W}{2l}\left\{[2(r_1 - r)l - cr_1(r_1+1)]rc + clr(r+1)\right\},$$
$$M_{r+1} = \frac{W}{2l}\left\{[2(r_1 - r)l - cr_1(r_1+1)](r+1)c + clr(r + 1)\right\},$$
$$\Delta M = M_{r+1} - M_r = \frac{Wc}{2n}\left\{2(r_1 - r)n - r_1(r_1 + 1)\right\}, \quad (70)$$

which is the first order of differences for the loaded part or end of the beam: and ΔM is an increasing function of r_1 for a given value of r, and will be greatest when r_1 is greatest;

that is, when $r_1 = n - 1$, its limit. But at this limit of r_1 the girder is loaded, and the positive differences on the left half will equal the negative differences for the corresponding intervals on the right half of the girder. Putting $n - 1$ for r_1 in (70), we have

$$\Delta M_r = \frac{Wc}{2}\left\{n - 1 - 2r\right\}, \qquad (71)$$

which gives the difference of simultaneous moments for each interval, c, of the beam fully loaded with $n - 1$ weights, W, applied at equal and all full intervals, c, or with n weights, W, when each end interval $= \frac{1}{2}c$.

Subtract (69), which is negative, from (71), whose positive values in one half-span are equal to its corresponding negative values in the other half-span, and the remainder is

$$\frac{Wc}{2n}\left\{(n - r)^2 - (n - r)\right\},$$

which is positive, since $n > r$, and both n and r are integers.

Therefore the greatest negative difference computed by (71) for any interval is numerically less than the difference computed by (69) for the same interval in the second half-span; that is, both half-spans, if the uniform load is to travel either way. Consequently we use (69) in finding the greatest difference of simultaneous moments for any interval due a uniform discontinuous moving load.

It may be observed here that (69), for the unloaded end of the beam, gives a constant first difference, while (70), for the loaded end, gives a first difference which is not constant. By putting $r + 1$ for r in (70), and subtracting (70) from the resulting equation, we find the second difference,

$$\Delta(\Delta M) = -Wc, \qquad (72)$$

which is constant and negative, and may be conveniently employed in some computations.

EXAMPLE I. — Let a girder of 10 panels, each 10 feet, be laden with a permanent load of 4 tons at each panel point, and a discontinuous uniform rolling load of 8 tons to be applied at the same points as the load advances; required the greatest moments at these panel points, and the greatest difference of simultaneous moments at any two consecutive panel points, due to both these loads.

The greatest moments will occur when both loads cover the beam. We have, then, in equation (65), $W = 12$, $l = 100$, $n = 10$, $\frac{Wl}{2n} = 60$, and $r = 1, 2, 3$, etc., in succession, for the greatest moments.

The difference of moments at consecutive panel points due dead load is to be computed by (71), making $W = 4$, $c = 10$, $n = 10$, $\frac{Wc}{2} = 20$, and $r = 0, 1, 2, 3, 4$, etc., in succession.

And the greatest difference of simultaneous moments for each interval due live load is found by using equation (69), when $W = 8$, $c = 10$, $n = 10$, $\frac{Wc}{2n} = 4$, and $r = 1, 2, 3, 4$, etc., in succession.

COMPUTATION FOR GREATEST MOMENTS AND DIFFERENCES.

No. of the Panel Point, r.	0	1	2	3	4	5	6	7	8	9
Greatest moments $= 60(10 - r)r$	0	540	960	1260	1440	1500	1440	1260	960	540
Differences, dead load $= 20(9 - 2r)$	180	140	100	60	20	−20	−60	−100	−140	−180
Greatest differences, live load $= -4r(r + 1)$	0	−8	−24	−48	−80	−120	−168	−224	−288	−360
Total differences for both loads	180	132	76	12	−60	−140	−228	−324	−428	−540
Differences, load moving either way					−60	−140	−228	−324	−428	−540
	540	428	324	228	140	60				

If in equation (6), instead of the factor $(l - x)$, we write $(l - (x + c))$, and then subtract (6) from the resulting equation, we shall have

$$\Delta M = \frac{Wc}{2n}\left(v + \tfrac{1}{2}\right), \qquad (7)$$

which gives the greatest difference of simultaneous moments at any two consecutive points of division, the live load advancing by equal panel weights, when the two extreme panels have each but half the length of every intervening panel. Here observe that v takes the successive values $-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2} \ldots \frac{2n-1}{2}$ and that $c = \frac{l}{2n}$ for the two extreme panels, but $c = \frac{l}{n}$ for all others.

EXAMPLE 2. — Given the same loads and length of girder as in example 1, but the panel points being now at the distances 5, 15, 25, 35, etc., from either end; required the greatest moment at each of these points, and the greatest difference of simultaneous moments at any two consecutive panel points, for both live and dead loads.

Equation ([illegible]) gives the greatest moments if we make $l = 100$, $n = 10$, $W = 10\,\frac{[illegible]}{bn} = 15$, and $v = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}$, etc., in succession.

The differences for dead load are computed from ([illegible]) by putting $W = 4$, $c = 5$ in two-end panels, $c = 10$ in all others, $n = 10$ and $v = -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$, etc., $\frac{Wc}{2} = 10$ or 20.

The greatest differences for live load are found from (7), where $W = 8$, $c = 5$ or 10, $n = 10$ and $v = -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$, etc., $\frac{Wc}{2n} = 2$ or 4.

COMPUTATION FOR GREATEST MOMENTS AND GREATEST SIMULTANEOUS DIFFERENCES FOR EACH INTERVAL.

r.	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{7}{2}$	$\frac{9}{2}$	$\frac{11}{2}$	$\frac{13}{2}$	$\frac{15}{2}$	$\frac{17}{2}$	$\frac{19}{2}$
Greatest moments, $\frac{Wl}{8n}\left\{4r(n-r)+1\right\}$		300	780	1140	1380	1500	1500	1380	1140	780	300
Differences, dead load, $\frac{Wc}{2}\left(n-1-2r\right)$	100	160	120	80	40	0	−40	−80	−120	−160	−100
Differences, live load, $-\frac{Wc}{2n}\left(r+\frac{1}{2}\right)^2$	0	−4	−16	−36	−64	−100	−144	−196	−256	−324	−200
Total differences . .	100	156	104	44	−24	−100	−184	−276	−376	−484	−300
Differences to be used					−24	−100	−184	−276	−376	−484	−300
	300	484	376	276	184	100	24				

26. To determine the Point in any Girder simply supported at its Two Ends, and carrying any Partial or Complete Uniform Discontinuous Load, where the Moment due that Load is Greatest. — The required greatest moment will occur at a point within the loaded part of the girder, since for any partial load the simultaneous moments decrease from either end of the load to the corresponding end of the girder.

If, therefore, we put $x = \frac{rl}{n}$ in (63), and call the result M_r, then in M_r thus found put $(r + 1)$ for r, giving M_{r+1}, and equate $\Delta M_r = M_{r+1} - M_r$ to zero, we shall find

$$r = r_1 - \frac{(r_1 - r_2)(r_1 + r_2 + 1)}{2n}; \qquad (74)$$

and the panel point of greatest moment lies between rc and $(r + 1)c$, except when rc and $(r + 1)c$ are panel points.

Let us verify this statement by referring to example 4,

article 20. Taking r_1, r_2, r, and the greatest moment, from that example, we compute r by (74), and write as below:—

r_1.	r_2.	r.	M_{max}.	r by (74).
6.5	2.5	4.5 or 5.5	640	4.5
5.5	1.5	4.5	624	3.9
4.5	0.5	3.5	544	3.3
3.5	−0.5	3.5	416	2.7
2.5	−0.5	2.5	270	2.05
1.5	−0.5	1.5	136	1.3
0.5	−0.5	0.5	38	0.45

27. If in equation (63) we make $r = r_1$, $x = \dfrac{r_1 l}{n}$, $r_2 = r_1 - e$, e being the number of equal weights on the beam, we shall find, after putting $\Delta M_{r_1} = M_{r_1+1} - M_{r_1} = 0$,

$$r_1 = \frac{2n + e - 3}{4}. \tag{75}$$

But when the advancing load reaches back to the left end of the girder, we may not know how many weights will give a maximum moment at the foremost weight. In that case we deduce $\Delta M_r = M_{r+1} - M_r = 0$ from (64), and find

$$r = r_1 = \tfrac{3}{8}(n - 1) \tag{76}$$

for a girder of equal panels to receive an advancing load of equal weights applied at successive panel points.

And for a girder each of whose two extreme panels is one-half of any other, the advancing load to be applied at panel points, we derive $\Delta M_r = M_{r+1} - M_r = 0$, from (66), and get

$$r = r_1 = \tfrac{1}{8}(2n - 5 \pm \sqrt{4n^2 + 4n - 2}). \tag{77}$$

In all these cases the panel point at foremost end, having the greatest moment as the load advances, lies between $r_1 c$ and $(r_1 + 1)c$, except when $r_1 c$ and $(r_1 + 1)c$ are panel points.

r_1-r_2	r_2	r	r_1	$\frac{x}{l}$	$\frac{Wl}{2n}$	From Equation (63).	M_1	M_2	M_3	M_4	M_5	M_6	M_7	M_8	M_9
1	0	1	1	0.1	40	$(0 - 1\times 2)0.1 + 1\times 2$	72	64	56	48	40	32	24	16	8
2	0	1	2	0.1	40	$(20\times 1 - 2\times 3)0.1 + 1\times 2$	136								
2	0	2	2	0.2	40	$(0 - 6)0.2 + 2\times 3$		192	168	144	120	96	72	48	24
3	0	1	3	0.1	40	$(20\times 2 - 3\times 4)0.1 + 1\times 2$	192								
3	0	2	3	0.2	40	$(20\times 1 - 12)0.2 + 2\times 3$		304							
3	0	3	3	0.3	40	$(0 - 12)0.3 + 3\times 4$			336	288	240	192	144	96	48
3	1	1	4	0.1	40	$(20\times 3 - 3\times 6)0.1 + 0$	168	336							
3	1	3	4	0.3	40	$(20\times 1 - 18)0.3 + 2\times 5$			424						
3	1	4	4	0.4	40	$(0 - 18)0.4 + 3\times 6$				432	360	288	216	144	72
3	2	2	5	0.1	40	$(20\times 3 - 3\times 8)0.1 + 0$	144	288	432						
3	2	4	5	0.4	40	$(20\times 1 - 24)0.4 + 2\times 7$				496					
3	2	5	5	0.5	40	$(0 - 24)0.5 + 3\times 8$					480	384	288	192	96
3	3	3	6	0.1	40	$(20\times 3 - 3\times 10)0.1 + 0$	120	240	360	480					
3	3	5	6	0.5	40	$(20\times 1 - 30)0.5 + 2\times 9$					520				
3	3	6	6	0.6	40	$(0 - 30)0.6 + 3\times 10$						480	360	240	120
Maxima moments							192	336	432	496	520				
Maxima differences of simultaneous moments								−8	−24	−48	−72				
							+192	+168	+144	+120	+96				

We now give an example of an equal-panelled girder traversed by an odd number of equal apex weights, and will then illustrate the application of equations (75), (76), and (77).

EXAMPLE.—Girder of 10 panels 10 feet each; 3 weights, of 8 tons each, at intervals of 10 feet between consecutive weights. What are the simultaneous moments at all the panel points, as the foremost weight of this load passes each panel point in succession? Use equation (63), where, now, $W = 8$, $l = 100$, $n = 10$, $r_2 = 0, 1, 2, 3$, etc., in succession.

Moments within the same brace, simultaneous. (See table, p. 42.)

Formula gives only the first moment in any horizontal line of moments. For other moments in same line, add four times the parenthetic quantity to the moment immediately before the required moment.

For the greatest moment at foremost end of this moving load, we have from (75), where, now, $e = 3$,

$$r_1 = \frac{20 + 3 - 3}{4} = 5,$$

$$r_1 + 1 = 6,$$

which agrees with the above table; the moment being 480 when the foremost end of load passes either of these points.

Also, if $e = 4$, as in example 4, article 20, we have, from (75), $r_1 = \dfrac{20 + 4 - 3}{4} = 5.25$; and 5.50, which gives the greatest moment 576, at foremost end of load, lies between 5.25 and 6.25.

For a full load coming upon the panel points of a girder having 10 equal panels, (76) gives $r_1 = \frac{2}{3}(10 - 1) = 6$, $r_1 + 1 = 7$, a result in accord with the solution in article 21, where the moment at foremost end is greatest, and equals 672 at these two points.

Also, when $n = 10$, (77) gives $r_1 = 5.98$, $r_1 + 1 = 6.98$; and 6.5, giving 686 foot-tons (example of article 23), lies between 5.98 and 6.98.

28. To find the general expression for the point of greatest moment, on the loaded part of the beam, due to a uniform dead load, a weight, W, being applied at each of the $(n - 1)$ or n panel points, and to a uniform live load consisting of $r_1 - r_2$ equal weights, L, applied at consecutive panel points as the load advances, we employ equation (63), putting L for W, and $x = \frac{rl}{n} = rc$, and so have M_r. Then, substituting $r + 1$ for r, we get M_{r+1},

$$\therefore \quad \Delta M_r = M_{r+1} - M_r$$
$$= \frac{Ll}{2n^2}\left\{2n(r_1 - r) - (r_1 - r_2)(r_1 + r_2 + 1)\right\}$$

for the loaded part of the beam.

This expression added to (71), and the sum made equal to 0, gives

$$r = \frac{L}{2n(L + W)}\left\{n(n-1)\frac{W}{L} - (r_1 - r_2)(r_1 + r_2 + 1) + 2nr_1\right\} \quad (78)$$

to be used when the value of r lies between r_2 and r_1; and the panel point under the live load, having the greatest moment, lies between rc and $(r + 1)c$ when rc and $(r + 1)c$ are not panel points.

In a similar manner, putting $r = r_2$, and $x = r_2c$, $(r_2 - 1)c$, in succession, in (63), finding ΔM_{r_2} and adding it to ΔM in (71), we derive

$$r = \tfrac{1}{2}(n - 1) + \frac{L}{2nW}(r_1 - r_2)[2n - (r_1 + r_2 + 1)] \quad (79)$$

where r is not greater than r_2, that is, at the left of a partial live load. Also, when the point of greatest moment, consider-

ing dead and live loads, lies beyond the live load, we derive, from (63),

$$r = \tfrac{1}{2}(n - 1) - \frac{L}{2nW}(r_1 - r_2)(r_1 + r_2 + 1), \qquad (80)$$

where r is not less than $r_{,,}$ that is, beyond the live load.

29. We next assume that the uniform load, w' units of weight per linear unit of beam, advances by continuous increments, and not by leaps, or entire panel weights added at once; and require the moment at the foremost end of a load which is thus uniformly distributed continuously from its foremost end to the left end of the girder.

Equation (56) applies here if we make $x = b =$ length of uniform load measured from the left support; and we have

$$M_{bw'} = \frac{w'b^2}{2l}(l - b). \qquad (81)$$

And, if w is the unit weight of the dead load, we have from (49), by putting $x = b$,

$$M_{bw} = \tfrac{1}{2}wb(l - b), \qquad (82)$$

where M_{bw} is the moment of a beam, at the distance b from one extremity, due to the unit weight, w, covering the entire beam.

For the total moment due to live and dead loads at the foremost end of bw', we take the sum of (81) and (82), and have

$$M_{bw'+bw} = \frac{\tfrac{1}{2}b(l - b)}{l}(w'b + lw). \qquad (83)$$

EXAMPLE. — Let $l = 100$, $w = 0.4$ ton, $w' = 0.8$ ton; and find the moments at the foremost end of the moving load, bw', at intervals of 10 feet as it advances. From (83), —

b.	$\frac{1}{2}b\frac{l-b}{l}$.	$w'b + lw$.	$M_{bw' + bw}$.
0	0	40	0
10	4.5	48	216
20	8.0	56	448
30	10.5	64	672
40	12.0	72	864
50	12.5	80	1000
60	12.0	88	1056
70	10.5	96	1008
80	8.0	104	832
90	4.5	112	504
100	0	120	0

Each of these moments is, as it should be, less than that found for apex loads by just the moment due $\frac{1}{2}w'$ at the point taken, since the point at the end of the continuously distributed live load sustains but half a panel weight of the live load.

30. If it be required to find the moment due to both dead load, lw, and live load, bw', at any point ahead of the latter, we use for the live load (57) by making $a = 0$, and for the dead load (49), and have

$$M_x = \frac{w'}{2l}b^2(l - x) + \tfrac{1}{2}w(l - x)x. \tag{84}$$

Or, for $(r + 1)$ intervals, each equal to $\frac{l}{n}$, we find, if $b = \frac{rl}{n}$, and $x = \frac{(r+1)l}{n}$,

$$M_{r+1} = \frac{w'l^2}{2n^3}r^2(n - r - 1) + \frac{wl^2}{2n^2}(r + 1)(n - r - 1). \tag{85}$$

EXAMPLE. — Let $l = 100$, $w' = 0.8$, $w = 0.4$, $n = 10$; and find the total moment at the distance $x = b + 10$, or at the end of the $(r + 1)^{\text{th}}$ interval, $\frac{l}{n}$.

First computation, using (84), —

b.	x.	$\frac{w'}{2l}$.	b^2.	$l - x$.	First Term.	$\frac{1}{2}w$.	$l - x$.	x.	Second Term.	M_x.
0	10	0.004	0	90	0	0.2	90	10	180	180
10	20	0.004	100	80	32	0.2	80	20	320	352
20	30	0.004	400	70	112	0.2	70	30	420	532
30	40	0.004	900	60	216	0.2	60	40	480	696
40	50	0.004	1600	50	320	0.2	50	50	500	820
50	60	0.004	2500	40	400	0.2	40	60	480	880
60	70	0.004	3600	30	432	0.2	30	70	420	852
70	80	0.004	4900	20	392	0.2	20	80	320	712
80	90	0.004	6400	10	256	0.2	10	90	180	436
90	100	0.004	8100	0	0	0.2	0	100	0	0

Second computation, using (85), —

r.	$\frac{w'l^2}{2n^3}$.	r^2.	$n-r-1$.	First Term.	$\frac{wl^2}{2n^2}$.	$r+1$.	$n-r-1$.	Second Term.	M_{r+1}.
0	4	0	9	0	20	1	9	180	180
1	4	1	8	32	20	2	8	320	352
2	4	4	7	112	20	3	7	420	532
3	4	9	6	216	20	4	6	480	696
4	4	16	5	320	20	5	5	500	820
5	4	25	4	400	20	6	4	480	880
6	4	36	3	432	20	7	3	420	852
7	4	49	2	392	20	8	2	320	712
8	4	64	1	256	20	9	1	180	436
9	4	81	0	0	20	10	0	0	0

This second computation will, in general, be found more convenient than the first, since n and r are usually integers, and not very large.

31. When a uniform continuous load is coming upon one end of a girder, we may find the position of the foremost end of the load at the instant the moment at that end reaches its maximum value, by differentiating (81) with respect to b, and putting $\frac{dM}{db} = 0$. Thus,

$$\frac{dM}{db} = \frac{w'}{2l}(2lb - 3b^2) = 0,$$

$$\therefore \quad b = \tfrac{2}{3}l. \qquad (86)$$

32. The position of the foremost end of the live load when the moment is a maximum there for combined dead and live loads, is determined by differentiating (83), and making $\frac{dM}{db} = 0$.

Thus, $$\frac{dM}{db} = \frac{w}{2l}(2elb - 3eb^2 + l^2 - 2lb) = 0,$$

$$\therefore \quad b = \frac{l}{3e}\left\{e - 1 \pm (e^2 + e + 1)^{\frac{1}{2}}\right\}, \qquad (87)$$

where $e = \frac{w'}{w}$, and $b =$ length of live load on one end of the girder.

Equation (87) is illustrated by the example in article 29, where $e = \frac{0.8}{0.4} = 2$; and (87) gives $b = 60.76$, while (83) gives the corresponding moment 1056.31, a maximum.

33. Equation (55) expresses the moment due any uniform partial or complete continuous load, $w'b$, at any loaded point, x. By differentiating (55), and putting $\frac{dM}{dx} = 0$, we may find the value of x, which gives the maximum moment. Thus,

$$\frac{dM}{dx} = \frac{w'b}{l}(l - a - \tfrac{1}{2}b) - \tfrac{1}{2}w'(2x - 2a) = 0,$$

$$\therefore \quad x = a + b - \frac{b}{l}(a + \tfrac{1}{2}b), \qquad (88)$$

which is the point of greatest moment due $w'b$.

34. Also, from (49), we may make

$$\frac{dM}{dx} = \tfrac{1}{2}w(l - 2x) = 0,$$

and find

$$x = \tfrac{1}{2}l, \tag{89}$$

which is the point of greatest moment due full continuous uniform load, as in equation (52).

35. If we add $\frac{dM}{dx}$ from (55) to $\frac{dM}{dx}$ from (49), and equate the sum to zero, we shall find

$$x = \frac{1}{w + w'}\left\{(a + b)w' - (a + \tfrac{1}{2}b)\frac{w'b}{l} + \tfrac{1}{2}wl\right\}, \tag{90}$$

which is the point of greatest moment due both loads, $w'b$ and wl.

36. We will now consider the case of a girder having n panels, each $= c = \frac{l}{n}$; and we will suppose the live load to consist of weights not all equal, nor spaced so as to conform to the panel points. Such a case is presented by a locomotive and train of cars.

Making use of equations (40) and (43), let us arrange formulæ convenient for this case.

If in these equations we put nc for l, and for d' and x write the proper multiple of c, we have the simultaneous moments due each weight, W, in its position at a panel point, as indicated in the following tabular arrangement:—

SIMULTANEOUS MOMENTS.

Order.	W_1	W_2	W_3	W_4	W_5	W_6	. . .	W_{n-1}
M_{W_1}	$\frac{n-1}{n}cW_1$	$\frac{n-2}{n}cW_1$	$\frac{n-3}{n}cW_1$	$\frac{n-4}{n}cW_1$	$\frac{n-5}{n}cW_1$	$\frac{n-6}{n}cW_1$	. . .	$\frac{1}{n}cW_1$
M_{W_2}	$\frac{n-2}{n}cW_2$	$\frac{2(n-2)}{n}cW_2$	$\frac{2(n-3)}{n}cW_2$	$\frac{2(n-4)}{n}cW_2$	$\frac{2(n-5)}{n}cW_2$	$\frac{2(n-6)}{n}cW_2$	. . .	$\frac{2}{n}cW_2$
M_{W_3}	$\frac{n-3}{n}cW_3$	$\frac{2(n-3)}{n}cW_3$	$\frac{3(n-3)}{n}cW_3$	$\frac{3(n-4)}{n}cW_3$	$\frac{3(n-5)}{n}cW_3$	$\frac{3(n-6)}{n}cW_3$	. . .	$\frac{3}{n}cW_3$
M_{W_4}	$\frac{n-4}{n}cW_4$	$\frac{2(n-4)}{n}cW_4$	$\frac{3(n-4)}{n}cW_4$	$\frac{4(n-4)}{n}cW_4$	$\frac{4(n-5)}{n}cW_4$	$\frac{4(n-6)}{n}cW_4$	. . .	$\frac{4}{n}cW_4$
M_{W_5}	$\frac{n-5}{n}cW_5$	$\frac{2(n-5)}{n}cW_5$	$\frac{3(n-5)}{n}cW_5$	$\frac{4(n-5)}{n}cW_5$	$\frac{5(n-5)}{n}cW_5$	$\frac{5(n-6)}{n}cW_5$	. . .	$\frac{5}{n}cW_5$
M_{W_6}	$\frac{n-6}{n}cW_6$	$\frac{2(n-6)}{n}cW_6$	$\frac{3(n-6)}{n}cW_6$	$\frac{4(n-6)}{n}cW_6$	$\frac{5(n-6)}{n}cW_6$	$\frac{6(n-6)}{n}cW_6$	. . .	$\frac{6}{n}cW_6$
. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .
$M_{W_{n-1}}$	$\frac{1}{n}cW_{n-1}$	$\frac{2}{n}cW_{n-1}$	$\frac{3}{n}cW_{n-1}$	$\frac{4}{n}cW_{n-1}$	$\frac{5}{n}cW_{n-1}$	$\frac{6}{n}cW_{n-1}$	. . .	$\frac{n-1}{n}cW_{n-1}$

Since the above moments are simultaneous, we may sum them, and thus find the moment at each joint due all the weights in their respective positions.

SUM OF MOMENTS.

Equation (91).

Order.	W_1	W_2	W_3	W_4	W_5	W_6	. . .	W_{n-1}
$\Sigma M =$	$\frac{c}{n}$ $(n-1)W_1$	$\frac{2c}{n}$ $\frac{1}{2}(n-2)W_1$	$\frac{3c}{n}$ $\frac{1}{3}(n-3)W_1$	$\frac{4c}{n}$ $\frac{1}{4}(n-4)W_1$	$\frac{5c}{n}$ $\frac{1}{5}(n-5)W_1$	$\frac{6c}{n}$ $\frac{1}{6}(n-6)W_1$	. . .	$\frac{c}{n}$ W_1
	$+(n-2)W_2$	$+(n-2)W_2$	$+\frac{2}{3}(n-3)W_2$	$+\frac{2}{4}(n-4)W_2$	$+\frac{2}{5}(n-5)W_2$	$+\frac{2}{6}(n-6)W_2$	. . .	$+2W_2$
	$+(n-3)W_3$	$+(n-3)W_3$	$+(n-3)W_3$	$+\frac{3}{4}(n-4)W_3$	$+\frac{3}{5}(n-5)W_3$	$+\frac{3}{6}(n-6)W_3$	. . .	$+3W_3$
	$+(n-4)W_4$	$+(n-4)W_4$	$+(n-4)W_4$	$+(n-4)W_4$	$+\frac{4}{5}(n-5)W_4$	$+\frac{4}{6}(n-6)W_4$	. . .	$+4W_4$
	$+(n-5)W_5$	$+(n-5)W_5$	$+(n-5)W_5$	$+(n-5)W_5$	$+(n-5)W_5$	$+\frac{5}{6}(n-6)W_5$	. . .	$+5W_5$
	$+(n-6)W_6$	$+(n-6)W_6$	$+(n-6)W_6$	$+(n-6)W_6$	$+(n-6)W_6$	$+(n-6)W_6$	. . .	$+6W_6$
	$+(n-7)W_7$	$+(n-7)W_7$	$+(n-7)W_7$	$+(n-7)W_7$	$+(n-7)W_7$	$+(n-7)W_7$	. . .	$+7W_7$
	$+(n-8)W_8$	$+(n-8)W_8$	$+(n-8)W_8$	$+(n-8)W_8$	$+(n-8)W_8$	$+(n-8)W_8$	. . .	$+8W_8$
	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .
	$+3W_{n-3}$	$+3W_{n-3}$	$+3W_{n-3}$	$+3W_{n-3}$	$+3W_{n-3}$	$+3W_{n-3}$	. . .	$+(n-3)W_{n-3}$
	$+2W_{n-2}$	$+2W_{n-2}$	$+2W_{n-2}$	$+2W_{n-2}$	$+2W_{n-2}$	$+2W_{n-2}$	. . .	$+(n-2)W_{n-2}$
	$+W_{n-1}$	$+W_{n-1}$	$+W_{n-1}$	$+W_{n-1}$	$+W_{n-1}$	$+W_{n-1}$	. . .	$+(n-1)W_{n-1}$

Now, in this equation, (91), for the sum of the moments due all the weights, we may evidently put any weight in the place of any other, and suppose any number of the weights equal to zero.

Hence we may, by means of (91), find the momental effects of any load traversing the girder, at each of the equal intervals, c, in its progress.

EXAMPLE. — Let the span $= l =$ 100 feet; $n = 10 =$ number of panels, each $= c =$ 10 feet. Dead load $= w =$ 0.5 ton per linear foot $= cw =$ 5 tons at each panel point or apex, and $2\frac{1}{2}$ tons on each abutment. Live load consists of two locomotives, each of the following lengths and weights: —

Between bearings of truck wheels,	5.75 feet.
Between bearings of second truck wheels and first driver,	8.50 feet $= S_1$.
Between bearings of drivers,	7.75 feet $= S$.
Between bearings of second drivers and first tender,	7.25 feet $= S_2$.
Between bearings of first and second tenders,	4.00 feet.
Between bearings of second and third tenders,	7.25 feet.
Between bearings of third and fourth tenders,	4.00 feet.
Total wheel base,	44.50 feet.

Between bearings of first tender and truck of second engine, 8 feet.

Total weight of tender,	42,000 pounds =	21.00 tons.
Total weight of engine,	65,000 pounds =	32.50 tons.
Total weight on 2 pairs drivers,	42,000 pounds =	21.00 tons.
Total weight on 2 pairs truck,	23,000 pounds =	11.50 tons.
Weight on each pair truck wheels,		5.75 tons $= k$.
Weight on each pair drivers,		10.50 tons $= D$.
Weight on each pair tender wheels,		5.25 tons $= t$.

Suppose one girder carries these two locomotives.

We first find the greatest weight that can come upon a panel point or joint from the weights in adjacent panels.

Let $D_1 = D_2$, Fig. 10, be the equal weights on each pair of drivers, and take A, B, C, any three consecutive joints at the given interval, c feet; let $x = AE$, $S =$ space between bases of drivers, it being less than c.

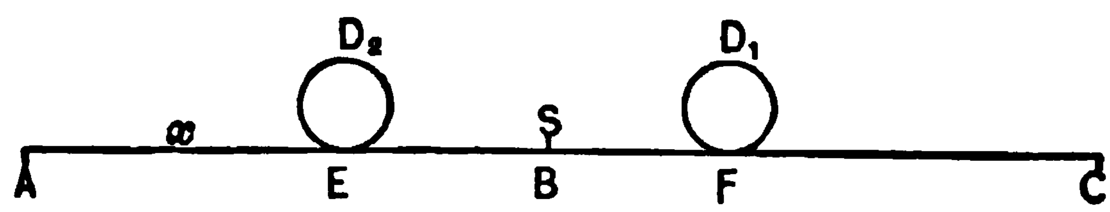


FIG. 10.

Then the weight at B, from drivers, is

$$\frac{x}{c}D_2 + \frac{2c - x - S}{c}D_1 = \frac{2c - S}{c}D, \qquad (92)$$

which is a constant, while the point B is anywhere in the space S.

If both drivers are between two consecutive joints, as AB, we have

$$\frac{x}{c}D_2 + \frac{x + S}{c}D_1 = \frac{2x + S}{c}D,$$

which is not a constant, but reaches its greatest value within the prescribed limits when $x = c - S$; that is, when

$$\frac{2x + S}{c}D = \frac{2(c - S) + S}{c}D = \frac{2c - S}{c}D.$$

Therefore $\frac{2c - S}{c}D$ is the greatest pressure that can come upon any joint from the drivers of one engine.

Now, when the foremost driver, D_1, is at B, the second truck wheel is between B and C; and when the second driver is at

B, the first tender wheel is between A and B. In the former case the increment of weight at B from the truck would be $\frac{c - S_1}{c}k$; in the latter case the increment at B from the tender would be $\frac{c - S_2}{c}t$. Therefore the first or second driver at B gives the greatest pressure at that point according as $\frac{c - S_1}{c}k$ is greater or less than $\frac{c - S_2}{c}t$.

In the present example,

$$\frac{c - S_1}{c}k = \frac{10 - 8.5}{10} \times 5.75 = 0.86250,$$

$$\frac{c - S_2}{c}t = \frac{10 - 7.25}{10} \times 5.25 = 1.44375.$$

We will, therefore, find the pressures at points whose intervals are equal to c, when the second driver of the foremost engine is at one of these points. Let this point be the third, counting from the right-hand pier.

Then Fig. 11 shows the positions of all the wheels with reference to the joints at the equal intervals; and a simple calcu-

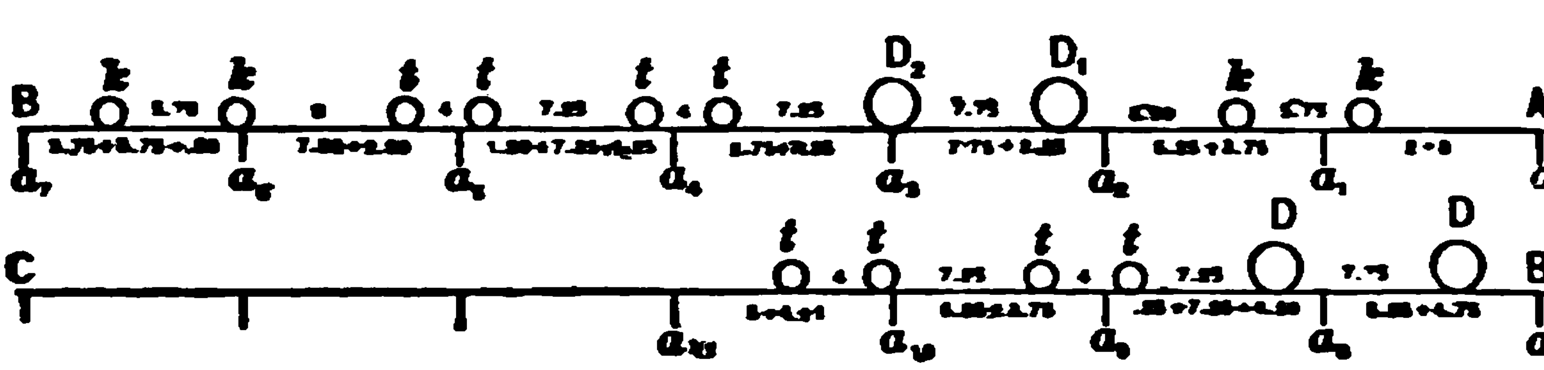


FIG. 11.

lation according to the principle involved in (38) and (39) gives the total pressure at each joint, which pressure is to be substituted for W in the equation (91).

In this position of the locomotives the pressures at the equal intervals due to the weights on the adjacent panels are—

1.	At A,	$0.200k$	$=$ 1.15000 tons.
2.	At a_1,	$1.425k$	$=$ 8.19375 tons.
3.	At a_2,	$0.375k + 0.775D$	$=$ 10.29375 tons.
4.	At a_3,	$1.225D + 0.275t$	$=$ 14.30625 tons.
5.	At a_4,	$1.750t$	$=$ 9.18750 tons.
6.	At a_5,	$1.725t$	$=$ 9.05625 tons.
7.	At a_6,	$0.250t + 1.325k$	$=$ 8.93125 tons.
8.	At a_7,	$0.675k + 0.525D$	$=$ 9.39375 tons.
9.	At a_8,	$1.225D + 0.025t$	$=$ 12.99375 tons.
10.	At a_9,	$0.250D + 1.600t$	$=$ 11.02500 tons.
11.	At a_{10},	$1.775t$	$=$ 9.31875 tons.
12.	At a_{11},	$0.600t$	$=$ 3.15000 tons.
		Total,	107.00000 tons.

Stopping with the second driver at a_3, we see that the hindmost tender truck has not yet come upon the girder. We compute, however, the moments due this load as it advances panel by panel, till the twelfth weight is upon the girder, and the first, second, and third have passed off; using indices M_1, M_{1-2}, . . . M_{3-11}, M_{4-12}, to denote, inclusively, what panel weights produce the simultaneous moments opposite M.

MOMENTS DUE LIVE LOAD.

JOINT.	1st.	2d.	3d.	4th.	5th.	6th.	7th.	8th.	9th.
M_1 . . .	10.350	9.200	8.050	6.900	5.750	4.600	3.450	2.300	1.150
M_{1-2} . .	82.944	83.950	73.457	62.962	52.469	41.975	31.481	20.988	10.494
M_{1-3} . .	166.244	229.550	210.918	180.788	150.656	120.526	90.396	60.264	30.131
M_{1-4} . .	275.362	407.662	436.023	384.449	320.374	256.299	192.214	128.150	64.075
M_{1-5} . .	324.107	556.339	645.508	631.741	536.031	428.826	321.619	214.413	107.207
M_{1-6} . .	362.482	634.402	814.447	851.429	785.470	637.576	478.182	318.789	159.394
M_{1-7} . .	390.676	692.040	902.242	1021.764	997.627	870.551	661.540	441.026	220.513
M_{1-8} . .	414.099	734.262	965.112	1115.400	1153.812	1059.161	861.575	582.051	291.026
M_{1-9} . .	460.532	791.226	1027.783	1175.130	1231.909	1196.815	1018.559	737.566	374.532
M_{2-10} .	476.251	842.252	1078.316	1220.441	1273.263	1235.504	1105.881	833.193	457.565
M_{3-11} .	466.739	840.289	1103.589	1236.953	1276.377	1224.690	1086.042	863.712	478.322
M_{4-12} .	400.582	769.664	1045.558	1211.099	1246.805	1188.575	1041.132	803.026	473.043
M max. .	476.251	842.252	1103.589	1236.953	1276.377	1235.504	1105.881	863.712	478.322
Mw . .	225.000	400.000	525.000	600.000	625.000	600.000	525.000	400.000	225.000
M total .	701.251	1242.252	1628.589	1836.953	1901.377	1835.504	1630.881	1263.712	703.322

Differences.

	ΔM_{0-1}	ΔM_{1-2}	ΔM_{2-3}					ΔM_{7-8}	ΔM_{8-9}
1	10.350	−1.150	− 1.150					− 1.150	− 1.150
2	82.944	1.006	−10.494					− 10.494	− 10.494
3	166.244	63.306	−18.632					− 30.131	− 30.131
4	275.362	132.300	28.361					− 64.075	− 64.075
5	324.107	232.232	89.165					−107.207	−107.207
6	362.482	271.920	180.045	36.982	−65.959	−147.894	−159.394	−159.394	−159.394
7	390.676	301.364	210.202	119.522	−24.137	−127.076	−209.011	−220.514	−220.513
8	414.099	320.163	230.850	150.288	38.412	− 94.651	−197.586	−279.524	−291.025
9	460.532	330.694	236.557	147.347	56.779	− 35.094	−178.256	−280.993	−363.034
10	476.251	366.001	236.064	142.125	52.822	− 37.759	−129.623	−272.688	−375.528
11	466.739	373.550	263.300	133.364	39.424	− 51.687	−138.648	−222.330	−385.390
12	400.582	369.082	275.894	165.541	35.706	− 58.230	−147.443	−238.106	−329.983

Maxima Differences of Moment.

Interval.	0-1	1-2	2-3	3-4	4-5	5-6	6-7	7-8	8-9	9-10
Live load + .	476.251	373.550	275.894	165.541	56.779	-	-	-	-	-
Live load − .	-	−1.150	−18.632	−51.574	−95.710	−147.894	−209.011	−280.993	−385.390	−478.322
Dead load. .	225.000	175.000	125.000	75.000	25.000	− 25.000	− 75.000	−125.000	−175.000	−225.000
Maximum + .	701.251	548.550	400.894	240.541	81.779	-	-	-	-	-
Maximum − .	-	-	-	-	−70.710	−172.894	−284.011	−405.993	−560.390	−703.322
Use . . .	-	-	-	-	−70.710	+ 70.710	-	-	-	-
	703.322	560.390	405.993	284.011	172.894	−172.894	−284.011	−405.993	−560.390	−703.322

The moments for dead load, given opposite Mw, have been computed by (65), whilst (91) has been used in finding the moments due the advancing load.

The differences are taken directly from the computed moments; and we must evidently use for each half-span the greatest difference due any interval, the load being supposed to travel either way.

37. Let us now suppose that this same live load of 107 tons is distributed uniformly over the 10 panels, so that W = panel weight = 10.7 tons. We then find by means of (65) the greatest moments due live load, and by means of (69) the greatest differences of moment due live load. Taking the moments due dead load as found above, we write:—

WEIGHT OF TWO LOCOMOTIVES UNIFORMLY DISTRIBUTED.

	DISTANCE FROM PIER.	10	20	30	40	50
	r.	-	1	2	3	4
1	$-\frac{Wc}{2n}(r+1)r$	-	−10.70	−32.10	−64.20	−107.00
2	Full live load, difference	481.50	374.50	267.50	160.50	53.50
3	Dead load, difference	225.00	175.00	125.00	75.00	25.00
4	Maximum difference + ...	706.50	549.50	392.50	235.50	78.50
5	Maximum difference −	-	-	-	-	−82.00
6	Live load, M	481.50	856.00	1123 50	1284.00	1337.50
7	Dead load, M	225.00	400.00	525.00	600.00	625.00
8	Total M maximum.	706.50	1256.00	1648.50	1884.00	1962.50

	DISTANCE FROM PIER.	60	70	80	90	100
	r.	5	6	7	8	9
1	$-\frac{Wc}{2n}(r+1)r$	−160.50	−224.70	−299.60	−385.20	−481.50
2	Full live load, difference	− 53.50	−160.50	−267.50	−374.50	−481.50
3	Dead load, difference	− 25.00	− 75.00	−125.00	−175.00	−225.00
4	Maximum difference + ...	-	-	-	-	-
5	Maximum difference −	−185.50	−299.70	−424.60	−560.20	−706.50
6	Live load, M.	1284.00	1123.50	856.00	481.50	-
7	Dead load, M	600.00	525.00	400.00	225.00	-
8	Total M maximum.	1884.00	1648.50	1256.00	706.50	-

A comparison of these maxima moments and differences with those just found for the natural distribution of the weights of these two locomotives, shows but one moment and one difference to be less for uniform load than for naturally distributed load. The extreme length of wheel base of these two united locomotives is $2 \times 44.5 + 8 = 97$ feet.

It will be observed, that, when the second driver of the second engine is at a joint, the weight at that joint is greater than the weight we have used in the calculation of moments. But, when the second driver of the second engine is at a joint, the second driver of the first engine is 2.5 feet (see Fig. 11) from a joint; so that, assuming the coupled locomotives to travel either way, our calculation is correct.

We will close this section with an example including every kind of loading contemplated herein.

EXAMPLE. — Let $W = 20.0$ tons, $d' = 50$ ft.
$w = 0.4$ tons, $l = 100$ ft.
$w' = 0.8$ tons, $b = 20$ ft., $a = 40$ ft.
$P = 10.0$ tons, $d'' = 50$ ft., $\alpha = 30°$.

Find the moments and differences of moment for every ten feet throughout the girder.

In this calculation we use equations (40), (43), (49), (53), (55), (57), (47), and (48), with the following result: —

DISTANCE.	10	20	30	40	50	60	70	80	90	100
For W, M	100	200	300	400	500	400	300	200	100	0
For w, M	180	320	420	480	500	480	420	320	180	0
For w', M	80	160	240	320	360	320	240	160	80	0
For P, M	−25	−50	−75	−100	−125	−100	−75	−50	−25	0
Total M	335	630	885	1100	1235	1100	885	630	335	0
Total dif.	335	295	255	215	135	−135	−215	−255	−295	−335

SECTION 3.

Horizontal Girder of One Span, with Fixed Ends. Effects of End Moments.

38. If we suppose the simple girder, Fig. 9, not only supported at its ends, but also fixed by being built into the walls, or by means of forces applied to the sections AB and CE, to keep them from changing place as the beam inclines to yield to the other applied pressures, we then have a moment developed at each extremity of the girder, which will manifestly affect the normal moment due all other applied pressures at every cross-section.

Let us now find expressions for the momental effects at any point of the girder due to the given end moments, without attempting at present to formulate the value of these end moments, nor to determine whether they are simply sufficient to "fix" the ends of the girder.

Let AB, Fig. 12, represent a beam whose end moments are M_1 and M_2.

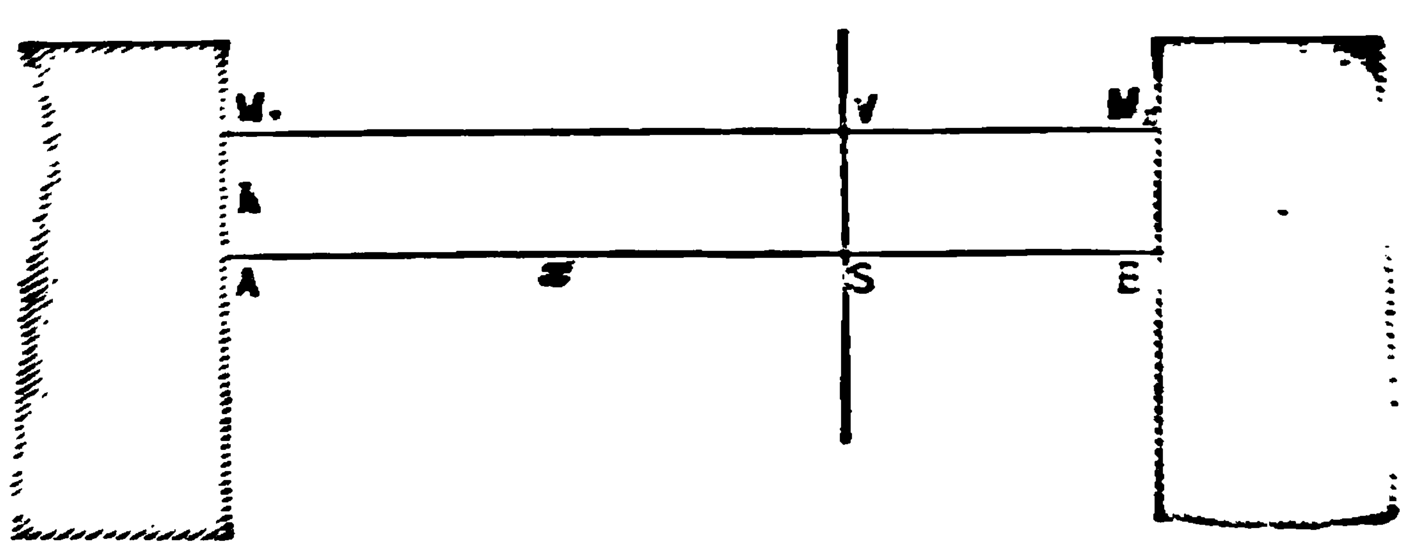


FIG. 12.

Call the length of clear span l, and take VS any vertical section at the horizontal distance x from the left abutment. Let v_1 = the vertical re-action, positive or negative, at E, due

to the moment M_1 at A; and let $v_2 =$ the vertical re-action, positive or negative, at A, due to the moment M_2 at B.

Then, taking moments about A and B, we have

$$M_1 = lv_1, \quad \therefore \quad v_1 = M_1 \div l,$$

$$M_2 = lv_2, \quad \therefore \quad v_2 = M_2 \div l.$$

Call the moment at VS,

$$\text{For } v_1, \; -v_1(l - x) = -\frac{l - x}{l}M_1;$$

$$\text{For } v_2, \quad v_2x \quad = \frac{x}{l}M_2.$$

Hence the moment at VS due to the two end moments acting in opposite directions is

$$M_x = \frac{l - x}{l}M_1 + \frac{x}{l}M_2 = \frac{M_2 - M_1}{l}x + M_1, \qquad (93)$$

where both end moments tend to diminish the normal moment at the section VS, and are negative.

If, therefore, we apply the correction (93) to the moment found at any cross-section of a girder with free ends, which we have called the normal moment, we shall have the total moment, including the influence of the end or pier moments.

39. When $c = l \div n =$ one of the equal panel lengths of the girder whose end moments are M_1 and M_2, we may find the momental difference for any interval, c, due to M_1 and M_2, by putting $x + c$ for x in equation (93), and subtracting. Thus,

$$M_{x+c} = \frac{M_2 - M_1}{l}(x + c) + M_1,$$

$$\therefore \quad \Delta M_c = M_{x+c} - M_x = (M_2 - M_1)\frac{c}{l}. \qquad (94)$$

By means of (94) we may correct the normal difference of moments for the influence of the given pier moments.

The pier moments M_1 and M_2 are here supposed to be constant. The cases of their variation will be considered hereafter, when we come to formulate their values.

EXAMPLE 1. — Let us suppose that the girder for which we have computed the maxima moments and differences of moment, in article 25, example 1, had, in addition to the pressures there given, been subjected to these end moments; viz.,

$$M_1 = -400 \text{ foot-tons},$$
$$M_2 = -500 \text{ foot-tons}.$$

From (93) we find decrements of moment:

$$\begin{aligned}
\frac{400 - 500}{100}x - 400 &= -410 \text{ when } x = 10 \\
&= -420 \text{ when } x = 20 \\
&= -430 \text{ when } x = 30 \\
&= -440 \text{ when } x = 40 \\
&= -450 \text{ when } x = 50 \\
&= -460 \text{ when } x = 60 \\
&= -470 \text{ when } x = 70 \\
&= -480 \text{ when } x = 80 \\
&= -490 \text{ when } x = 90 \\
&= -500 \text{ when } x = 100
\end{aligned}$$

From (94), or from the decrements just found, we have the constant decrement of difference,

$$\frac{400 - 500}{100} \times 10 = -10.$$

Applying these corrections to the tabulated maxima moments and differences in article 25, example 1, there results:—

x.	0	10	20	30	40	50	60	70	80	90	100
M . . .	−400	130	540	830	1000	1050	980	790	480	50	−500
Difference .		530	418	314	218	+130	+50				
						−70	−150	−238	−334	−438	−550

EXAMPLE 2. — Let us suppose that the girder of example in article 37 has its right end extended 20 feet beyond the point of support, and has a weight, $W_{,}$ = 10 tons applied at that extremity. What is the pier moment developed by the 10 tons and by the girder's own uniform weight, $w = 0.4$ ton per linear foot? And what is the effect of this pier moment on the normal moments and differences already found for the given pressures?

From (22), moment due W is $-Wl = -10 \times 20 = -200.$

From (25), moment due w is $-\frac{1}{2}wl^2 = -\frac{0.4 \times 20^2}{2} = -80.$

Moment at right pier $= M_2 = -280.$

Moment at left pier $= M_1 = 0.$

Whence, by (93), we have corrections of moment,

$$
\begin{aligned}
\frac{0-280}{100}x + 0 &= -28 \text{ when } x = 10 \\
&= -56 \text{ when } x = 20 \\
&= -84 \text{ when } x = 30 \\
&= -112 \text{ when } x = 40 \\
&= -140 \text{ when } x = 50 \\
&= -168 \text{ when } x = 60 \\
&= -196 \text{ when } x = 70 \\
&= -224 \text{ when } x = 80 \\
&= -252 \text{ when } x = 90 \\
&= -280 \text{ when } x = 100
\end{aligned}
$$

From (94), correction for differences,

$$(0 - 280) \times \tfrac{10}{100} = -28.$$

Applying these corrections to the computed normal moments and differences, we find:—

x.	0	10	20	30	40	50	60	70	80	90	100
M . .	0	307	574	801	988	1095	932	689	406	83	−280

Dif. + . .	307	267	227	187	107					
Dif. − . .						−163	−243	−283	−323	−363

CHAPTER IV.

STRAINS IN FRAMED OR BUILT GIRDERS, DEDUCED FROM THE MOMENTS OF THE EXTERNAL FORCES AND FROM THE SHEARING-FORCES, AND FROM THESE COMBINED.

40. By the definition of statical moment, as given in article 9, it is the product of two numbers, — one representing the length of a straight line, the other the amount of force conceived to be applied at either end of the given straight line or lever arm, and to act in a line at right angles to that arm. If, therefore, H is the force or strain, and h the lever arm, the moment is

and the strain

$$\left.\begin{aligned} M &= Hh, \\ H &= M \div h, \end{aligned}\right\} \qquad (95)$$

whose line of action is perpendicular to the arm h.

It hence appears that strains are deducible from moments; and, in order to apply this method to the determination of strains in girders of the most general description, let Fig. 13 represent one end of a framed girder, consisting of triangular panels, as ABD, CDF, EFH, etc., or of quadrilateral panels, $ABDC$, $CDFE$, $EFHG$, etc., whichever we choose to conceive them. Let the horizontal projection of each panel length, AC, CE, EG, etc., BD, DF, FH, etc., of both top and bottom chords, be equal to $2c$; and let the apices, A, C, E, G, etc., be horizontally projected at the centres of the horizontal projections of

BD, *DF*, *FH*, etc., respectively. Let the inclinations of the segments of the top chord, *AC*, *CE*, *EG*, etc., to the horizon be α_1, α_2, α_3, etc.; those of the segments of the bottom chord, *BD*, *DF*, *FH*, etc., to the horizon be β_1, β_2, β_3, etc.; the inclination to the horizon of a *Y* web member, as *BA*, *DC*, *FE*, etc., be ϕ_1, ϕ_2, ϕ_3, etc.; and the inclination to the horizon of a *Z* web member, as *AD*, *CF*, *EH*, etc., be θ_1, θ_2, θ_3, etc.: each angle of inclination of chord, α, β, to be measured from the horizontal drawn through the left end of the chord segment, and each angle, ϕ, θ, to be measured from the horizontal through the lower extremity of the web member, as in trigonometrical notation.

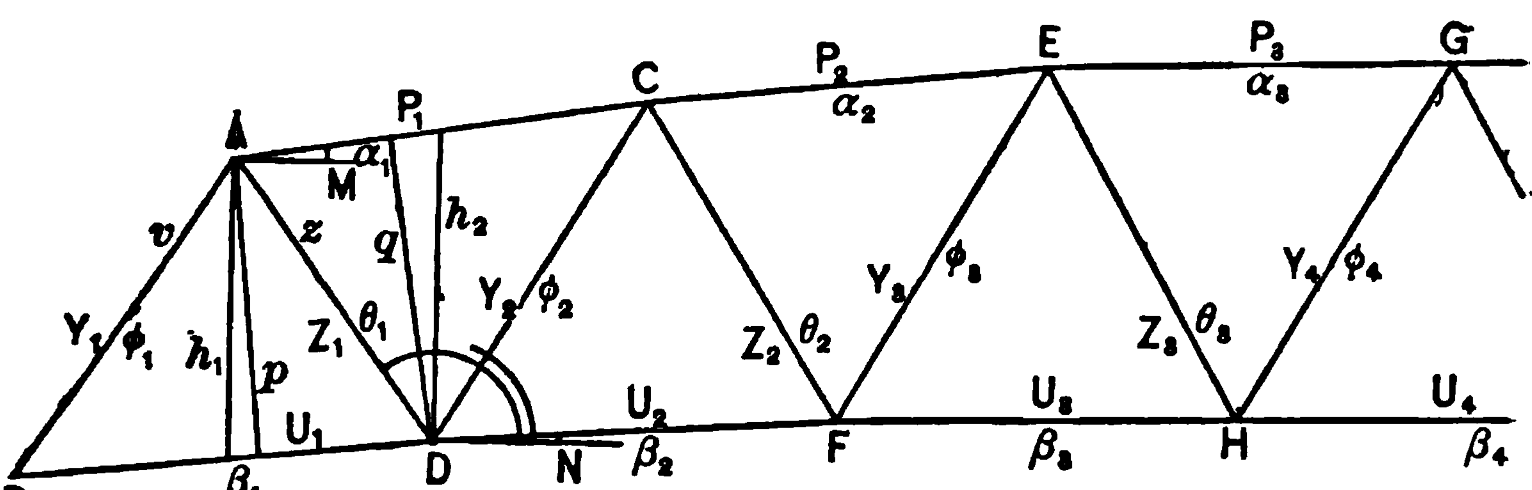


FIG. 13.

Assume, further, that each member of the structure is capable of resisting the strain that may come upon it, either of tension or compression; and, for distinction, call strains in compression positive, and tensile strains negative.

Also, the simultaneous forces acting at each apex are supposed to be in equilibrium, and the structure at rest. All the dimensions of the skeleton girder, as Fig. 13, are given, and may be varied so as to represent all the usual forms of girder, as illustrated below.

Let *P* symbolize the strain along any segment of the top chord; *U* the strain along any segment of the bottom chord; *Y* the strain along a *Y* web member whose slope is ϕ, as defined

above, and length v; and Z the strain along a Z web member whose slope is θ, and length z. Therefore $p = -v \sin(\phi - \beta)$, and $q = z \sin(180 - \theta + \alpha) = z \sin(\theta - \alpha)$, where p is negative, and represents the line drawn from any upper apex perpendicular to the chord opposite it, and q is positive, and denotes the length of the perpendicular drawn from any lower apex to the chord opposite.

Let H_r and H_{r+1} denote the two simultaneous horizontal strains at consecutive apices, whose difference, ΔH, is the greatest of all differences of simultaneous horizontal strains for that interval; and let M_r, M_{r+1}, be the corresponding moments, and h_r, h_{r+1}, the heights or vertical distances from those apices to the axis of each chord opposite. Then $H = M \div h$, and ΔH is the horizontal component of the strain developed in the diagonal or web member for the interval to which ΔH belongs.

41. Suppose that the greatest moments due the given loading have been computed for the vertical section at right angles to the plane of the girder through each point, B, A, D, C, F, etc., Fig. 13; that is, at intervals each equal to c: and call these moments M_1, M_2, M_3, etc., at these consecutive intervals. Also, if the simultaneous moments which yield the greatest difference of horizontal strains at any two consecutive apices are different from these greatest moments, as may be the case for rolling loads, suppose such moments known.

We then have, from the figure and from the principles of articles 10 and 3, —

$$\left.\begin{array}{llll} \text{Strain along } BD, & U_1 = M_2 \div p_1; & \text{along } AC, & P_1 = M_3 \div q_1. \\ DF, & U_2 = M_4 \div p_2; & CE, & P_2 = M_5 \div q_2. \\ FH, & U_3 = M_6 \div p_3; & EG, & P_3 = M_7 \div q_3. \\ \text{Generally} & U = M_r \div p; & & P = M_{r+1} \div q. \end{array}\right\} \quad (96)$$

$$\left.\begin{array}{llll} \text{Strain along } AB, & Y_1 = \Delta_1 H \div \cos\phi_1; & \text{along } AD, & Z_1 = \Delta_2 H \div \cos\theta_1. \\ CD, & Y_2 = \Delta_3 H \div \cos\phi_2; & CF, & Z_2 = \Delta_4 H \div \cos\theta_2. \\ \text{Generally} & Y = \Delta_r H \div \cos\phi; & & Z = \Delta_{r+1} H \div \cos\theta. \end{array}\right\} \quad (97)$$

42. Shearing Forces and Strains. — If through any material body or structure we conceive a plane to pass, dividing the body into any two parts whatsoever, and assume either one of the two parts to be fixed in position, while the other part slides or tends to slide in any direction along this plane, then the force acting in a line parallel to the dividing-plane, and causing this sliding or tendency to slide, is called a *shearing-force*, and the strain on the particles of the body lying in this plane, resisting or tending to resist the shearing-force, is called the *shearing-strain*. The amount of shearing-strain per unit of the shearing-surface is its intensity; and the intensity at the instant of rupture (that is, at the beginning of actual sliding of one part of the body over the other along the shearing-plane) is the breaking shearing-strain, and is, in general, peculiar to each kind of material, and must be determined by experiment.

The published results of trustworthy experiments for determining the ultimate resistance of materials to shearing, are very meagre; and the following table, compiled from the works of two of the best authorities I know, viz., Professor Rankine and Mr. Bindon B. Stoney, is probably as worthy of confidence as any published records of the kind.

TABLE I.

ULTIMATE RESISTANCE OF MATERIALS TO SHEARING, IN POUNDS, PER SQUARE INCH.

MATERIAL.	Resistance to Shearing.	Remarks.
Metals.		
Cast-iron . .	27700 R.	Tensile strength ranges from 13400 to 29000. R.
Cast-iron . .	17920 to 20160 S.	"Substantially its tensile strength." S.
Wrought-iron .	55059 S.	Mean of 5 tests by Mr. Jones, punching-plates.
Wrought-iron .	50400 S.	Mean of 2 tests, Mr. Little, hammered scrap, inch punch.
Wrought-iron .	43456 S.	Mean of 4 tests, Mr. Little, hammered scrap, two-inch punch.
Wrought-iron .	50848 S.	Bar, 0.5×3 inches, punched both ways, Mr. Little, mean.
Wrought-iron .	48160 S.	2 bars, 1×3 inches, punched both ways, Mr. Little, mean.
Wrought-iron .	46144 S.	Flanged tire, 1.8×5 inches, edgewise, by Mr. Little.
Wrought-iron .	52192 S.	Rivet, $\frac{7}{8}$ inch, Mr. Clark. Tensile strength 53760.
Wrought-iron .	45696 S.	Rivet, $\frac{7}{8}$ inch, 2 plates, Mr. Clark.
Wrought-iron .	49952 S.	Rivet, $\frac{7}{8}$ inch, 3 plates, Mr. Clark.
Wrought-iron .	50000 R.	
Steel	63796 S.	Kirkaldy, rivet steel, tensile strength 86450.
Timber.		
Fir	592 S.	In direction of grain, Barlow.
Fir, red pine .	500 to 800 R.	
Fir, spruce . .	600 R.	
Fir, larch . .	970 to 1700 R.	
Oak	2300 R.	
Oak	4000 S.	Across grain, Rankine's deduction from Parsons's tests of English oak treenails.
Ash and elm .	1400 R.	

Abbreviations: R., Rankine; S., Stoney.

Instead of a plane cutting the body into two parts, we may conceive it cut into two separate parts by any cylindrical surface, and may suppose the sliding, or tendency to slide, to be in the direction of the generating line of the cylindrical surface, as in the case of a cylindrical punch.

43. From the definition of shearing-force, it follows, that if any girder, as Fig. 13, be cut by a vertical plane at right angles to its own plane, then the shearing force or strain at this vertical section is equal to the algebraic sum of the vertical components of all the forces impressed upon either side of this vertical plane. And these two algebraic sums of the vertical components of the forces impressed upon the opposite sides of this vertical plane will have contrary signs, and be numerically equal, except when a vertical force or weight is applied in the vertical plane itself, in which case the shearing-strains on opposite sides of the shearing-plane will differ by the value of this weight applied in the vertical plane.

Since the resultant of parallel forces is simply their algebraic sum, if the external forces applied to a girder are all vertical (that is, made up of the applied weights and the consequent vertical resistances of the piers), the shearing-force on either side of a vertical shearing-plane is merely the difference between the sum of the weights and the re-action of the pier on that side.

If, therefore, S denotes the shearing-force on either side of the shearing-plane, W being positive and denoting any weight, and V being the vertical re-action and negative, on the side chosen, we then have

$$S = V + \Sigma_0^x W, \qquad (98)$$

where $\Sigma_0^x W$ is the sum of all the weights between the shearing-plane and the point of support having the re-action V; that is, of all the weights on the length x.

In case of the semi-beam for the free end, $V = 0$, and

$$\left.\begin{array}{ll} \text{Sum of weights on } l - x, & S = \Sigma_0^{l-x} W, \\ \text{Sum of weights on } l, & S = \Sigma_0^{l} \ W; \end{array}\right\} \qquad (99)$$

x being measured from the fixed end.

When the girder is supported at both ends, the re-actions due to a single weight, W, applied at the distance d', Fig. 9, from the left support, are, by equations (38) and (39),

$$\text{At left support,} \quad V_1 = -W\frac{l - d'}{l};$$

$$\text{At right support,} \quad V_2 = -W\frac{d'}{l};$$

calling them negative.

And for any number of different weights applied at different points,

$$\left.\begin{array}{l} V_1 = -\Sigma\left(W\dfrac{l - d'}{l}\right) \\ V_2 = -\Sigma\left(W\dfrac{d'}{l}\right). \end{array}\right\} \qquad (100)$$

Therefore for this case the shearing-strain at a vertical section distant x from the left support is

$$\left.\begin{array}{l} S = -\Sigma\left(W\dfrac{l - d'}{l}\right) + \Sigma_0^{x} \quad W, \\ \text{or} \\ S = -\Sigma\left(W\dfrac{d'}{l}\right) \quad + \Sigma_0^{l-x} W, \end{array}\right\} \qquad (101)$$

If upon the girder supported at both ends there are $n - 1$

equal weights, W, at equal intervals, $c = \frac{l}{n}$, and $\frac{1}{2}W$ upon each end of the girder on a pier, we have

$$\left.\begin{aligned} V_1 &= -\frac{n}{2}W = V_2, \\ S &= -\frac{n}{2}W + (r + \tfrac{1}{2})W = \tfrac{1}{2}W(2r - n + 1)\,; \end{aligned}\right\} \quad (102)$$

the shearing-plane being at the r^{th} point of division counted from the left.

For a uniform continuous load, lw, upon a girder supported at its extremities,

$$\text{At any point, } x, \qquad \left.\begin{aligned} V_1 &= -\tfrac{1}{2}lw = V_2. \\ S &= -\tfrac{1}{2}lw + wx. \end{aligned}\right\} \quad (103)$$

For a uniform continuous load, lw, upon a semi-girder at any point, x, measured from the fixed end, the shearing-strain is

$$\text{and when } x = 0, \qquad \left.\begin{aligned} S &= (l - x)w\,; \\ S &= lw. \end{aligned}\right\} \quad (104)$$

For any partial uniform continuous load, bw', Fig. 9, on a beam simply supported at its two ends, the re-actions of the piers are

$$\left.\begin{aligned} V_1 &= -\frac{bw'}{l}(l - a - \tfrac{1}{2}b), \\ V_2 &= -\frac{bw'}{l}(a + \tfrac{1}{2}b)\,; \end{aligned}\right\} \quad (105)$$

which re-actions are identical with the shearing-strains for the unloaded parts of the beam.

But for the loaded part b, the shearing-strain is

$$\left.\begin{aligned} S &= -\frac{bw'}{l}(l - a - \tfrac{1}{2}b) + w'(x - a), \\ \text{or} \quad S &= -\frac{bw'}{l}(a + \tfrac{1}{2}b) \qquad + w'(a + b - x). \end{aligned}\right\} \quad (106)$$

If $a = 0$, and $x = b$, (106) becomes

$$S = \pm\frac{w'b^2}{2l}, \qquad (107)$$

which is the shearing-strain at the foremost end of a uniform continuous load reaching to the left end of the beam supported at both ends. And equation (107) gives the greatest positive and the greatest negative value of S for this kind of load; since, in equations (106), x cannot be greater than b, and in the first of those equations S is an increasing function of x, while in the second S is a decreasing function of x.

The shearing-strain at any point, x, of the partial uniform continuous load on a semi-beam, Fig. 8, is

$$S = w'(a + b - x); \qquad (108)$$

x being measured from the fixed end, and not being greater than $a + b$, nor less than a.

In order to simplify the application of equations (101) for the important case of a partial or complete uniform discontinuous moving-load, L, to be applied at equal intervals, $c = \frac{l}{n}$, along the girder, we proceed as in article 20, where we found the moments due such a load on a beam supported at both ends. Let $r_1 - r_2 =$ the number of weights, L, on the beam

at any instant. Take r not less than r_2 nor greater than r_1. Then, in the first of equations (101), we have

$$\Sigma a^0 = r_1 - r_2$$

and

$$\Sigma a' = c[(r_2 + 1) + (r_2 + 2) + (r_2 + 3) + \ldots r_1]$$
$$= \tfrac{1}{2}c(r_1 - r_2)(r_1 + r_2 + 1)$$

for the first term. But in the second term, $\Sigma_0^x L$, we must take L no times for the left unloaded end of the beam, $r - r_2$ times for the loaded part, and $r_1 - r_2$ times for the unloaded part on the right end. Therefore

$$S = -\frac{L}{l}[(r_1 - r_2)l - \tfrac{1}{2}c(r_1 - r_2)(r_1 + r_2 + 1)] \qquad (109)$$

for the shearing-strain left of the load.

$$S = -\frac{L}{l}[(r_1 - r_2)l - \tfrac{1}{2}c(r_1 - r_2)(r_1 + r_2 + 1) - (r - r_2)l], \qquad (110)$$

which is the shearing-strain between the points rc and $(r + 1)c$.

$$S = \frac{Lc}{2l}(r_1 - r_2)(r_1 + r_2 + 1) = \frac{L}{2n}r_1(r_1 + 1) \qquad (111)$$

if $r_2 = 0$, and $c = \frac{l}{n}$; and this is the shearing-strain at and beyond the foremost end of a uniform discontinuous load reaching back to the left end of the beam.

44. The influence of end moments on the normal shearing-strains may be regarded as operating upon that term only of the shearing-strain which expresses the re-action of the pier.

Now, the pier moment M_2, acting at the right-hand pier, will affect the re-action V_1 of the left pier by the amount $-\frac{M_2}{l}$; and the pier moment M_1, acting at the left pier, will affect the

reaction V_2 of the right pier by the amount $-\frac{M_1}{l}$. But, by the principles of article 10, the force $-\frac{M_1}{l}$, acting at the right end of the lever arm, l, induces a re-action, $+\frac{M_1}{l}$, at the left end of that arm, that is, in this case, at the left pier, where, consequently,

$$\left.\begin{aligned} \Delta S = \Delta V_1 &= \frac{M_1 - M_2}{l}. \\ \text{Similarly} \qquad \Delta S = \Delta V_2 &= \frac{M_2 - M_1}{l}, \end{aligned}\right\} \tag{112}$$

which are the increments of the shearing-strains due to the end moments, and are to be added algebraically to the shearing-strains found for the given load on the same beam simply supported at its two ends.

The values of M_1 and M_2 are here arbitrary, but will be determined for particular cases in subsequent chapters of this work.

45. To find the Shearing-Strain at any Vertical Section of a Girder (Fig. 13) in Terms of the Vertical Components of the Forces which are impressed upon the Shearing-Plane through the Members of the Girder cut by that Plane.—Using the notation already given in article 40, Fig. 13, and equations (3), we have, as the vertical component resulting from all the pressures on the left of each odd vertical plane, or the planes through the lower apices, B, D, F, etc.,

$$\begin{array}{ll} \text{Left of } B, & S_0 = V_1. \\ \text{Left of } D, & S_2 = -P_1 \sin\alpha_1 + Z_1 \sin\theta_1 - U_1 \sin\beta_1; \\ \text{Left of } F, & S_4 = -P_2 \sin\alpha_2 + Z_2 \sin\theta_2 - U_2 \sin\beta_2; \\ \cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \end{array}$$

$$\text{Left of } (2r+1)^{\text{th}} \text{ apex}, \; S_{2r} = -P_r \sin\alpha_r + Z_r \sin\theta_r - U_r \sin\beta_r; \tag{113}$$

counting r on P, Z, and U.

And on the left of each even vertical plane, or those through the upper apices, A, C, E, etc.,

$$\begin{aligned}
&\text{Left of } A,\ S_1 = -P_0 \sin\alpha_0 - Y_1 \sin\phi_1 - U_1 \sin\beta_1;\\
&\text{Left of } C,\ S_3 = -P_1 \sin\alpha_1 - Y_2 \sin\phi_2 - U_2 \sin\beta_2;\\
&\text{Left of } E,\ S_5 = -P_2 \sin\alpha_2 - Y_3 \sin\phi_3 - U_3 \sin\beta_3;\\
&\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot
\end{aligned}$$

Left of $(2r)^{\text{th}}$ apex,

$$S_{2r-1} = -P_{r-1} \sin\alpha_{r-1} - Y_r \sin\phi_r - U_r \sin\beta_r; \quad (114)$$

counting r on Y and U.

These values of S may be used to verify solutions by equations (96) and (97), as will be illustrated in some of the examples below.

46. Strains in all Members of a Girder determined from the Given Shearing-Forces.—Equilibrium of the system requires that at each apex, Fig. 13, the sum of the horizontal forces, as well as the sum of the vertical forces, shall vanish. Therefore at any lower apex we have

$$U_{r-1} \cos\beta_{r-1} - Z_{r-1} \cos\theta_{r-1} - Y_r \cos\phi_r - U_r \cos\beta_r = 0, \quad (115)$$

and

$$P_{r-1} \cos\alpha_{r-1} + Z_r \cos\theta_r \qquad + Y_r \cos\phi_r - P_r \cos\alpha_r = 0 \quad (116)$$

at any upper apex.

The four equations, (113), (114), (115), (116), enable us to determine the four quantities, U_r, P_r, Y_r, Z_r, in terms of P_{r-1}, and the given shearing-strains, S_{2r-1} and S_{2r}, if we use the auxiliary equation,

$$-P_{r-1} \cos\alpha_{r-1} = U_{r-1} \cos\beta_{r-1} - Z_{r-1} \cos\theta_{r-1}, \quad (117)$$

expressing the equality of the horizontal strains at any lower apex, and at the point directly above it in the top chord.

Therefore, after solving and reducing,

$$U_r = \frac{S_{2r-1}\cos\phi_r - P_{r-1}\sin(\phi_r - \alpha_{r-1})}{\sin(\phi_r - \beta_r)}, \qquad (118)$$

$$P_r = \frac{S_{2r}\cos\theta_r - U_r\sin(\theta_r - \beta_r)}{\sin(\theta_r - \alpha_r)}, \qquad (119)$$

$$Y_r = \frac{-P_{r-1}\sin\alpha_{r-1} - U_r\sin\beta_r - S_{2r-1}}{\sin\phi_r}, \qquad (120)$$

$$Z_r = \frac{P_r\sin\alpha_r + U_r\sin\beta_r + S_{2r}}{\sin\theta_r}. \qquad (121)$$

Now, if we begin at the left end of the girder, Fig. 13, to compute, U_r becomes U_1, and P_{r-1} is zero; therefore (118) gives U_1: and with this value of $U_r = U_1$ we at once find $P_r = P_1$, by (119), and similarly follow Y_r and Z_r from (120) and (121). A repetition of this process, putting the value of P_r just found, in the place of P_{r-1}, may be continued through the girder.

47. We will now give examples illustrating the determination of strains in open girders, first by the method of moments, and second by the method of shearing-strains, and will verify the solutions by equations (113) and (114).

EXAMPLE I.—Let B, Fig. 13, represent the unsupported end of a semi-girder, whose fixed end coincides with the vertical plane passing through E, and at right angles to the plane of the girder. Let the horizontal distance between consecutive apices, B, A, D, C, etc., $= c =$ 10 feet, and the elevation of the apices, in feet, above the point B be as shown in the first line of the solution below. These elevations, with the horizontal distance $c =$ 10 feet, furnish all the angles and lines required. If any apex is below B, its elevation is negative; and all angles and trigonometric functions follow the ordinary trigonometrical laws. At each apex, top and bottom, of this semi-beam, let a weight, $W = 1$ ton, be applied. Required the strains due this load in every member of the girder. The moment M is given by (35), where $n = 5$, $l = 50$, $W = 1$, and r takes the values 4, 3, 2, 1, 0, in succession.

LOGARITHMIC SOLUTION FOR DIMENSIONS AND STRAINS. — SEMI-BEAM.

Method of Moments by Equations (96) and (97).

APEX.	A.	D.	C.	F.	E.	H.
Elevation above B	18	1	20	2.5	21	2.5
tan β	-	0.05	-	0.075	-	0
tan α	-	-	0.1	-	0.05	-
log tan β	-	8.6989700	-	8.8750613	-	-
log sin β	-	8.6984422	-	8.8738446	-	-
β	-	2° 51′ 45″	-	4° 17′ 21″	-	0
log tan α	-	-	9	-	8.6989700	-
log sin α	-	-	8.9978997	-	8.6984422	-
α	-	-	5° 42′ 41″	-	2° 51′ 45″	-
tan φ	1.8	-	1.9	-	1.85	-
tan θ	-	−1.7	-	−1.75	-	-
log tan φ	0.2552725	-	0.2787536	-	0.2671717	-
log sin φ	9.9415891	-	9.9460040	-	9.9443377	-
log cos φ	9.6863166	-	9.6681466	-	9.6771666	-
φ	60° 56′ 43″	-	62° 14′ 30″	-	61° 36′ 25″	-
log tan θ	-	0.2304489n	-	0.2430380n	-	-
log sin θ	-	9.9354741	-	9.9386408	-	-
log cos θ	-	9.7050252n	-	9.6956018n	-	-
180° − θ	-	59° 32′ 4″	-	60° 15′ 18″	-	-
φ − β	58° 4′ 58″	-	57° 57′ 9″	-	61° 36′ 25″	-
180° − θ + α	-	65° 14′ 45″	-	63° 7′ 3″	-	-
c = 10. log 10	1	1	1	1	1	-
log v	1.3136834	-	1.3318534	-	1.3228334	-
v = c ÷ cos φ	-	-	-	-	-	-
log sin (φ − β)	9.9288119	-	9.9281953	-	9.9443377	-
log ρ	1.2424953n	-	1.2600487n	-	1.2671711n	-
ρ = v sin (φ − β)	-	-	-	-	-	-

LOGARITHMIC SOLUTION FOR DIMENSIONS AND STRAINS. — SEMI-BEAM. — *Concluded.*

APEX.	A.	D.	C.	F.	E.	H.
$\log s$	-	1.2949748	-	1.3043982	-	-
$s = c \div \cos\theta$	-	-	-	-	-	-
$\log \sin(180° - \theta + a)$	-	9.9581397	-	9.9503336	-	-
$\log q$	-	1.2531145	-	1.2547318	-	-
$q = s \sin(180° - \theta + a)$	-	-	-	-	-	-
h	17.5	18	18.25	18	18.5	-
$M = -5(5 - r + 1)(5 - r)$	−10	−30	−60	−100	−150	-
$\log M$	1 *n*	1.4771213*n*	1.7781513*n*	2 *n*	2.1760913*n*	-
$\log U_r = \log(M \div p)$	9.7575047	-	0.5181026	-	0.9089202	-
U_r	0.5721	-	3.2969	-	8.1081	-
$\log P_r = \log(M \div q)$	-	0.2240068*n*	-	0.7452682*n*	-	-
P_r	-	−1.6749	-	−5.5625	-	-
$\log h$	1.2430380	1.2552725	1.2612629	1.2552725	1.2671717	-
$\log(M \div h) = \log H$	9.7569620*n*	0.2218488*n*	0.5168884*n*	0.7447275*n*	0.9089196*n*	-
H	−0.57143	−1.66666	−3.28767	−5.55555	−8.10810	-
ΔH	−0.57143	−1.09523	−1.62101	−2.26788	−2.55255	-
$\log \Delta H$	9.7569620*n*	0.0395014*n*	0.2097856*n*	0.3556202*n*	0.4069743*n*	-
$\log(\Delta H \div \cos\phi) = \log Y_r$	0.0706454*n*	-	0.5416390*n*	-	0.7298077*n*	-
Y_r	−1.1766	-	−3.4805	-	−5.3679	-
$\log(\Delta H \div \cos\theta) = \log Z_r$	-	0.3344762	-	0.6600184	-	-
Z_r	-	2.1601	-	4.5711	-	-

Equations (113), (114). — *Proof.* — *Shearing-Strains.* — *Tons.*

	S_1	S_2	S_3	S_4	S_5
$-P_{r-1} \sin a_{r-1} - Y_r \sin\phi_r - U_r \sin\beta_r$	1	-	3	-	5
$-P_r \sin a_r + Z_r \sin\theta_r - U_r \sin\beta_r$	-	2	-	4	-

Logarithmic Solution by Equations (118) to (121).

Method of Shearing-Strains.

$r.$	1	2	3
ϕ_r	60° 56′ 43″	62° 14′ 30″	61° 36′ 25″
α_{r-1}	0	5° 42′ 41″	2° 51′ 45″
$\phi_r - \alpha_{r-1}$	60° 56′ 43″	56° 31′ 49″	58° 44′ 40″
β_r	2° 51′ 45″	4° 17′ 21″	0
$\phi_r - \beta_r$	58° 4′ 58″	57° 57′ 9″	61° 36′ 25″
$180° - \theta_r$	59° 32′ 4″	60° 15′ 18″	-
$180° - (\theta_r - \beta_r)$	62° 23′ 49″	64° 32′ 39″	-
$180° - (\theta_r - \alpha_r)$	65° 14′ 45″	63° 7′ 3″	-
S_{2r-1}	1	3	5
$\log S_{2r-1}$	0.	0.4771213	0.6989700
$\log \cos \phi_r$	9.6863166	9.6681466	9.6771666
$\log S_{2r-1} \cos \phi_r$	9.6863166	0.1452679	0.3761366
$S_{2r-1} \cos \phi_r$	0.48564	1.3972	2.3776
$\log P_{r-1}$	-	0.2240067n	0.7452619n
$\log \sin (\phi_r - \alpha_{r-1})$	-	9.9212585	9.9318958
$\log P_{r-1} \sin (\phi_r - \alpha_{r-1})$	-	0.1452652n	0.6771577n
$-P_{r-1} \sin (\phi_r - \alpha_{r-1})$	-	1.3972	4.7551
Numerator of (118)	0.48564	2.7944	7.1327
log num.	9.6863166	0.4462948	0.8532540
$\log \sin (\phi_r - \beta_r)$	9.9288119	9.9281953	9.9443377
$\log U_r$	9.7575047	0.5180995	0.9089163
U_r	0.5721	3.2969	8.1081
S_{2r}	2	4	-
$\log S_{2r}$	0.3010300	0.6020600	-
$\log \cos \theta_r$	9.7050252n	9.6956018n	-
$\log S_{2r} \cos \theta_r$	0.0060552n	0.2976618n	-
$S_{2r} \cos \theta_r$	−1.01404	−1.9845	-
$\log \sin (\theta_r - \beta_r)$	9.9475214	9.9556478	-
$\log U_r \sin (\theta_r - \beta_r)$	9.7050261	0.4737473	-
$-U_r \sin (\theta_r - \beta_r)$	−0.50702	−2.9768	-
Numerator of (119)	−1.52106	−4.9613	-
log num.	0.1821464n	0.6955955n	-
$\log \sin (\theta_r - \alpha_r)$	9.9581397	9.9503336	-
$\log P_r$	0.2240067n	0.7452619n	-
P_r	−1.6749	−5.5625	-
$\log \sin \alpha_{r-1}$	-	8.9978997	8.6984422

LOGARITHMIC SOLUTION BY EQUATIONS (118) TO (121). — *Concluded.*

$r.$	1	2	3
$\log P_{r-1} \sin a_{r-1}$	-	9.2219064n	9.4437041n
$P_{r-1} \sin a_{r-1}$	-	—0.16666	—0.27777
$\log \sin \beta_r$	8.6984422	8.8738446	-
$\log U_r \sin \beta_r$	8.4559469	9.3919441	0.9089163
$U_r \sin \beta_r$	0.02857	0.24657	0
Numerator of (120)	—1.02857	—3.0799	—4.7222
log num.	0.0122339n	0.4885380n	0.6741464n
$\log \sin \phi_r$	9.9415891	9.9469040	9.9443377
$\log Y_r$	0.0706448n	0.5416340n	0.7298087n
Y_r	—1.1766	—3.4805	—5.3679
From $P_{r-1} \sin a_{r-1}$ comes $P_r \sin a_r$,	—0.16666	—0.27777	-
Numerator of (121)	1.8619	3.9688	-
log num.	0.2699587	0.5986581	-
$\log \sin \theta_r$	9.9354741	9.9386408	-
$\log Z_r$	0.3344846	0.6600173	-
Z_r	2.1601	4.5711	-

N.B. — The method of shearing-strains, though applicable, is not conveniently used for live loads, since every change of load requires recomputation from the beginning.

48. Maxima Strains in the Web Members of an Open Girder, deduced from the Moments and Shearing-Forces combined, for Uniform Discontinuous Dead and Live Loads.

Let W = panel weight of dead load at $(n - 1)$ equidistant points,

L = panel weight of live load to be applied at the same points,

M_W = moment at each panel point due dead load, by equation (65),

M_L = moment due live load at its foremost end, by equation (64),

S_W = shearing-force at each panel point due dead load, by (102),
S_L = shearing-force at foremost end of live load due live load, by (111);

$\therefore \quad S_W + S_L$ = greatest shearing-force simultaneous with $M_W + M_L$,

$\frac{M_W + M_L}{h} = H$ = simultaneous horizontal chord strain.

Now, we have on the immediate right of any vertical plane through an upper apex (Fig. 13),

$$U \cos\beta = -H_r = -P\cos\alpha + Z\cos\theta,$$

$$-P\sin\alpha + Z\sin\theta - U\sin\beta = S_r.$$

Whence, after eliminating U and P, there results,

$$Z = \frac{-H\dfrac{\sin(\beta - \alpha)}{\cos\beta} + S\cos\alpha}{\sin(\theta - \alpha)}, \qquad (122)$$

where H and S belong to the vertical section through the upper extremity of the Z_θ member, which joins the left end of the P_α chord segment, to the right end of the U_β chord segment.

Similarly, on the immediate right of the vertical plane through the consecutive lower apex,

$$Y = \frac{H_1\dfrac{\sin(\beta_1 - \alpha)}{\cos\alpha} - S_1\cos\beta_1}{\sin(\phi - \beta_1)}, \qquad (123)$$

where H_1 and S_1 belong to the given vertical plane through the lower extremity of the Y member, which joins the left

end of the $U_{1\beta_1}$ chord segment, to the right end of the P_a chord segment of equation (122).

It may here be observed, that according to our notation (article 40, Fig. 13), in any symmetrical girder, θ in either half-span is the supplement of ϕ in the corresponding panel of the other half-span; also α and β in either half-span are respectively equal to $-\alpha$ and $-\beta$ of the corresponding panel of the other half-span. So s of the first half-span equals the corresponding v of the second.

EXAMPLE. — Uniform discontinuous dead and live loads. Let all that part of Fig. 13 which is on the left of the vertical line through E represent one of the equal half-spans of a girder supported at its two ends, B, and L not shown in the figure. Take the dimensions for each half-span the same as those already given in the example of article 47.

Let the dead load, $W_1 = 4$ tons, be applied at each apex, top and bottom; and the live load, $L_1 = 8$ tons, at the same points progressively. Each extreme apex may be supposed to bear $\frac{1}{2}(W + L)$ when fully loaded; but this will only affect the resistances V_1 and V_2, so far as the present method of computing strains is concerned. We may find greatest strains as follows: —

STRAINS DEDUCED FROM MOMENTS AND SHEARING-FORCES. — TONS.

APEX.	A.	D.	C.	F.	E.	H.	G.	J.	I.
By (65), M_{W+L}	540	960	1200	1440	1500	1440	1200	960	540
log M_{W+L}	2.7323938	2.9822712	3.0791812	3.1583625	3.1760913	-	-	-	-
log p	1.2424953n	-	1.2600487n	-	1.2671711n	-	-	-	-
log q	-	1.2531145	-	1.2547318	-	-	-	-	-
log $(M_{W+L} + p)$	1.4898985n	-	1.8191325n	-	1.9089208n	-	-	-	-
Minimum U	30.896	-	65.938	-	81.081	-	−65.938	-	−30.896
log $(M_{W+L} + q)$	-	1.7291567	-	1.9036307	-	-	-	-	-
Maximum P	-	53.599	-	80.097	-	80.097	-	53.599	-
By (65), M_W	180	320	420	480	500	480	420	320	180
By (64), M_L	72	192	336	480	600	672	672	576	360
$M_W + M_L$	252	512	756	960	1100	1152	1092	896	540
By (102), S_W	−14	−10	−6	−2	+2	+6	+10	+14	+18
By (111), S_L	0.8	2.4	4.8	8	12	16.8	22.4	28.8	36
$S_W + S_L$	−13.2	−7.6	−1.2	+6	+14	+22.8	+32.4	+42.8	+54
h	17.5	18	18.25	18	18.5	18	18.25	18	17.5
log $(M_W + M_L)$	2.4014005	2.7092700	2.8785218	2.9822712	3.0413927	3.0614525	3.0382226	2.9523080	2.7323938
log h	1.2430380	1.2552725	1.2612629	1.2552725	1.2671717	1.2552725	1.2612629	1.2552725	1.2430380
log H	1.1583625	1.4539975	1.6172589	1.7269987	1.7742210	1.8061800	1.7769597	1.6970355	1.4893558
β	2° 51′ 45″	-	4° 17′ 21″	-	0	-	−4° 17′ 21″	-	−2° 51′ 45″
α	-	5° 42′ 41″	-	2° 51′ 45″	-	−2° 51′ 45″	-	−5° 42′ 41″	-
$\beta - \alpha$	−2° 50′ 56″	-	1° 25′ 36″	-	2° 51′ 45″	-	1° 25′ 20″	-	−2° 51′ 45″
log sin $(\beta - \alpha)$	8.6961719n	-	8.3961550	-	8.6984428	-	8.3948002	-	8.6984422n
log cos β, a. c.	0.0005423	-	0.0012181	-	0.	-	0.0012181	-	0.0005423
log $[H \sin(\beta - \alpha) + \cos\beta]$	9.8552787n	-	0.0146320	-	0.4726632	-	0.1729780	-	0.1883403n
1st term of (122)	+0.7166	-	−1.0343	-	−2.9694	-	−1.4893	-	+1.5429
log $(S_W + S_L)$	1.1205739n	0.8808136n	0.0791812n	0.7781513	1.1461280	1.3579348	1.5105450	1.6314438	1.7323938
log cos α	9.9978387	-	9.9994578	-	9.9994578	-	9.9978387	-	0.
log $(S_W + S_L)$ cos α	1.1184126n	-	0.0786390n	-	1.1455858	-	1.5083837	-	1.7323938

STRAINS DEDUCED FROM MOMENTS AND SHEARING-FORCES. — TONS. — *Concluded.*

APEX.	A.	D.	C.	F.	E.	H.	G.	J.	I.
2d term of (122)	−13.1345	-	−1.1985	-	+13.9825	-	+32.2398	-	+54
Numerator of (122)	−12.4179	-	−2.2328	-	11.0131	-	30.7409	-	55.5409
log numerator	1.0940482n	-	0.3488498n	-	1.0419095	-	1.4878438	-	1.7446083
$180° - \theta$	-	59° 32′ 4″	-	60° 15′ 18″	-	61° 36′ 25″	-	62° 14′ 30″	-
$180° - (\theta - \alpha)$	65° 14′ 45″	-	63° 7′ 3″	-	58° 44′ 40″	-	56° 31′ 49″	-	60° 56′ 41″
log sin $(\theta - \alpha)$	9.9581397	-	9.9503335	-	9.9318958	-	9.9212584	-	9.9415891
log Z	1.1359085n	-	0.3985163n	-	1.1100137	-	1.5665854	-	1.8030304
Z	−13.674	-	−2.503	-	+12.883	-	+36.863	-	+63.538
$\beta_1 - \alpha$	-	−1° 25′ 20″	-	−2° 51′ 45″	-	−1° 25′ 36″	-	+2° 50′ 56″	-
log sin $(\beta_1 - \alpha)$	-	8.3948002n	-	8.6984422n	-	8.3961550n	-	8.6961719	-
log cos α, a. c.	-	0.0021613	-	0.0005422	-	0.0005422	-	0.0021613	-
log $[H \sin(\beta_1 - \alpha) \div \cos\alpha]$,	-	9.8509590n	-	0.4259831n	-	0.2028772n	-	0.1055707	-
1st term of (123)	-	−0.7095	-	−2.6667	-	−1.5954	-	+1.4864	-
log cos β_1	-	9.9987819	-	0.	-	9.9987819	-	9.9994977	-
log $(S_W + S_L) \cos\beta_1$	-	0.8795955n	-	0.7781513	-	1.3567167	-	1.6309015	-
2d term of (123)	-	+7.5787	-	−6.	-	22.7361	-	42.7466	-
Numerator of (123)	-	6.8692	-	−8.6667	-	−24.3315	-	−40.2698	-
log numerator	-	0.8369062	-	0.9378538n	-	1.3861689n	-	1.6048760n	-
ϕ	-	-	62° 14′ 30″	-	61° 36′ 25″	-	60° 15′ 18″	-	59° 18′ 4″
$\phi - \beta_1$	-	-	57° 57′ 9″	-	61° 36′ 25″	-	64° 32′ 39″	-	60° 23′ 40″
log sin $(\phi - \beta_1)$	-	9.9281953	-	9.9443376	-	9.9556478	-	9.9475214	-
log Y	-	0.9087109	-	0.9935162n	-	1.4305211n	-	1.6573546n	-
Y	-	+8.105	-	−9.852	-	−26.948	-	45.431	-
Maximum $Y+$	63.538	-	36.863	-	12.883	-	-	-	-
Maximum $Y-$	-	-	-	-	−9.852	-	−26.948	-	45.431
Maximum $Z+$	-	-	-	-	12.883	-	36.863	-	63.538
Maximum $Z-$	−45.431	-	−26.948	-	−9.852	-	-	-	-

STRAINS FOUND FROM MOMENTS, FOR DEAD AND LIVE LOADS.

APEX.	A.	D.	C.	F.	E.	H.	G.	J.	I.
By (65), M_W	180	320	420	480	500	480	420	320	180
By (64), M_L	72	192	336	480	600	672	672	576	360
By (68), M_{+1L}	-	64	168	288	400	480	504	448	288
$M_W + M_L$	252	512	756	960	1100	1152	1092	896	540
$M_W + M_{+1L}$	-	384	588	768	900	960	924	768	468
$\log (M_W + M_L)$	2.4014005	2.7092700	2.8785218	2.9822712	3.0413927	3.0614525	3.0382226	2.9523080	2.7323938
$\log (M_W + M_{+1L})$	-	2.5843312	2.7693773	2.8853612	2.9542425	2.9822712	2.9656720	2.8853612	2.6702459
$\log h$	1.2430380	1.2552725	1.2612629	1.2552725	1.2671717	1.2552725	1.2612629	1.2552725	1.2430380
$\log \frac{M_W + M_L}{h}$	1.1583625	1.4539975	1.6172589	1.7269987	1.7742210	1.8061800	1.7769597	1.6970355	1.4893558
$\log \frac{M_W + M_{+1L}}{h}$	-	1.3290587	1.5081144	1.6300887	1.6870708	1.7269987	1.7044091	1.6300887	1.4272079
$\frac{M_W + M_L}{h} = H_r$	14.400	28.4444	41.4247	53.3333	59.4595	64.	59.8356	49.7778	30.8572
$\frac{M_W + M_{+1L}}{h} = H_{+1}$	-	21.3333	32.2192	42.6666	48.6486	53.3333	50.6301	42.6666	26.7429
$\Delta H = H_{+1} - H_r$	6.9333	3.7748	1.2419	−4.6847	−6.1262	−13.3699	−17.1690	23.0349	30.8572
$\log \Delta H$	0.8409400	0.5768939	0.0940866	0.6706818n	0.7871912n	1.1261282n	1.2347450n	1.3623864n	1.4893558n
$\log \cos\theta$	9.7050252n	-	9.6956018n	-	9.6771666n	-	9.6681466n	-	9.6863166n
$\log (\Delta H + \cos\theta)$	1.1359148n	-	0.3984848n	-	1.1100246	-	1.5665984	-	1.8030392
Z	−13.674	-	−2.503	-	+12.883	-	+36.863	-	+63.538
$\log \cos\phi$	-	9.6681466	-	9.6771666	-	9.6956018	-	9.7050252	-
$\log (\Delta H + \cos\phi)$	-	0.9087473	-	0.9935152n	-	1.4305264n	-	1.6573612n	-
Y	-	+8.105	-	−9.852	-	−26.948	-	−45.431	-

The chord strains, U and P, are to be found as before; their values being greatest when the two uniform loads cover the beam.

In the second line of this last solution, M_L is the moment due live load at its foremost end as that end passes the successive apices.

In the third line, M_{+1L} is the moment one interval beyond the foremost end, and simultaneous with M_L.

It is manifest, from what precedes, that we need compute the moments M_W, M_L, and M_{+1L}, only when h is not constant; as, when h does not vary, we may find ΔM_W and ΔM_L by (71) and (69), whence $\Delta H = \Delta M \div h$.

49. We now proceed to classify girders according to the form which the general equations assume when particular values are assigned to one or more of their variables; first, recapitulating the general equations of the *method of moments*, and of the *method of moments and shearing-forces*.

From equations (95), (96), (97), (122), (123), we arrange

GENERAL FORMULÆ.

METHOD OF	
Moments.	Moments and Shearing-Forces.
$p = -v\sin(\phi-\beta) = -h_r\cos\beta.$	$p = -v\sin(\phi-\beta) = -h_r\cos\beta.$
$q = s\sin(\theta-\alpha) = h_{r+1}\cos\alpha.$	$q = s\sin(\theta-\alpha) = h_{r+1}\cos\alpha.$
$H = \pm M \div h.$	$H = \pm(M_W + M_L) \div h.$
$\Delta H = H_{r+1} - H_r$, or	$S = S_W + S_L.$
$\Delta H = \dfrac{M_{r+1}}{h_{r+1}} - \dfrac{M_r}{h_r}.$	$P = M_{(W+L)(r+1)} \div q = H_{(W+L)(r+1)} \div \cos\alpha.$
$P = \dfrac{M_{r+1}}{q} = \dfrac{H_{r+1}}{\cos\alpha}.$	$U = M_{(W+L)_r} \div p = H_{(W+L)_r} \div \cos\beta.$
$U = \dfrac{M_r}{p} = \dfrac{-H_r}{\cos\beta}.$	$Y = \dfrac{H_{r+1}\sin(\beta_1-\alpha) - S_{r+1}\cos\alpha\cos\beta_1}{\cos\alpha\sin(\phi-\beta_1)}.$
$Y = \Delta_r H \div \cos\phi.$	
$Z = \Delta_{r+1} H \div \cos\theta.$	$Z = \dfrac{-H_r\sin(\beta-\alpha) + S_r\cos\alpha\cos\beta}{\cos\beta\sin(\theta-\alpha)}.$

v = length of the Y web member making the angle ϕ with the horizon, Fig. 13.

z = length of the Z web member making the angle θ with the horizon.

p = length of perpendicular drawn from any upper vertex to the lower chord.

q = length of perpendicular drawn from any lower vertex to the upper chord.

h = height of girder at any apex.

α = inclination of any segment of the upper chord to the horizon, as angle *CAM*.

β = inclination of any segment of the lower chord to the horizon, as angle *FDN*.

ϕ = inclination to horizon of any Y web member, as angle *CDN*.

θ = inclination to horizon of any Z web member, as angle *ADN*.

W = panel weight of dead load.

L = panel weight of live load.

M_W = moment due dead load.

M_L = moment due live load at its foremost end.

M_{W+L} = moment due dead load and full live load; that is, greatest moment for uniform loads.

H = horizontal component of chord strain at a joint or apex.

ΔH = difference of simultaneous horizontal components of chord strains at consecutive apices when this difference is greatest.

S_W = shearing-force due dead load on the immediate right of the shearing-plane.

S_L = shearing-force due live load at any point beyond its foremost end.

P = strain in any segment of top chord.

U = strain in any segment of bottom chord.

Y = strain in any Y web member.

Z = strain in any Z web member.

Count r always from the left, as indicated in the figures.

Now, although we have thus far considered each upper vertex to be horizontally projected midway between the horizontal projections of the lower vertices, this restriction is by no means necessary in the application of these equations, provided we compute the moments and the shearing-strains in accordance with the distribution of the loads, whatever that may be.

THE TWELVE CLASSES OF GIRDERS OF SINGLE SYSTEM.

Length.	*a.*		*v.*		*s.*		*c.*	
Strain.	*P.*		*Y.*		*Z.*		*U.*	
Class.	Top Chord.	α.	Tension Web Member.	ϕ.	Compression Web Member.	θ.	Bottom Chord.	β.
I. . .	Inclined .	α	Inclined .	ϕ	Inclined .	θ	Inclined .	β
II. . .	Inclined .	α	Inclined .	ϕ	Inclined .	θ	Horizontal,	0
III. . .	Horizontal,	0	Inclined .	ϕ	Inclined .	θ	Inclined .	β
IV. . .	Horizontal,	0	Inclined .	ϕ	Inclined .	θ	Horizontal,	0
V. . .	Inclined .	α	Inclined .	ϕ	Vertical .	90°	Inclined .	β
VI. . .	Inclined .	α	Vertical .	90°	Inclined .	θ	Inclined .	β
VII. .	Inclined .	α	Inclined .	ϕ	Vertical .	90°	Horizontal,	0
VIII. .	Horizontal,	0	Inclined .	ϕ	Vertical .	90°	Inclined .	β
IX. . .	Horizontal,	0	Inclined .	ϕ	Vertical .	90°	Horizontal,	0
X. . .	Inclined .	α	Vertical .	90°	Inclined .	θ	Horizontal,	0
XI. . .	Horizontal,	0	Vertical .	90°	Inclined .	θ	Inclined .	β
XII. .	Horizontal,	0	Vertical .	90°	Inclined .	θ	Horizontal,	0

The conditions yielding the twelve classes may be briefly stated thus: —

With regard to α and β we may have $\left\{\begin{array}{l}\text{neither,}\\ \text{the one,}\\ \text{the other,}\\ \text{both,}\end{array}\right\} = 0$; 4 conditions.

With regard to θ and ϕ we may have $\left\{\begin{array}{l}\text{neither,}\\ \text{the one,}\\ \text{the other,}\end{array}\right\} = 90°$; 3 conditions.

Combining these conditions gives twelve classes and no more.

Class I.—All Members but One inclined.

Use General Formulæ.

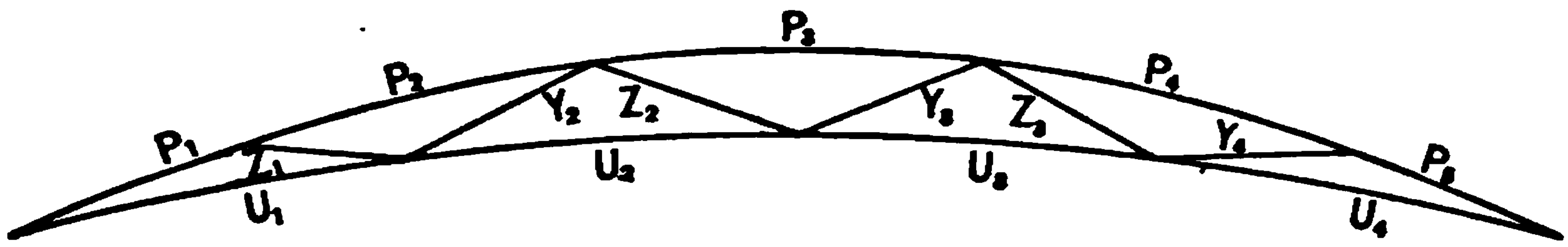


Fig. 14.—The Crescent Girder.

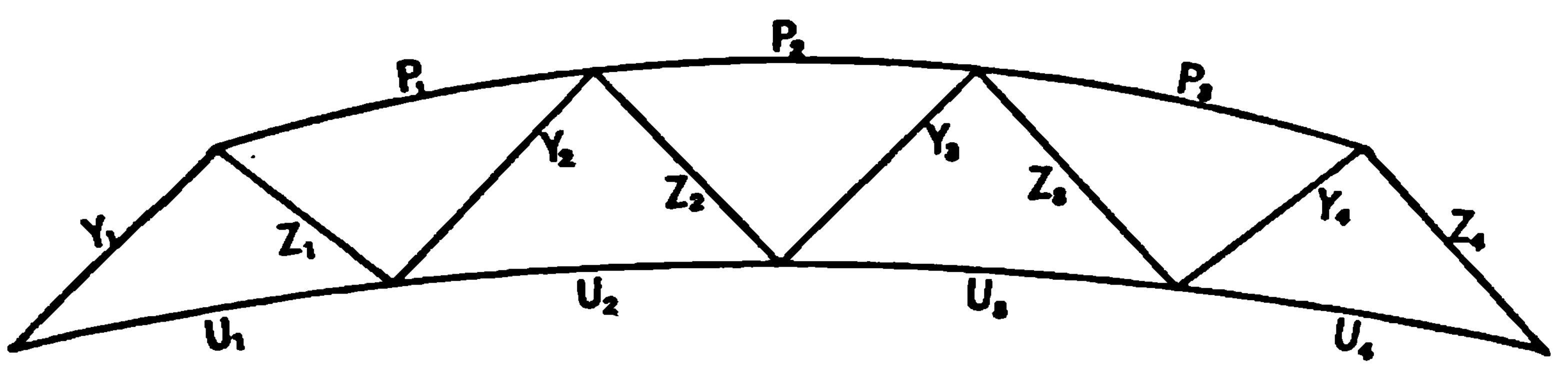


Fig. 15.—The Truncated Crescent.

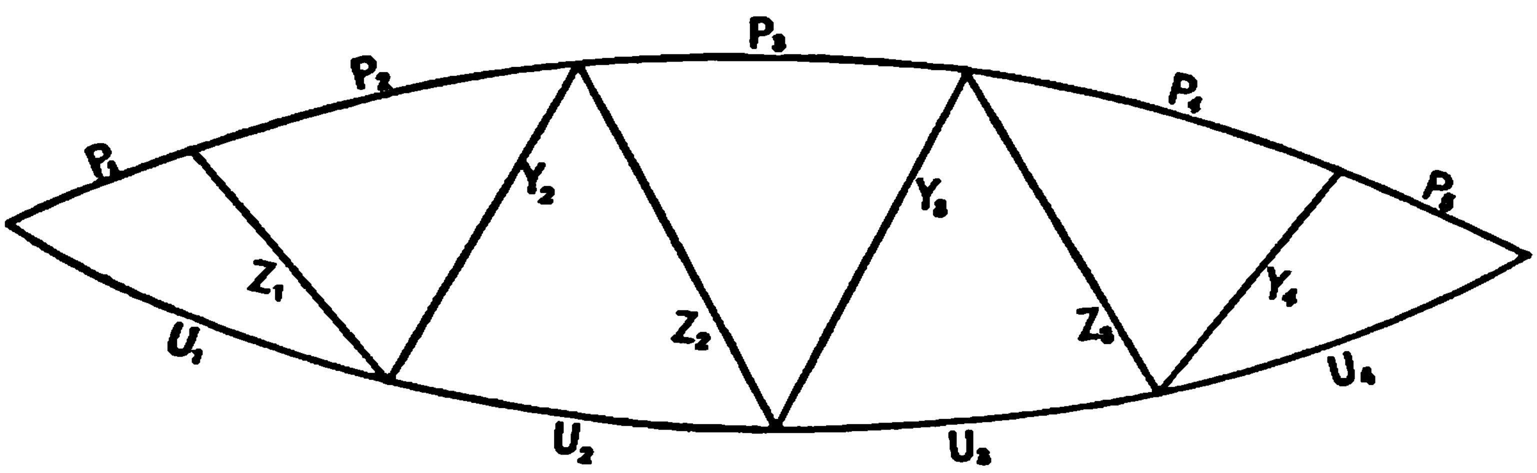


Fig. 16.—The Double Bow, or Brunel Girder.

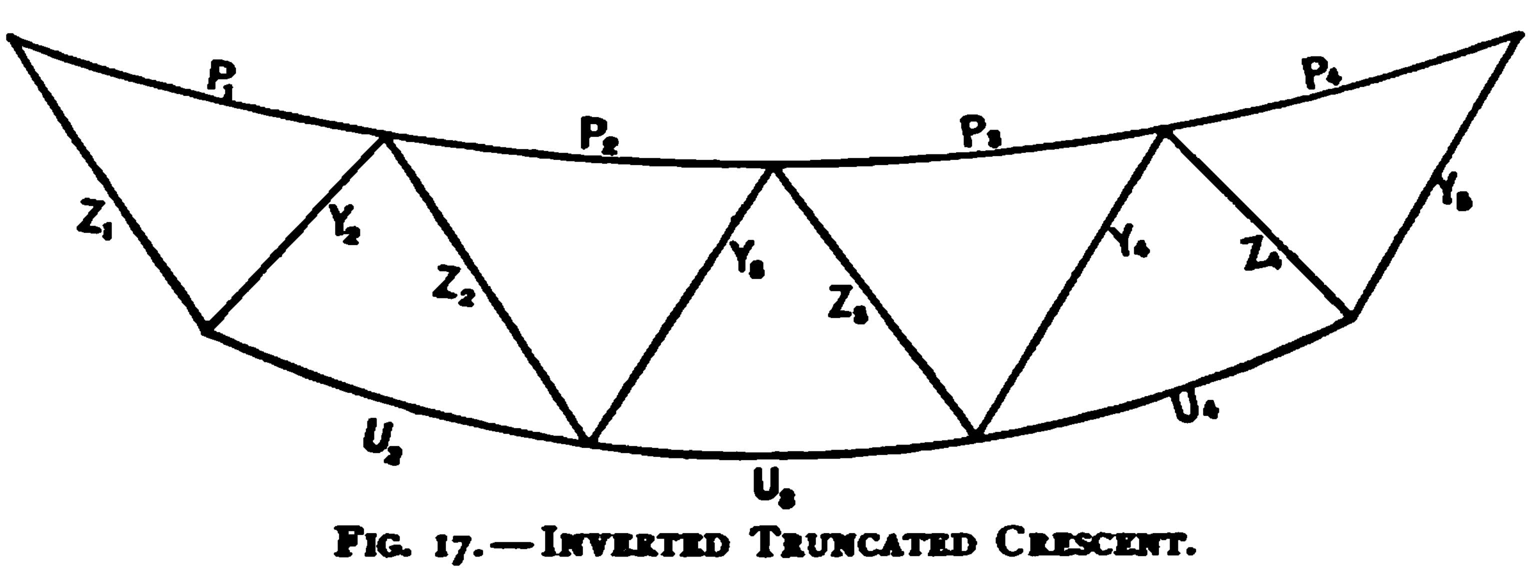


Fig. 17.—Inverted Truncated Crescent.

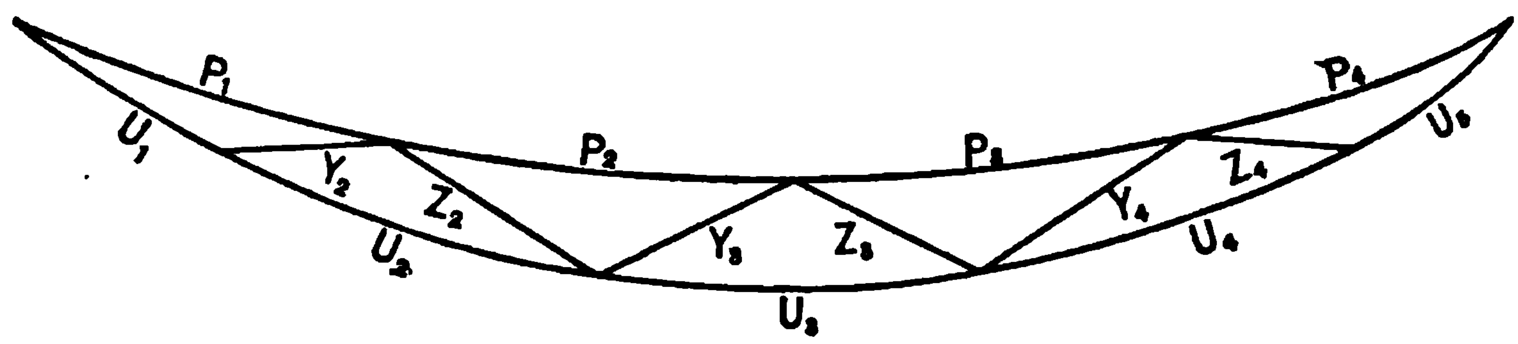


FIG. 18. — INVERTED OR SUSPENDED CRESCENT.

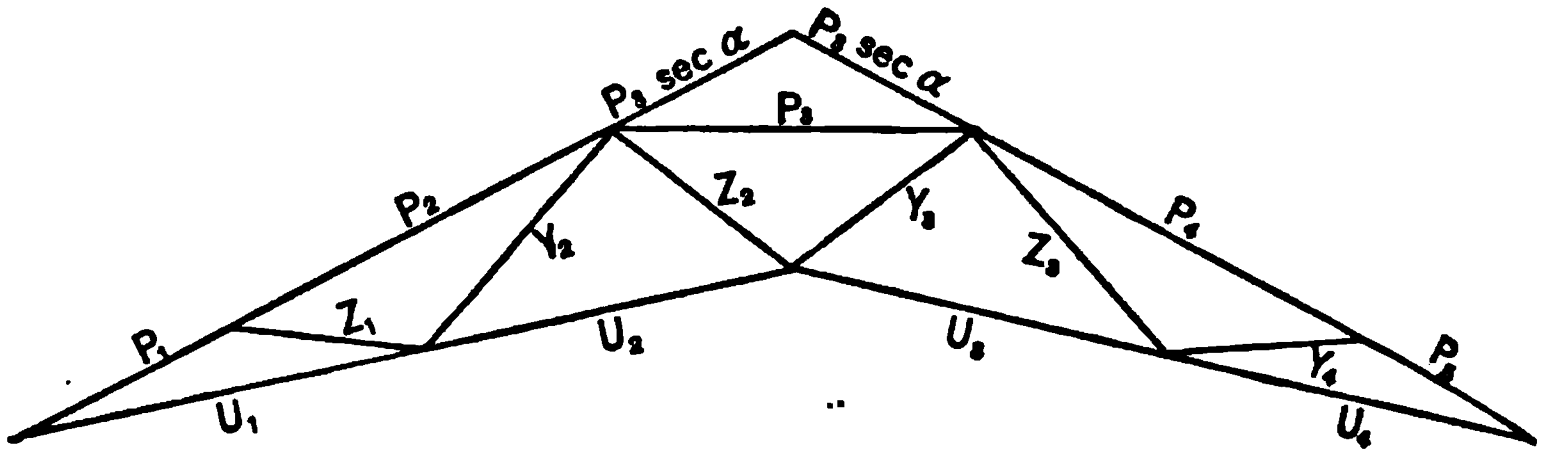


FIG. 19. — ROOF PRINCIPAL.

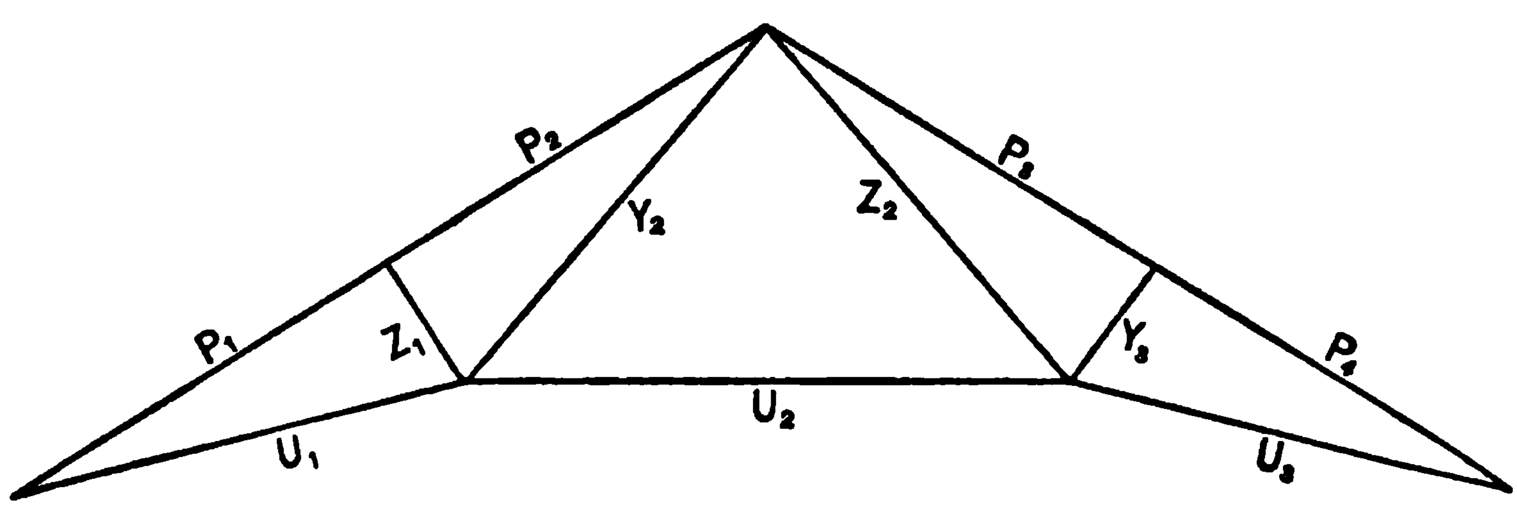


FIG. 20. — ROOF PRINCIPAL.

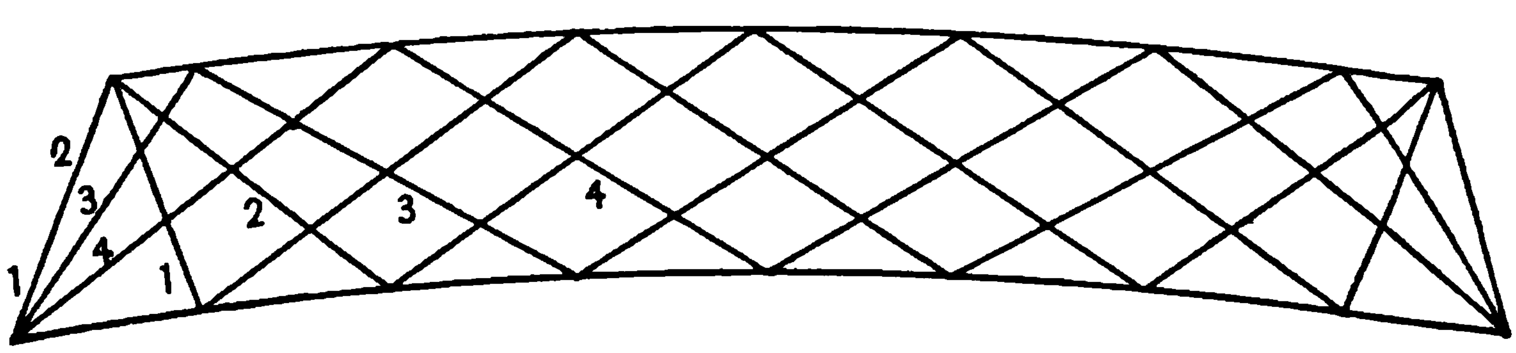


FIG. 21. — BENT GIRDER OF FOUR SYSTEMS.

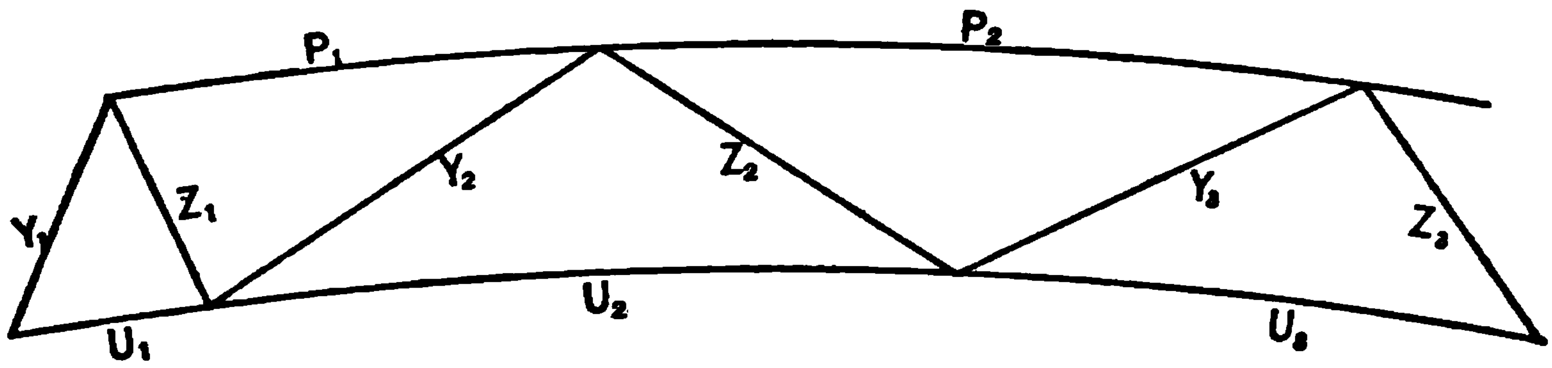


FIG. 21*a*.—FIRST SYSTEM.

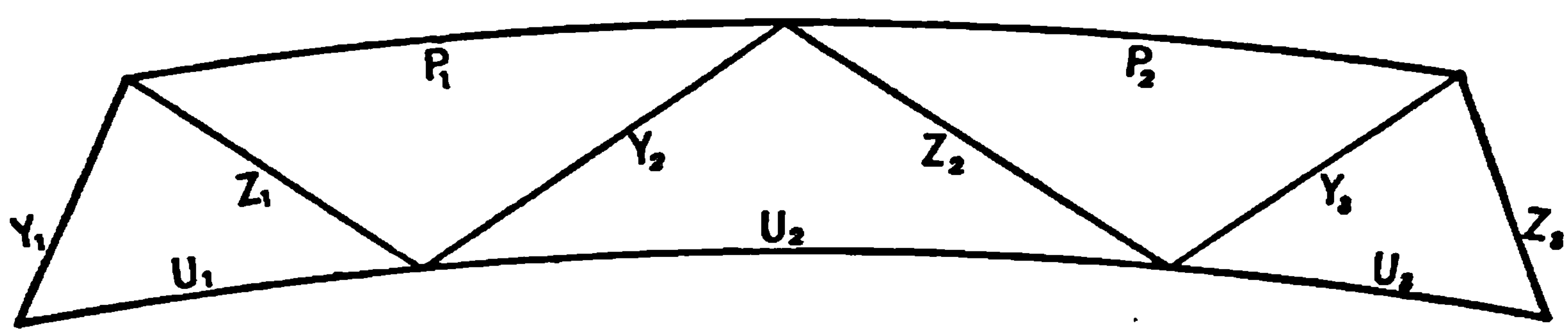


FIG. 21*b*.—SECOND SYSTEM.

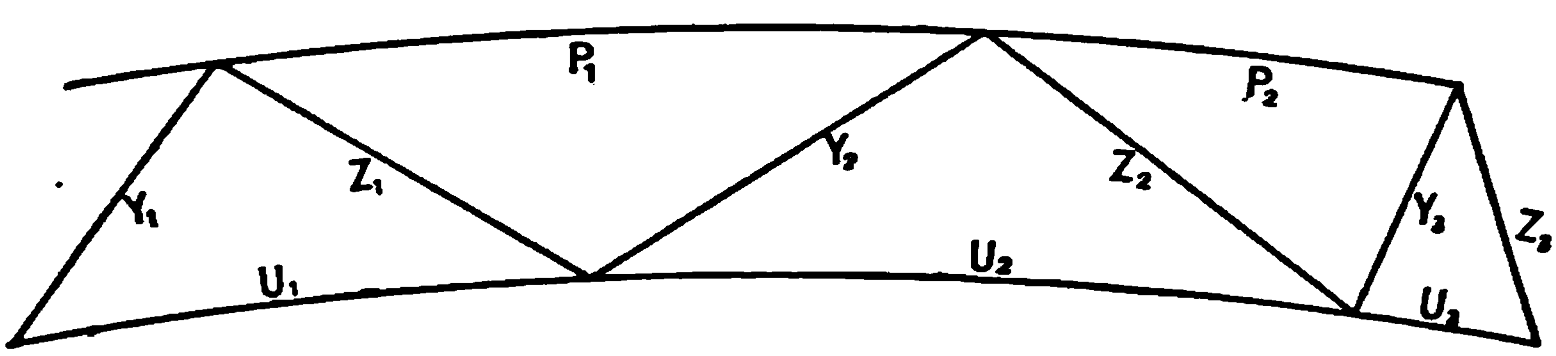


FIG. 21*c*.—THIRD SYSTEM.

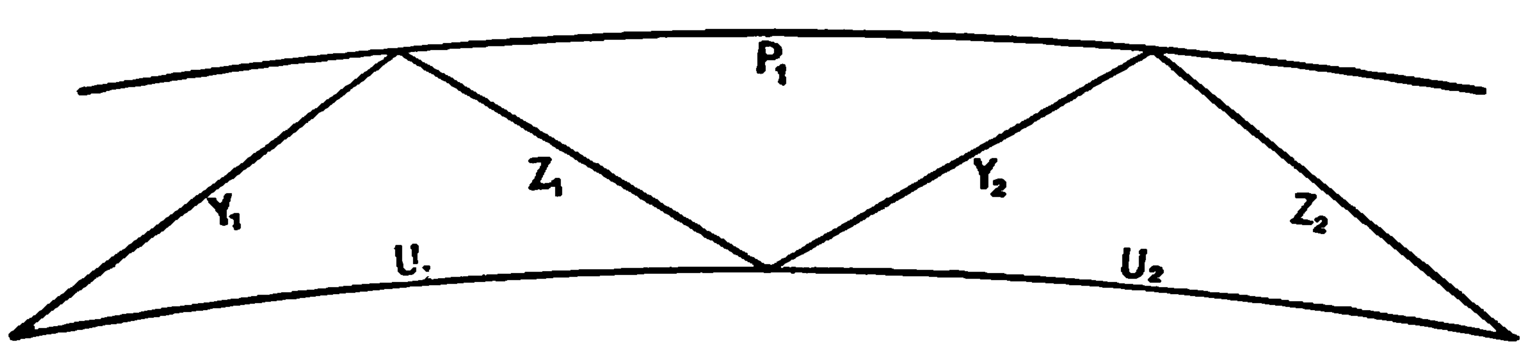


FIG. 21*d*.—FOURTH SYSTEM.

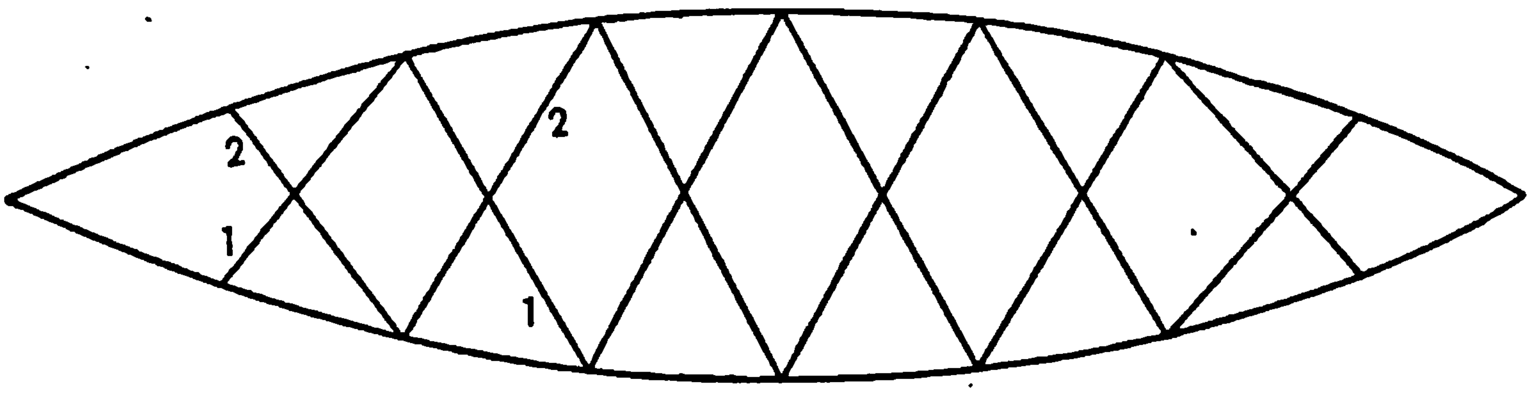


FIG. 22. — DOUBLE BOW OF TWO SYSTEMS.

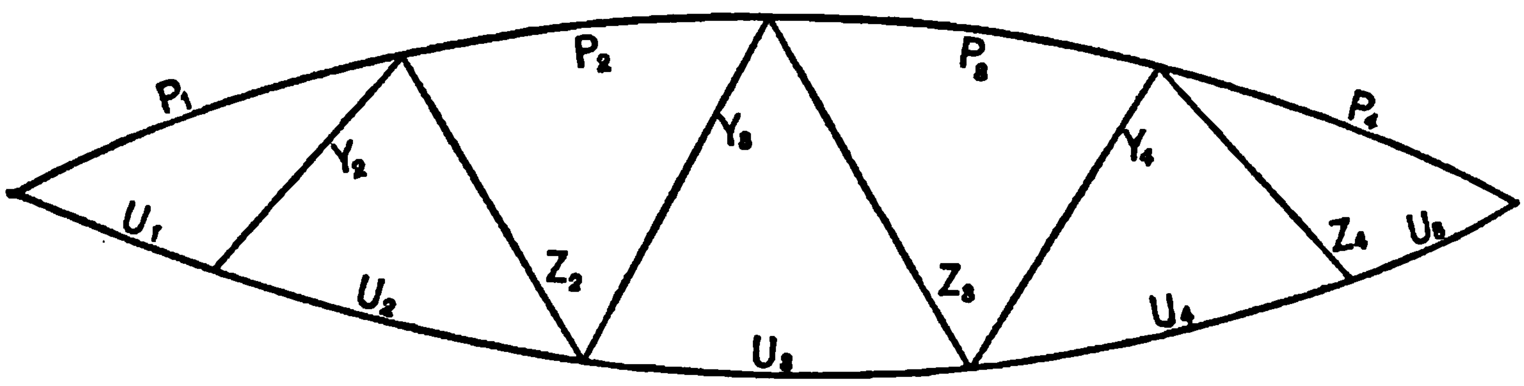


FIG. 22a. — FIRST SYSTEM. FIG. 16 SHOWS THE SECOND SYSTEM.

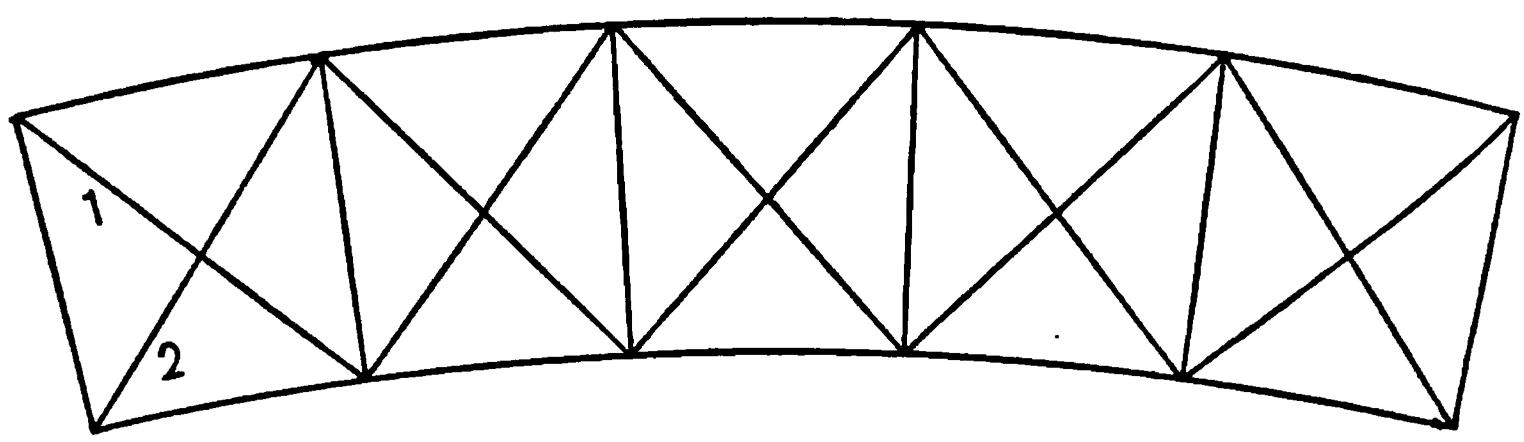


FIG. 23. — SEGMENT OF ROOF PRINCIPAL. (SEE FIG. 71.)

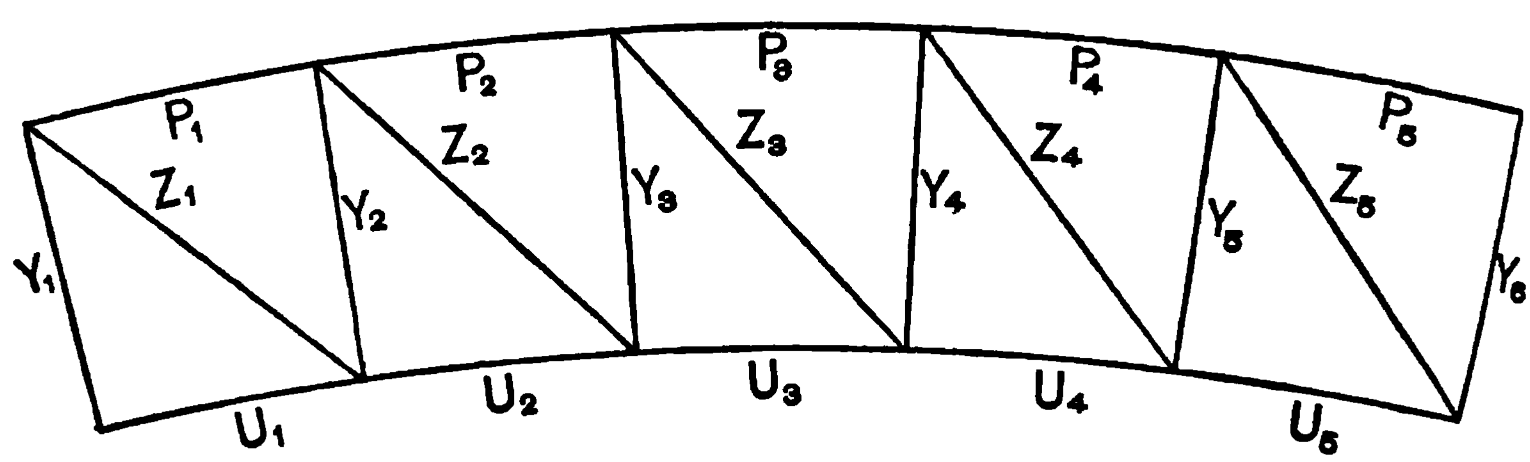


FIG. 23a. — DIAGONALS IN COMPRESSION.

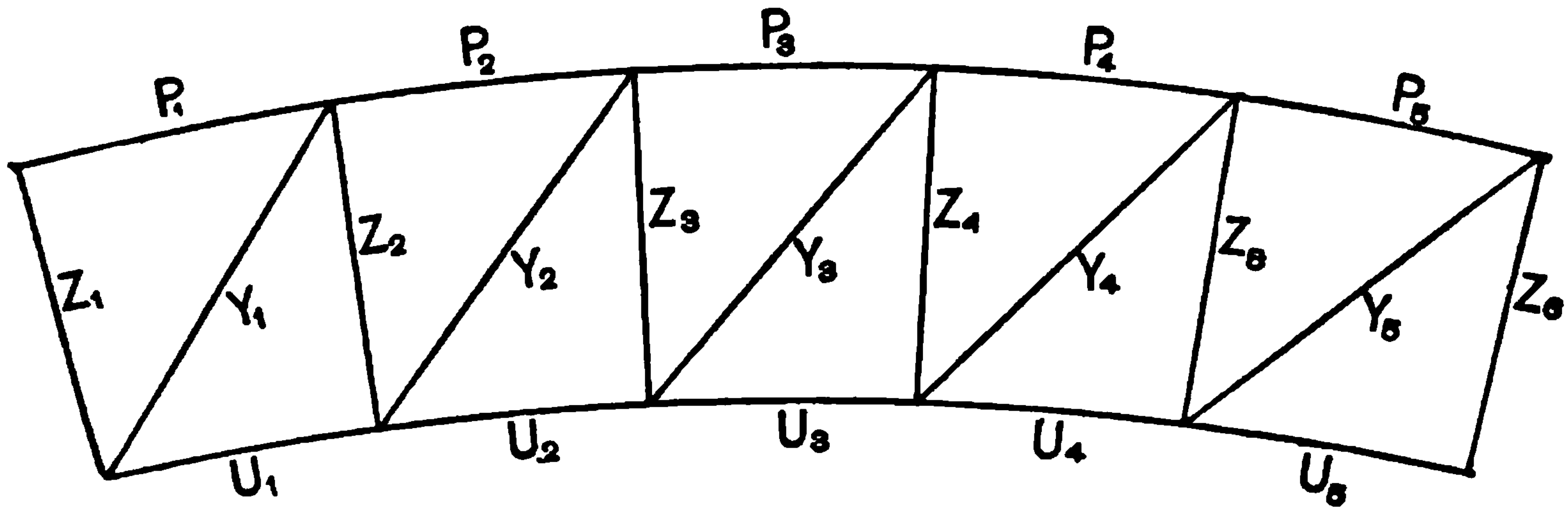


FIG. 23*b*.—DIAGONALS IN TENSION.

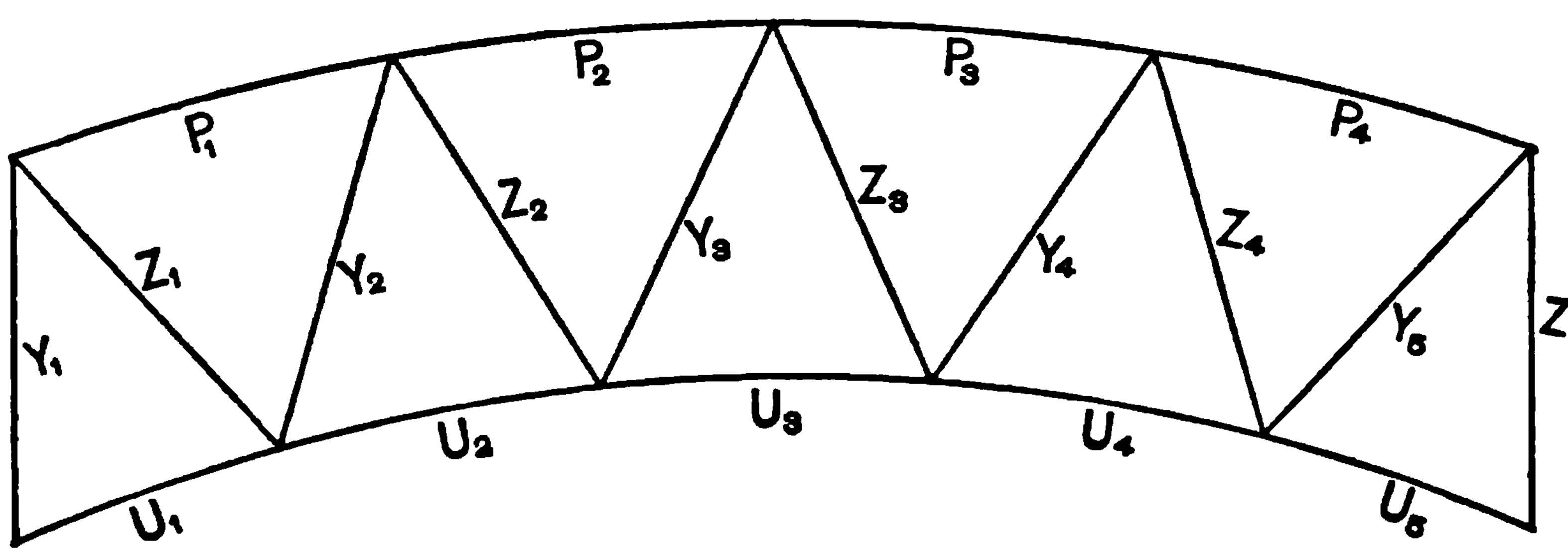


FIG. 24.—PARALLEL CHORDS. TRIANGULAR.

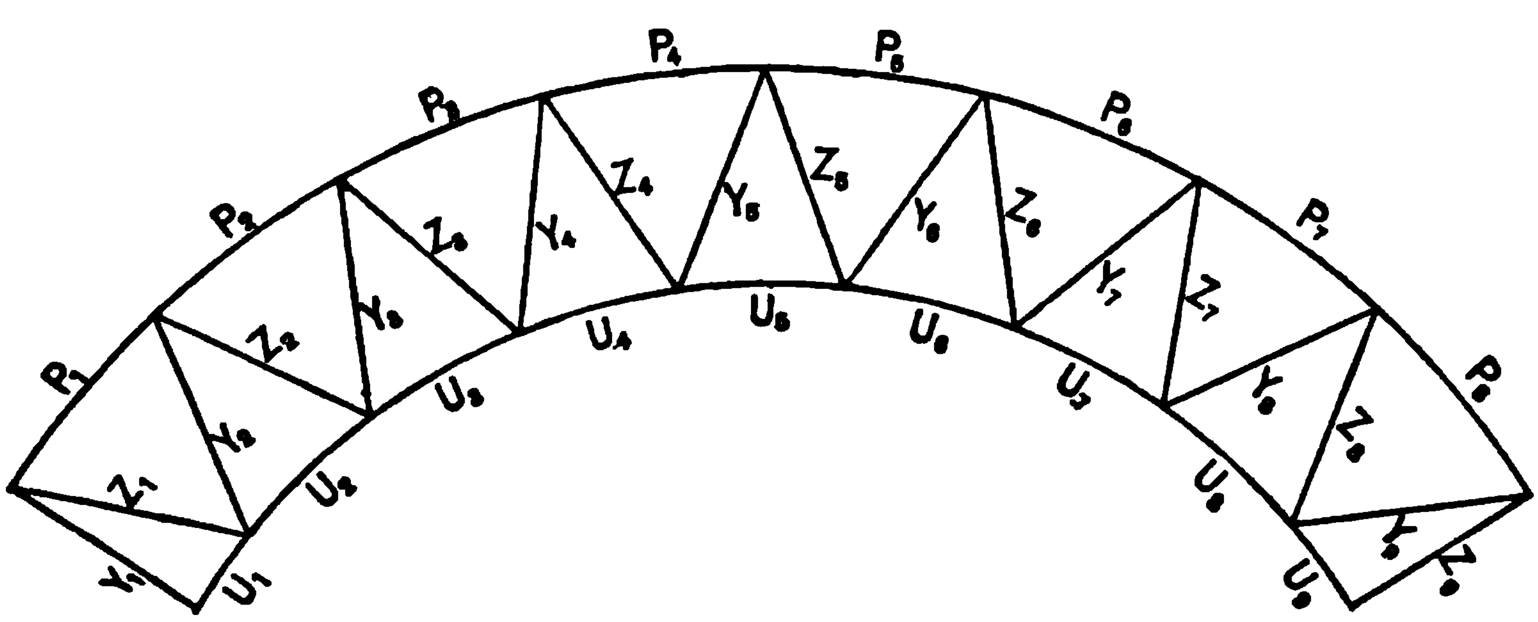


FIG. 25.—THE BRACED ARCH. ST. LOUIS BRIDGE SYSTEM. (SEE ARTICLE.)

Although we have supposed the linear dimensions of the girder known, we will now give a mode of finding them from the known length l, and central height h, in case of girders having either chord, or both chords, circular or parabolic.

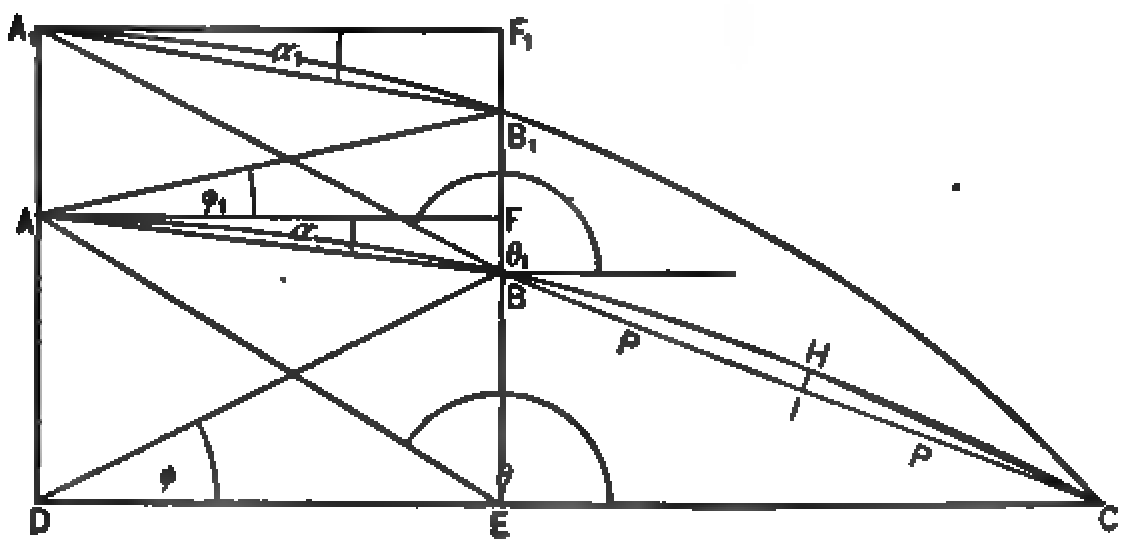


Fig. 26.

1st, Lower chord horizontal, and upper chord circular, as $ABCED$, Fig. 26. Let $l = 2DC$ = length of bottom chord, $h = AD$ = central height of girder. Then the radius

$$R = \frac{l^2 + 4h^2}{8h}. \tag{124}$$

Take D, the centre of the chord of any arc, as ABC or A_1B_1C, for the origin of rectangular co-ordinates; the axis of x being horizontal, and that of y being vertical. Then the equation to the curve ABC is

$$(y + R - h)^2 = R^2 - x^2,$$

$$BE = y = h - R \pm \sqrt{(R + x)(R - x)}, \tag{125}$$

s the height of the bowstring girder at any point, E, listance DE from the origin is x.

Thus, if $l = 100$, and $h = 10$, we have $R = \dfrac{100^2 + 4 \times 10^2}{8 \times 10}$ $= 130$; and from (125) we find, —

$$\begin{array}{llll}
\text{When } x = & 0, & y = & 10 = h. \\
x = & 10, & y = & 9.6148. \\
x = & 20, & y = & 8.4523. \\
x = & 30, & y = & 6.4912. \\
x = & 40, & y = & 3.6931. \\
x = & 45, & y = & 1.9631. \\
x = & 50 = \frac{1}{2}l, & y = & 0.
\end{array}$$

The length of any diagonal, DB, whose inclination to the horizon $= \phi$, is

$$v = \frac{BE}{\sin\phi} = \frac{DE}{\cos\phi};$$

or, in general,

$$v = y_r \div \sin\phi = \Delta x \div \cos\phi, \qquad (126)$$

Δx being one panel length.

The length of any diagonal, AE, whose inclination to the horizon $= \theta$, not acute, is

$$s = \frac{AD}{\sin\theta} = -\frac{DE}{\cos\theta};$$

or, in general,

$$s = y_{r-1} \div \sin\theta = -\frac{\Delta x}{\cos\theta}. \qquad (127)$$

The length of any chord, AB, whose inclination to the horizon $= \alpha$, is

$$a = \frac{DE}{\cos\alpha} = \frac{FB}{\sin\alpha};$$

or, in general,

$$a = \frac{\Delta x}{\cos\alpha} = \frac{\Delta y}{\sin\alpha}. \qquad (128)$$

From (126),

$$\frac{\sin\phi}{\cos\phi} = \quad y_r \quad \div \Delta x = \tan\phi. \tag{129}$$

From (127),

$$\frac{\sin\theta}{\cos\theta} = -y_{r-1} \div \Delta x = \tan\theta. \tag{130}$$

From (128),

$$\frac{\sin\alpha}{\cos\alpha} = \quad \Delta y \quad \div \Delta x = \tan\alpha. \tag{131}$$

Whence ϕ, θ, and α, and their sines and cosines, may be taken from a table of natural or logarithmic circular functions.

To determine the length of any part of the circular arc ABC, or of the whole arc, find the angle at the centre corresponding to the given chord of the required arc; then the required length of arc is to the whole circumference as the angle at the centre is to four right angles. Thus, if C denotes the angle at the centre whose chord is the span $l = 100$, we have

$$\sin(\tfrac{1}{2}C) = \frac{\frac{1}{2}l}{R} = \frac{50}{130} = \sin 22^\circ\, 37'\, 11''.5,$$

$$\therefore \quad C \quad = 45^\circ\, 14'\, 23'' = 45^\circ.239722.$$

Circumference = 360 degrees.

Length of circumference $= 2\pi R = 2 \times 3.14159 \times 130 = 816.8134$,

$$\therefore \quad \text{Required arc } 2ABC = \frac{45.239722}{360} \times 816.8134 = 102.645.$$

Or, the length of the arc subtended by a given chord may be from a table constructed for the purpose.

, Both chords or flanges circular, as $ABCB_1A_{11}$ Fig. 26, the chords meet at the ends of the girder; or, as Figs. d 17, where the chords do not meet at the ends.

If the curves meet, as in Figs. 14, 16, 18, and 26, then l in equation (124) will be common to both arcs, ABC, A_1B_1C; but the central heights of the two arcs will be $h = AD$, and $h_1 = A_1D$. Hence, for the upper curve, the radius

$$R_1 = \frac{l^2 + 4h_1^2}{8h_1}.$$

If, as in Fig. 15, the curves do not intersect at the ends, then, for each curve, l will be the chord subtended by the arc, h will be the central height of each arc above its chord, and the origin of co-ordinates for each curve will be at the centre of its own chord.

The ordinates y, corresponding to the same values of x, are to be found for each curve by (125); and if $y_1 =$ an ordinate to the upper curve, and $y =$ the corresponding ordinate to the lower curve, and $e =$ the difference in height of the two origins, then $y_1 + e - y =$ the height of girder at any point, x. And when $e = 0$, as in Fig. 26, the height of the girder $ABCB_1A_1$ at any point, B, is $BB_1 = y_1 - y$.

$$FB \;= BE \;- AD \;= \Delta y, \text{ in general.}$$

$$F_1B_1 = B_1E - A_1D = \Delta y_1, \text{ in general.}$$

$$DE \;= AF \;= A_1F_1 = \Delta x, \text{ in general.}$$

$$\tan\alpha = \frac{\Delta y}{\Delta x}, \quad \tan\alpha_1 = \frac{\Delta y_1}{\Delta x}.$$

$$AA_1 + BF = y_{1_r} - y_{r+1},$$

$$B_1F \qquad = BB_1 - FB = y_{1_{r+1}} - y_r.$$

$$\tan\theta = -\frac{y_{1_r} - y_{r+1}}{\Delta x}, \quad \tan\phi = \frac{y_{1_{r+1}} - y_r}{\Delta x}.$$

From these tangents, α, α_1, θ, ϕ, and their sines and cosines, are to be found as before. We then have

$$v = AB_1 = \frac{y_{1_{r+1}} - y_r}{\sin\phi} = \frac{\Delta x}{\cos\phi}, \qquad (132)$$

$$s = A_1B = \frac{y_{1_r} - y_{r+1}}{\sin\theta} = -\frac{\Delta x}{\cos\theta}, \qquad (133)$$

$$a = AB = \frac{\Delta y}{\sin\alpha} = \frac{\Delta x}{\cos\alpha}, \qquad (134)$$

$$a_1 = A_1B_1 = \frac{\Delta y_1}{\sin\alpha_1} = \frac{\Delta x}{\cos\alpha_1}. \qquad (135)$$

3d, When the curvature of one or both chords of the girder is parabolic, we proceed as in case of the circular chords just discussed, except in finding the ordinates and length of the curve, which only, therefore, we need now determine.

Let the curves, Fig. 26, now be parabolas, whose vertices are at A and A_1 respectively. Take the origin of rectangular co-ordinates, as before, at D; the axis of x being horizontal, and that of y vertical. Then the equation to the curve ABC is

$$y = \left(1 - \frac{4x^2}{l^2}\right)h; \qquad (136)$$

to the curve $A_1B_1C_1$

$$y_1 = \left(1 - \frac{4x_1^2}{l^2}\right)h_1. \qquad (137)$$

For the same value of x, (136) and (137) give

$$h_x = y_1 - y = \left(1 - \frac{4x^2}{l^2}\right)(h_1 - h), \qquad (138)$$

which is the height of the girder at any point whose distance is x from the centre or origin.

Thus, if $l = 100$, and $h = 10$, (136) gives, —

When $x = 0$,	$y = 10 = h$.
$x = 10$,	$y = 9.6$.
$x = 20$,	$y = 8.4$.
$x = 30$,	$y = 6.4$.
$x = 40$,	$y = 3.6$.
$x = 45$,	$y = 1.9$.
$x = 50 = \frac{1}{2}l$,	$y = 0$.

And if $h_1 = A_1D = 20$, $h = AD = 10$, and $l = 2DC = 100$, equation (138) gives the heights of girder ACA_1 as below:—

When $x = 0$,	$h_0 = 10 = h_1 - h$;
$x = 10$,	$h_{10} = 9.6$;
$x = 20$,	$h_{20} = 8.4$;
$x = 30$,	$h_{30} = 6.4$;
$x = 40$,	$h_{40} = 3.6$;
$x = 45$,	$h_{45} = 1.9$;
$x = 50 = \frac{1}{2}l$,	$h_{50} = 0$;

which are the same as the heights of the girder ADC just found, since $h_1 = 2h$, and, from (136) and (137),

$$\frac{y}{y_1} = \frac{h}{h_1}; \tag{139}$$

that is, the ordinates to the two curves, for the same value of x, are proportional to their central heights.

The length of the parabolic arc, in terms of the chord l and the central height h, is

$$S = \tfrac{1}{2}(l^2 + 16h^2)^{\frac{1}{2}} + 0.287823\frac{l^2}{h}\log\frac{4h + (l^2 + 16h^2)^{\frac{1}{2}}}{l}, \tag{140}$$

where *log* means the *common* logarithm.

If l = 100 feet = span, and h = 10 feet = central height, of parabolic arc, then (140) gives S = 102.606 feet.

From these examples it appears, that, when the curvature is small, there is but little difference between the ordinates and arc of the circular and the ordinates and arc of the parabolic girder of the same central height and span.

Instead of these exact determinations of the linear dimensions of a girder, the figure may be drawn to a scale, and the length of each member measured, where greater accuracy is not required.

It is proper to observe here, that, in all cases of curved flange, the line of action of the flange strain, P or U, is the chord of the arc between adjacent apices, and not the arc itself. When, therefore, either flange of a girder is curved, and not polygonal, there is developed midway between adjacent apices in the same flange a deflecting force tending to increase the curvature of a compressed flange, and to diminish the curvature of a flange in tension.

For the amount of this deflecting force F we have, if P is the strain along the chord BIC, Fig. 26,

$$F = 2P \tan HCI. \qquad (141)$$

Or, if C is the angle at the centre of the circle whose chord is BC, then

$$\tan HCI = \frac{\text{ver-sin}\,\frac{1}{2}C}{\sin\frac{1}{2}C} = \frac{1 - \cos\frac{1}{2}C}{\sin\frac{1}{2}C} = \operatorname{cosec}\tfrac{1}{2}C - \cot\tfrac{1}{2}C,$$

$$\therefore \quad F = 2P(\operatorname{cosec}\tfrac{1}{2}C - \cot\tfrac{1}{2}C); \qquad (142)$$

and the strain along the chord HC of each half of the arc BC is

$$P' = P \div \cos HCI. \qquad (143)$$

Similarly, in cases like P_3, Fig. 19, there is a deflecting force generated at the ridge equal to

$$F = 2P_3 \tan \alpha;$$

and the strain along the upper segment of each rafter is, as indicated, $P_3 \div \cos \alpha$.

The bending-moment due to this deflecting force is given by equation (46),

$$M = \tfrac{1}{4}Fa, \qquad (144)$$

where a is the length of the chord BC.

The amount of material required to neutralize this moment will be determined in the sequel. But it is already manifest that the least amount of resisting material will be required when the line of pressure coincides with the axis of the resisting member.

Multiple or compound web systems, as those represented in Figs. 21 and 22, may be separated into the single systems of which they are composed, when the sum of all the strains found for the same member will be the strain sought for that member.

Class II.—Bottom Chord horizontal, other Members inclined. $\beta = 0$.

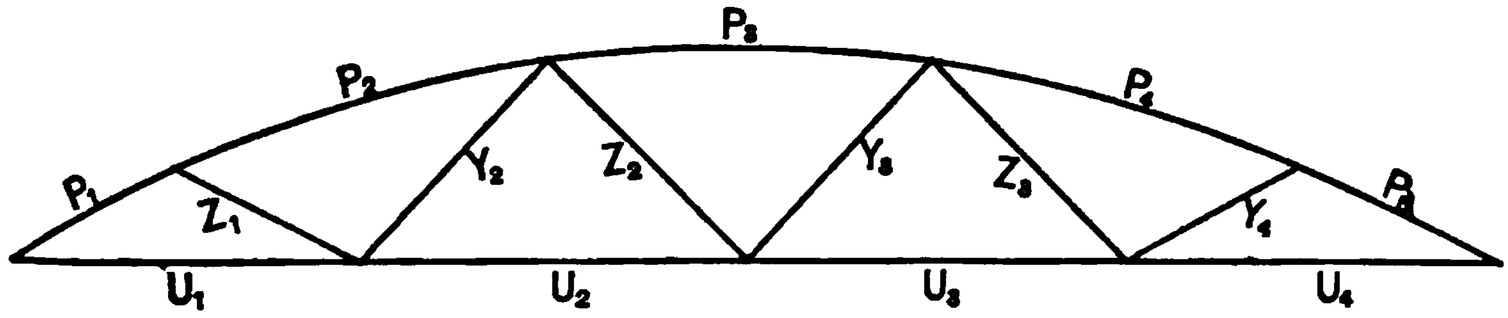


Fig. 27.—The Parabolic or Circular Bowstring.

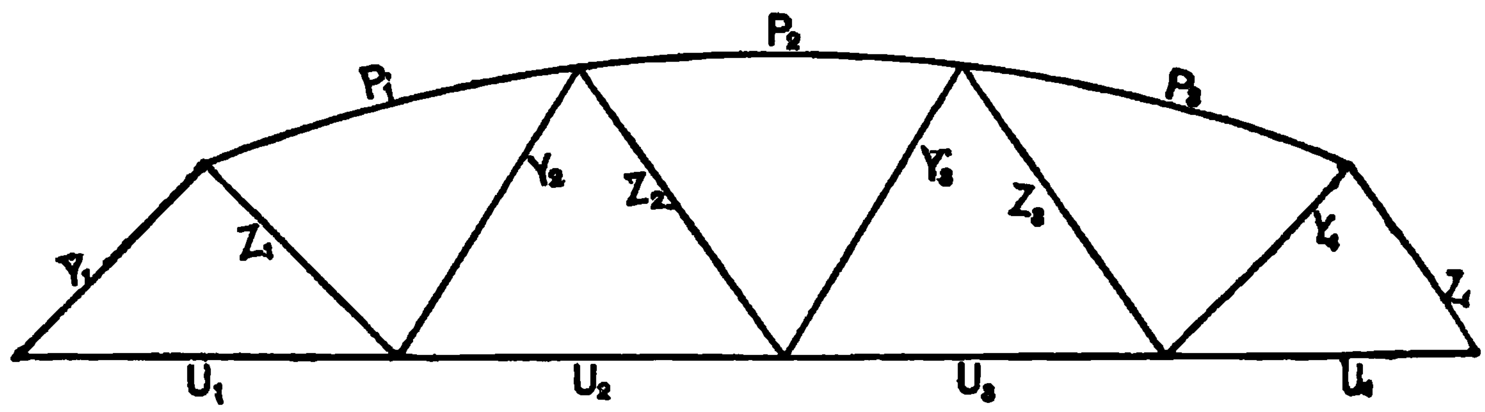


FIG. 28.—THE TRUNCATED BOWSTRING.

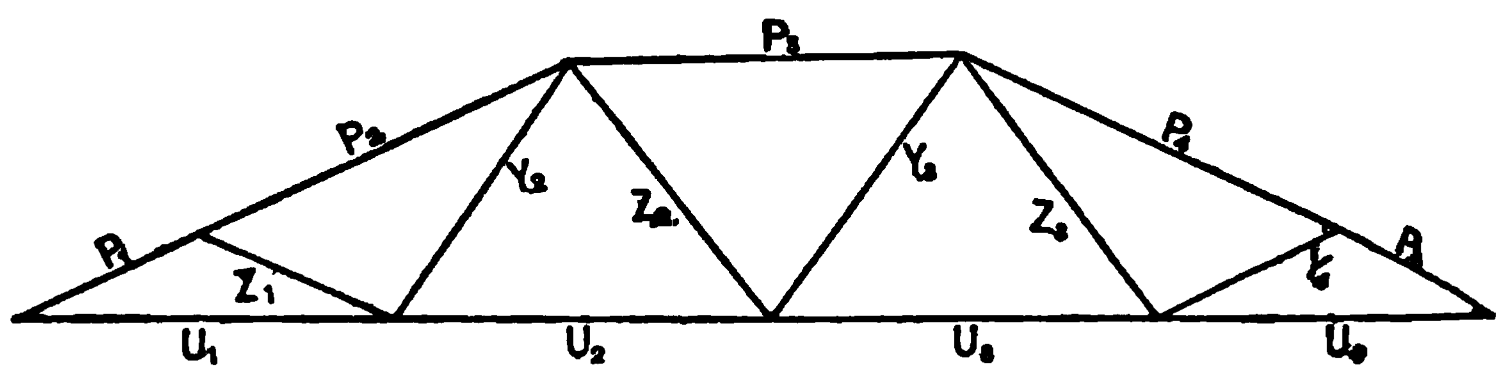


FIG. 29.—ROOF PRINCIPAL.

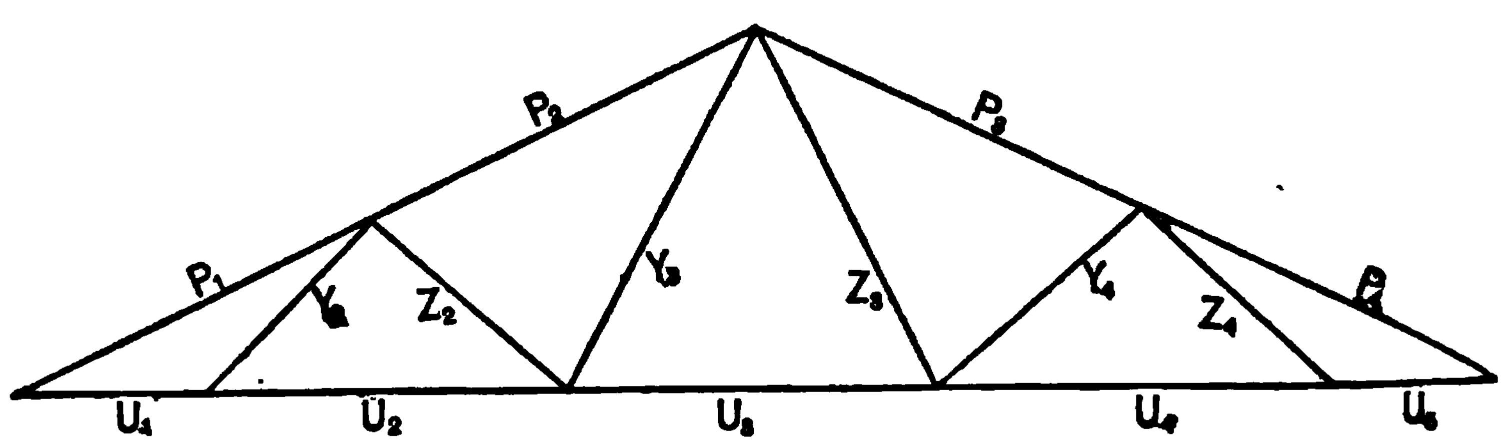


FIG. 30.—ROOF PRINCIPAL.

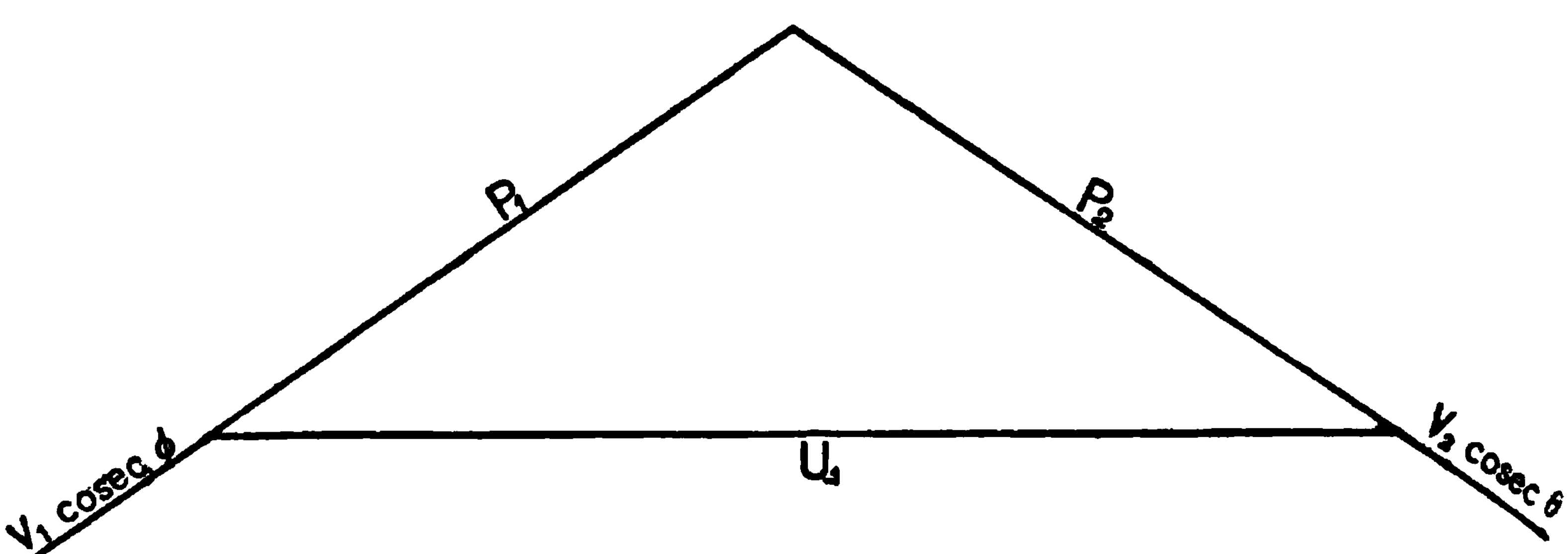


FIG. 31.—TIED RAFTERS.

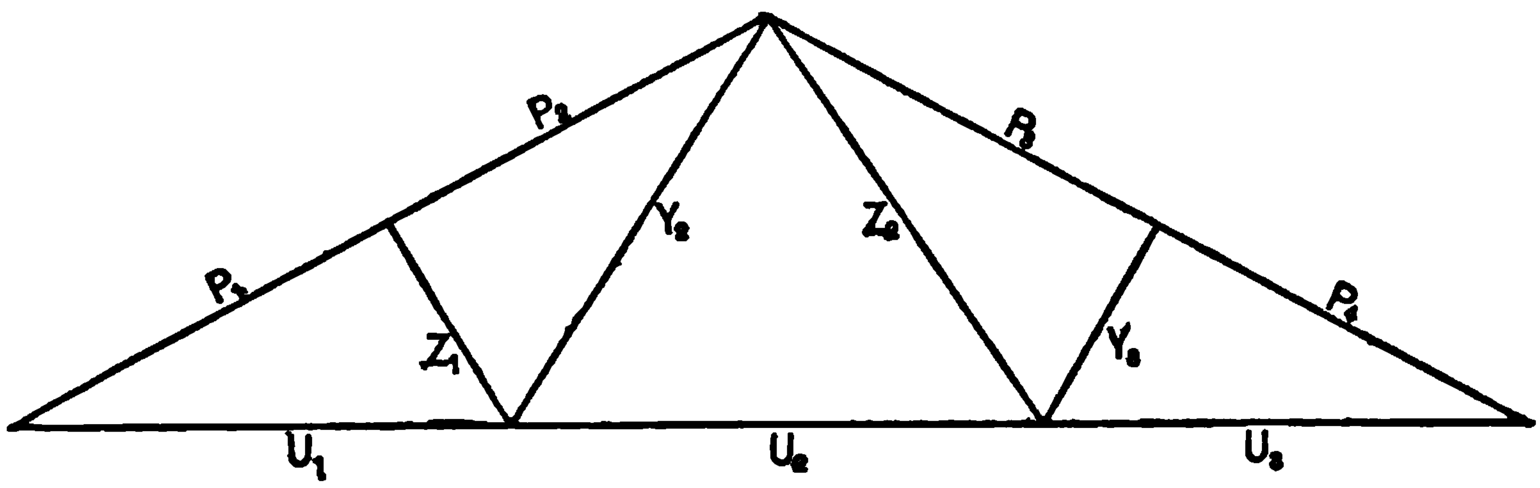


FIG. 32.—ROOF PRINCIPAL.

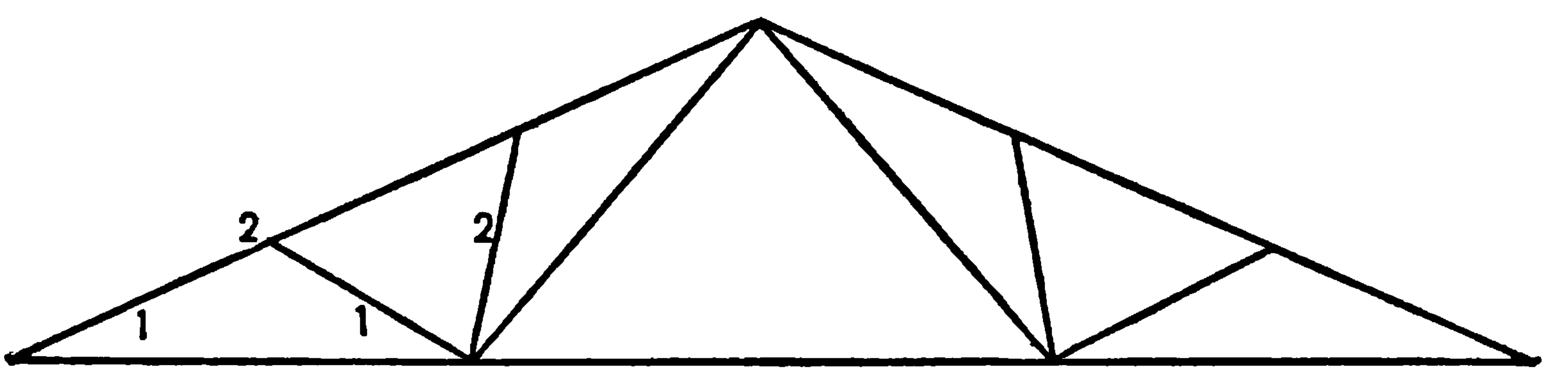


FIG. 33.—ROOF PRINCIPAL OF TWO SYSTEMS.

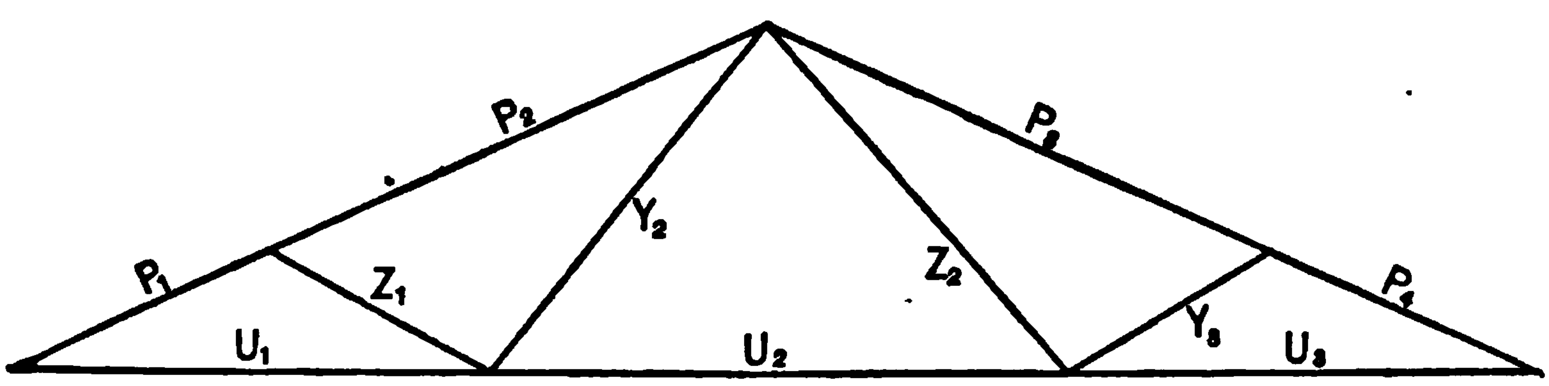


FIG. 33*a*.—FIRST SYSTEM.

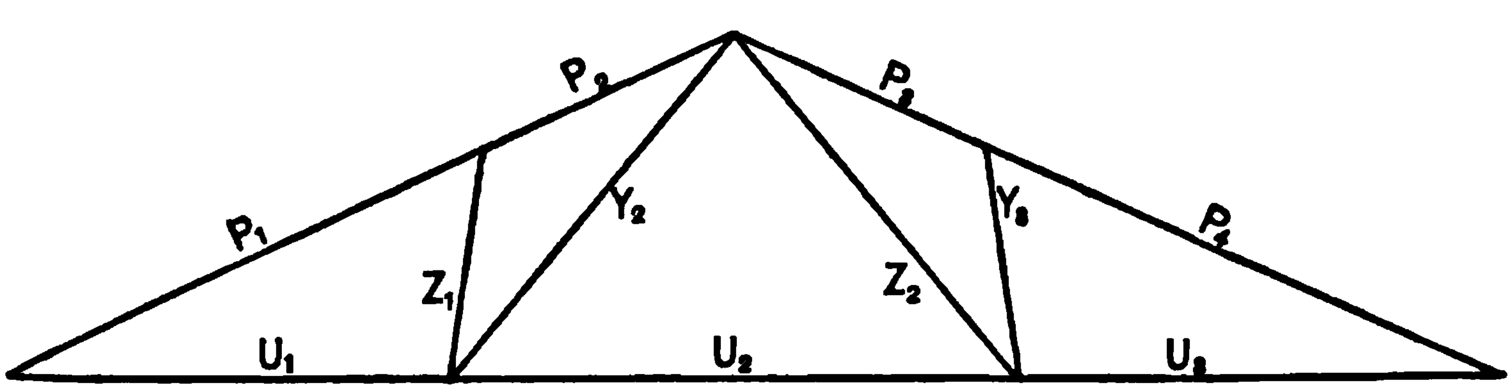


FIG. 33*b*.—SECOND SYSTEM.

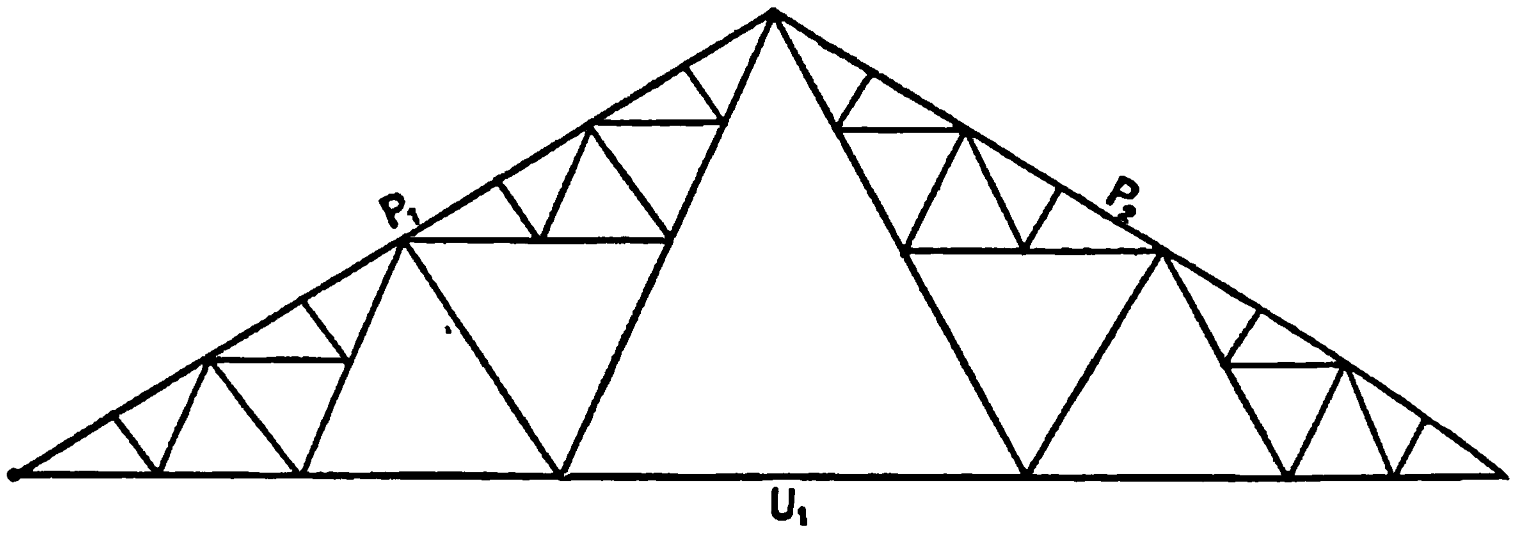


FIG. 34.—ROOF MAIN, COMPOUND SYSTEM. (SEE FIG. 70, CLASS VIII.)

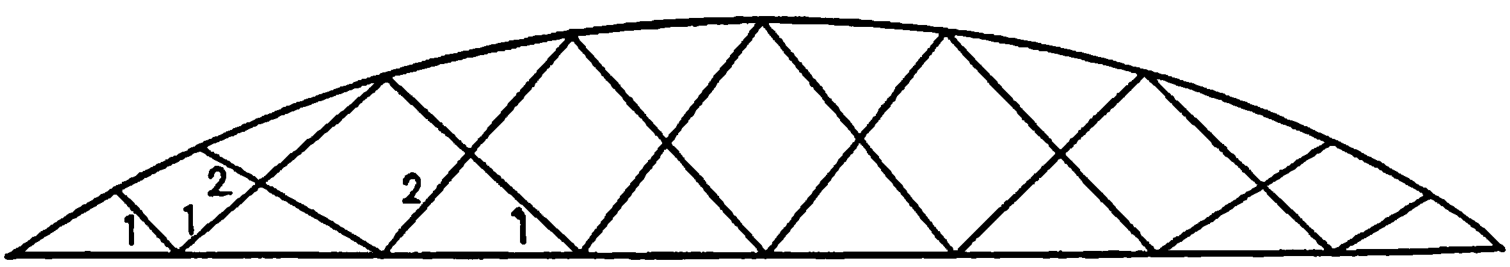


FIG. 35.—BOWSTRING OF TWO SYSTEMS. TRIANGULAR.

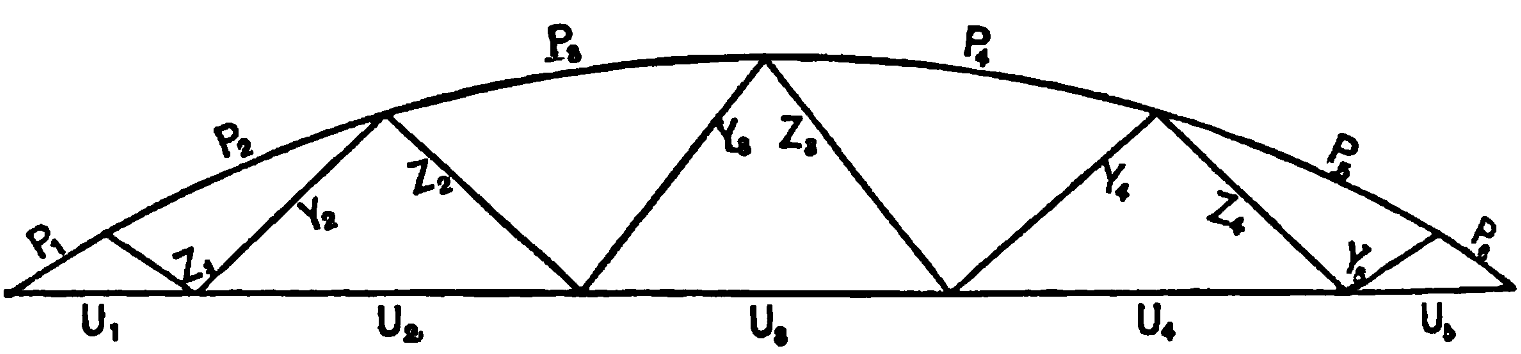


FIG. 35*a*.—FIRST SYSTEM. FIG. 27 SHOWS THE SECOND SYSTEM.

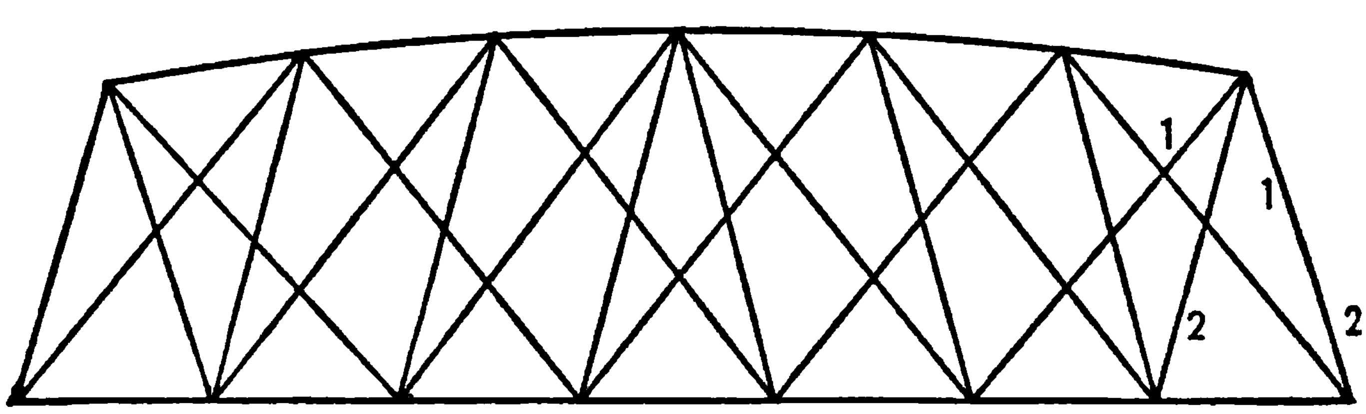


FIG. 36.—THE POST TRUSS WITH CURVED TOP.

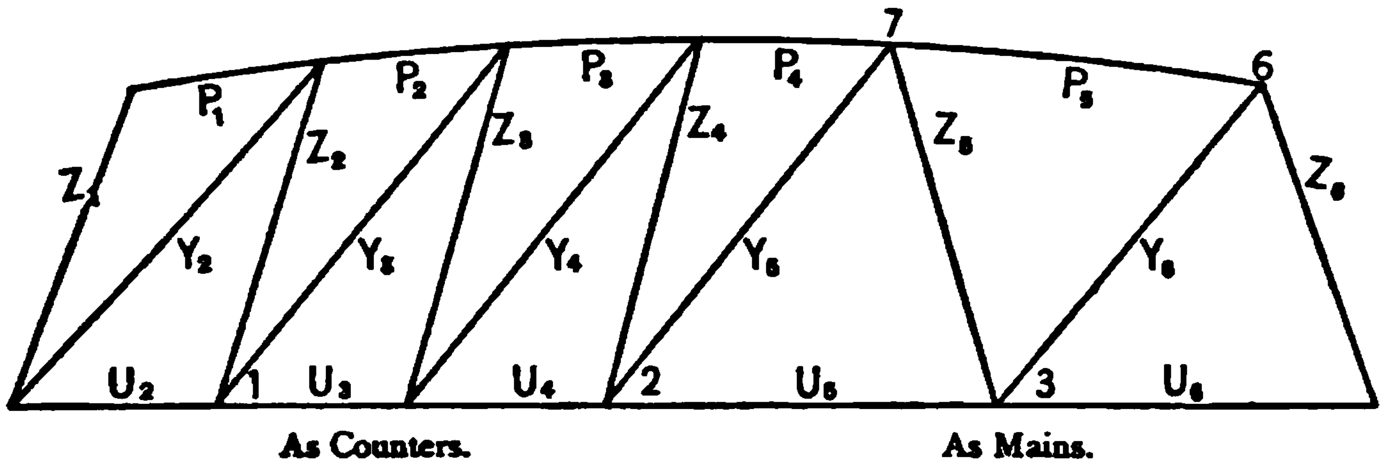


FIG. 36*a*. — FIRST SYSTEM.

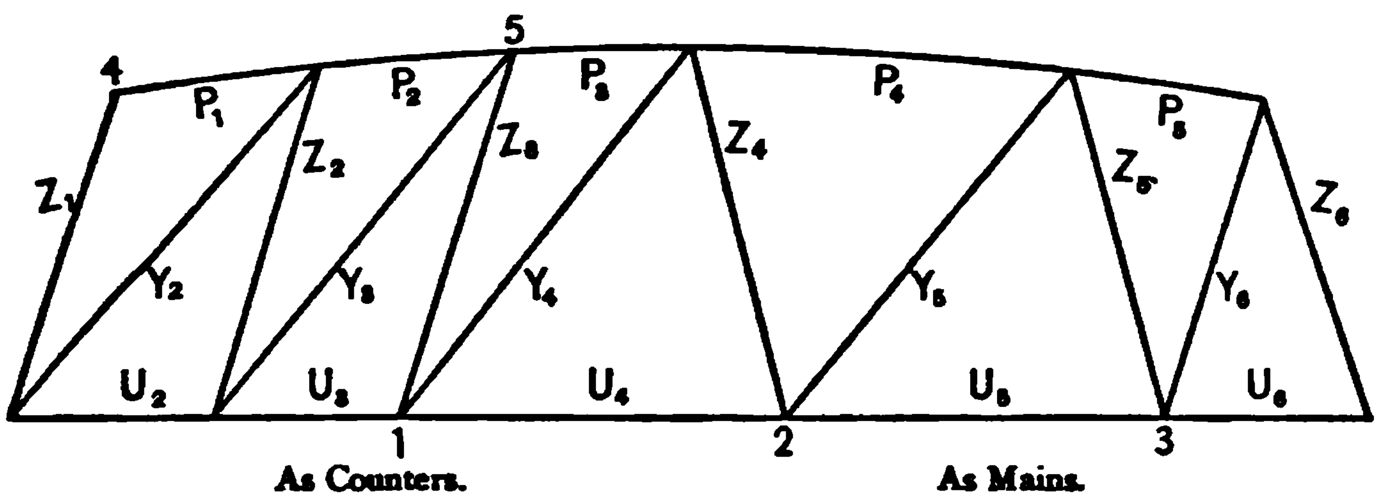


FIG. 36*b*. — SECOND SYSTEM.

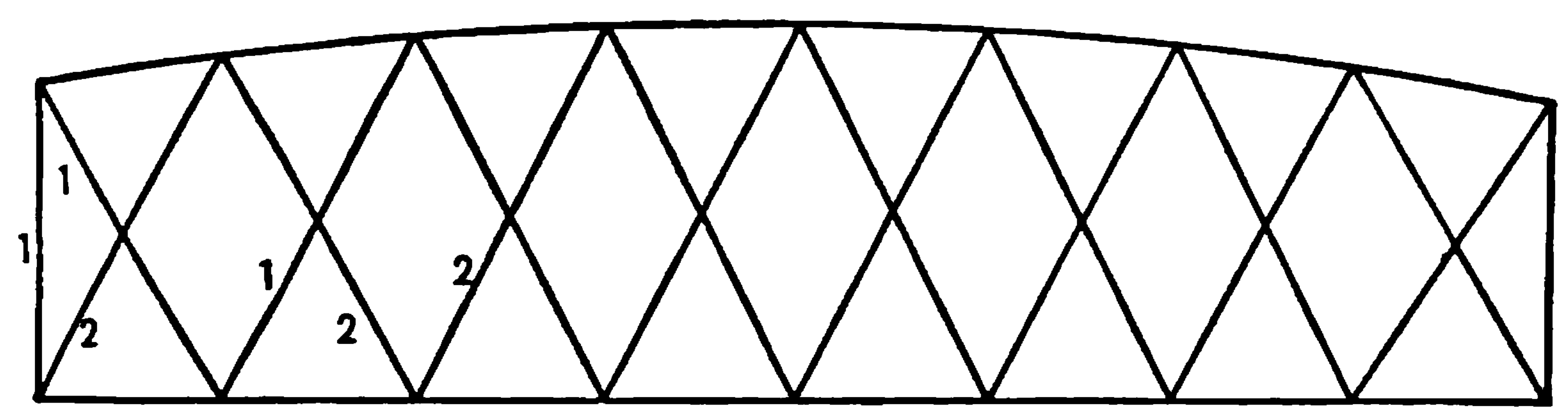


FIG. 37. — DOUBLE TRIANGULAR TRUNCATED BOWSTRING. SYSTEM OF KANSAS-CITY BRIDGE.

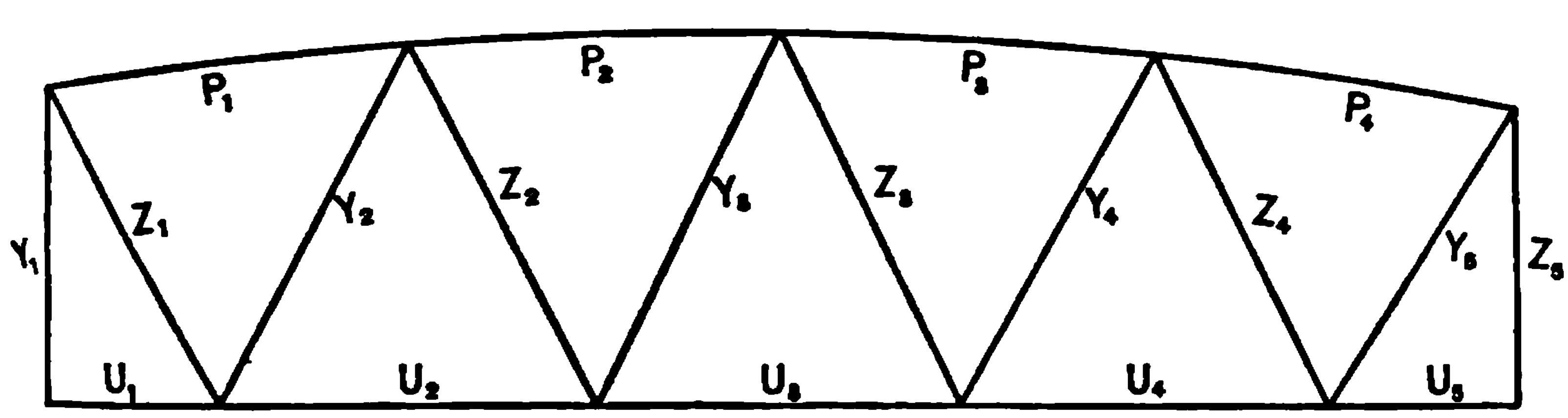


FIG. 37*a*. — FIRST SYSTEM.

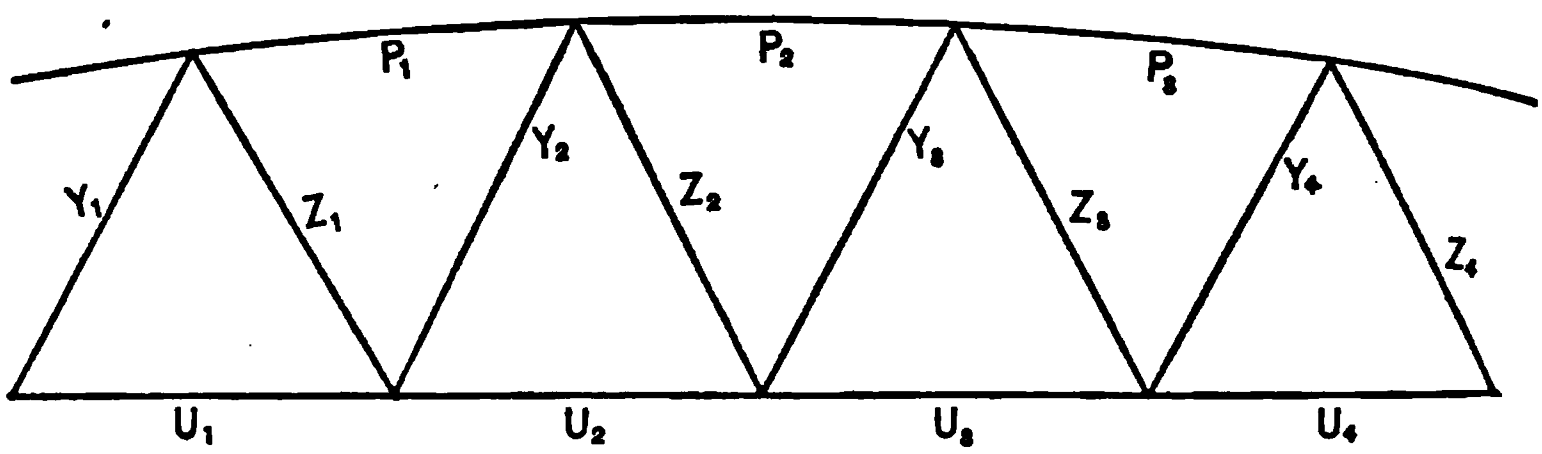


FIG. 37*b*.—SECOND SYSTEM.

FORMULÆ FOR CLASS II. $\beta = 0$.

Method of Moments.

$$H = M \div h.$$

$$\Delta H = \frac{M_{r+1}}{h_{r+1}} - \frac{M_r}{h_r}.$$

$$P = \frac{H_{r+1}}{\cos\alpha}.$$

$$U = -H_r.$$

$$Y = \frac{\Delta_r H}{\cos\phi}.$$

$$Z = \frac{\Delta_{r+1} H}{\cos\theta}.$$

In case of vertical end posts, as in Fig. 37*a*, where Y and Z become indeterminate by the above equations, we have

$$Y_1 = Z_1 \sin\theta_1 + P_1 \sin\alpha_1, . \qquad Z_5 = Y_5 \sin\phi_5 + P_4 \sin\alpha_4.$$

Where a vertical section through any apex cuts a web member, as in Fig. 33*b* for $Z_1 = Y_3$, and in Figs. 36*a* and 36*b* for P_4 and the counters Y and Z, we do not have H, the horizontal component of chord strain, equal to $M \div h$, but may proceed as follows:—

Find the moments M_1, M_2, M_3, etc., at vertical planes through consecutive apices. Call the heights above the bottom chord at which the cut diagonal meets these vertical planes in each panel, a and b, as in Fig. 38.

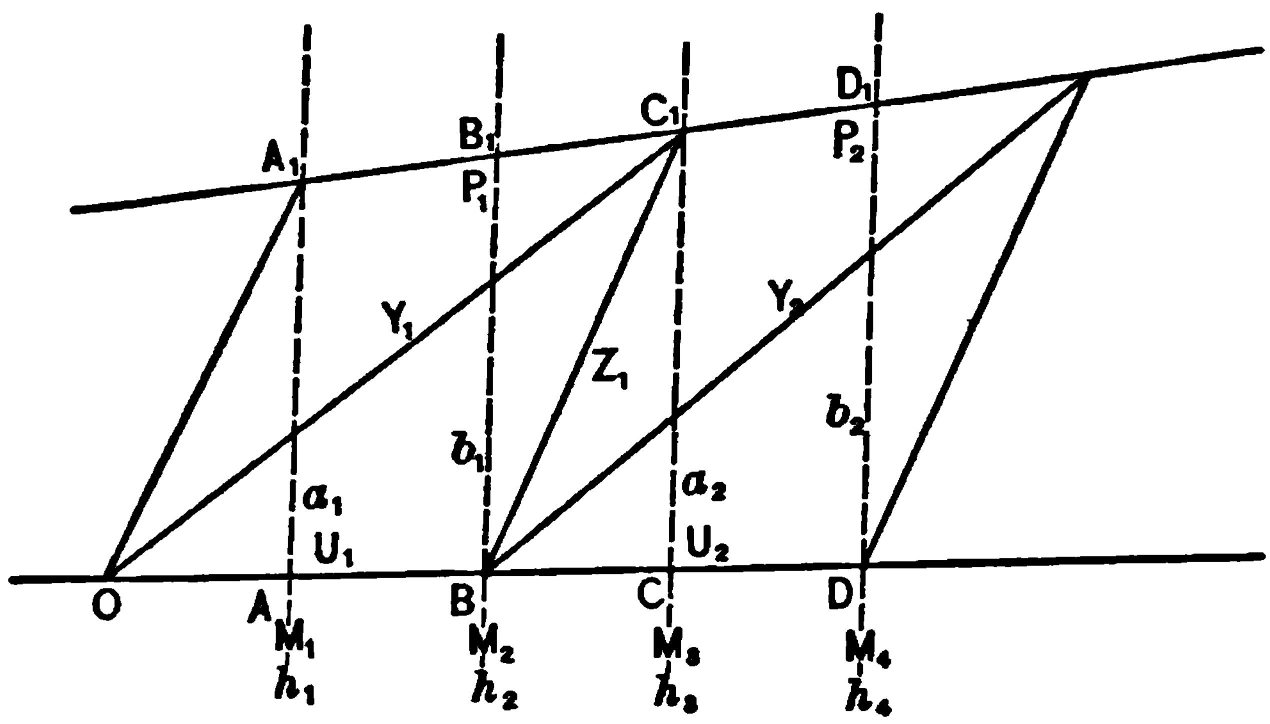


FIG. 38.

Then, taking moments about A and B, we have

$$M_1 = h_1 P_1 \cos \alpha_1 + a_1 Y_1 \cos \phi_1, \tag{145}$$

$$M_2 = h_2 P_1 \cos \alpha_1 + b_1 Y_1 \cos \phi_1, \tag{146}$$

$$\therefore \quad P_1 = \frac{M_2 a_1 - M_1 b_1}{(a_1 h_2 - b_1 h_1)\cos \alpha_1}, \tag{147}$$

$$Y_1 = \frac{M_1 h_2 - M_2 h_1}{(a_1 h_2 - b_1 h_1)\cos \phi_1}. \tag{148}$$

Taking moments about A_1 and B_1, there results

$$M_1 = -h_1 U_1 \cos \beta_1 - (h_1 - a_1) Y_1 \cos \phi_1, \tag{149}$$

$$M_2 = -h_2 U_1 \cos \beta_1 - (h_2 - b_1) Y_1 \cos \phi_1, \tag{150}$$

$$\therefore \quad U_1 = \frac{M_2(h_1 - a_1) - M_1(h_2 - b_1)}{(a_1 h_2 - b_1 h_1)\cos \beta_1}. \tag{151}$$

Similarly, or by increasing the indices of a, b, α, and ϕ by 1, and those of M and h by 2, we find

$$P_2 = \frac{M_4 a_2 - M_3 b_2}{(a_2 h_4 - b_2 h_3)\cos\alpha_2},$$

$$Y_2 = \frac{M_3 h_4 - M_4 h_3}{(a_2 h_4 - b_2 h_3)\cos\phi_2},$$

$$U_2 = \frac{M_4(h_3 - a_2) - M_3(h_4 - b_2)}{(a_2 h_4 - b_2 h_3)\cos\beta_2}.$$

Then, taking moments about C gives

$$M_3 = h_3(P_1\cos\alpha_1 + Y_1\cos\phi_1 + Z_1\cos\theta_1) + a_2 Y_2\cos\phi_2,$$

$$\therefore\quad Z_1 = \left\{\frac{M_3}{h_3} - \frac{a_2}{h_3}Y_2\cos\phi_2 - Y_1\cos\phi_1 - P_1\cos\alpha_1\right\}\frac{1}{\cos\theta_1}. \qquad (152)$$

Now, in case of the Post Truss, we need only the counter-strains Y, since Z, P, and U have their greatest values as main strains.

And when both chords are horizontal, the horizontal projection of the Z member, or strut, is one-third of the horizontal projection of the Y member, or tie, as usually built; hence $a = \frac{1}{2}b = \frac{1}{8}h$,

$$\therefore\quad Y_1 = \frac{3(M_2 - M_1)}{h\cos\phi} = \frac{3\Delta_1 H}{\cos\phi}, \qquad (153)$$

which, it will be seen, is the same thing as $Y_1 = \dfrac{\Delta H}{\cos\phi}$, provided ΔH is taken for the interval equal to the horizontal projection of the Y member, while $\Delta_1 H$ belongs to one-third of that interval; since the foremost end of the live load, at the instant the value of Y is here sought, is at the foot of the Y member, being applied either directly at the lower apex, or

indirectly at the upper apex, and reaching the bottom chord through the *Z* member terminating there. In other words, when the counter-strain *Y*, due live load, is greatest, there is no part of the live load applied on the right of the foot of the *Y* member, or of the top of the *Z* member above.

In multiple systems, where the chords are not straight lines, in finding total chord strains, care should be taken to reduce all strains that are to be added, so that their lines of action will be parallel; horizontal, for instance.

CLASS III.—TOP CHORD HORIZONTAL, OTHER MEMBERS INCLINED. $\alpha = 0$.

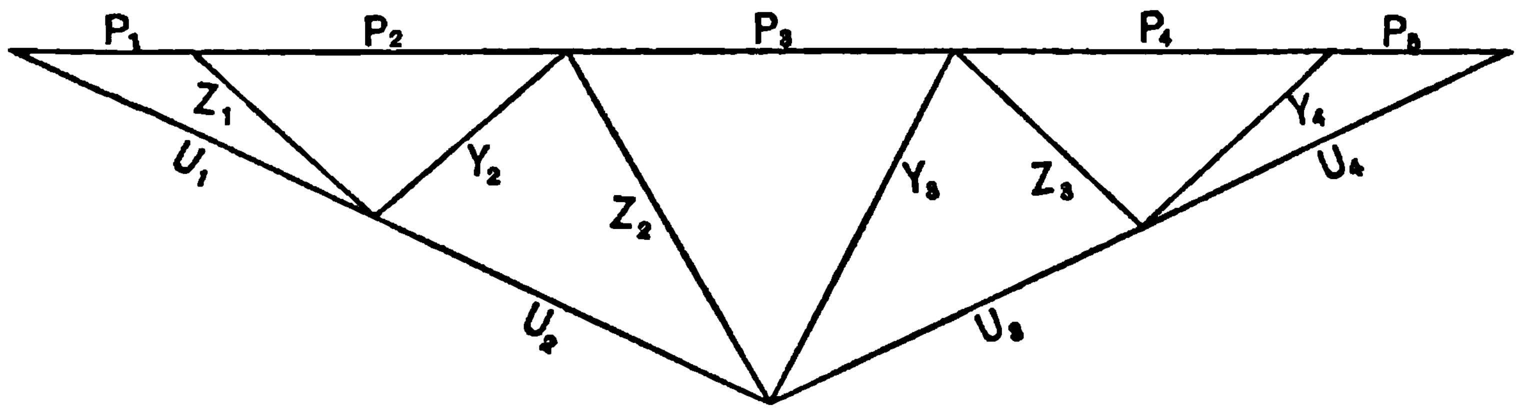


FIG. 39.—TRUSSED BEAM.

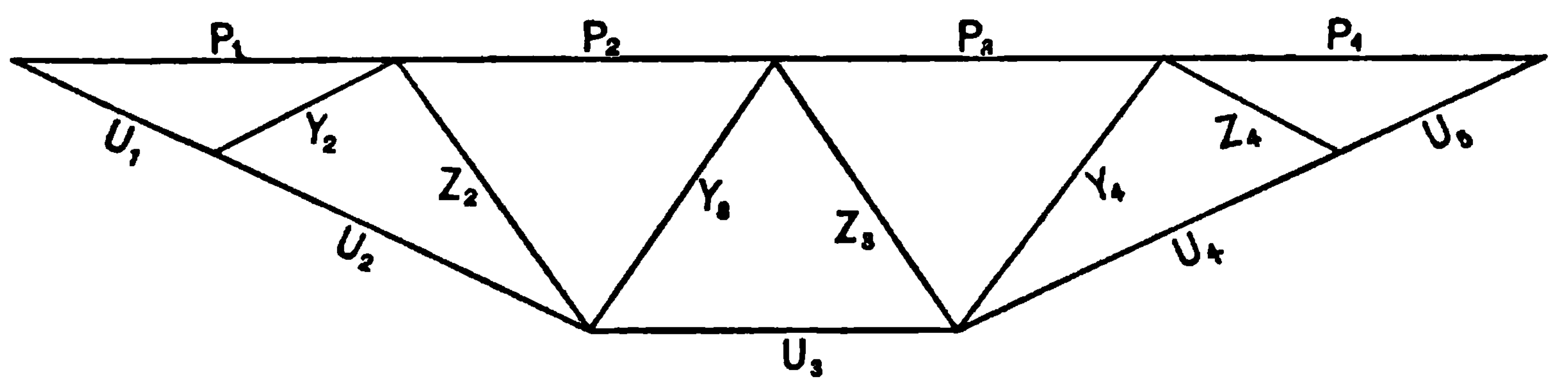


FIG. 40.—TRUSSED BEAM.

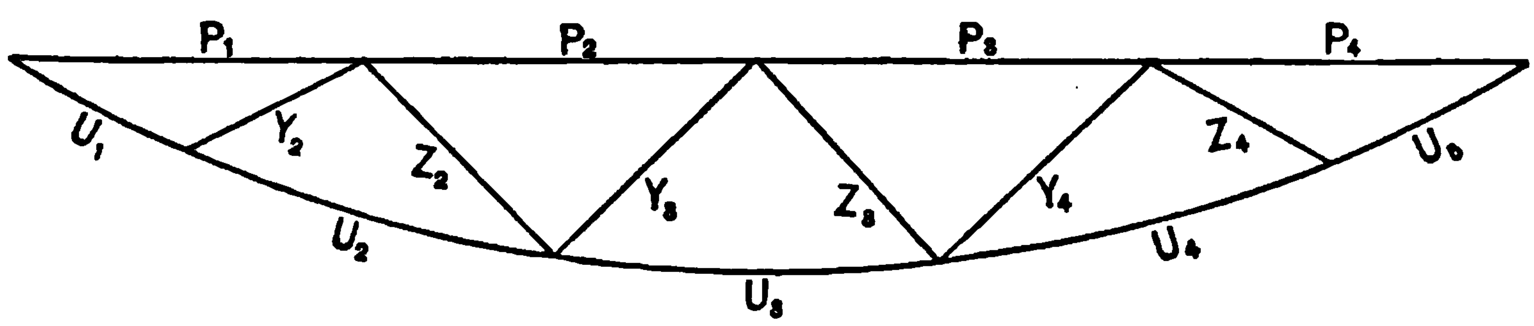


FIG. 41.—INVERTED BOWSTRING.

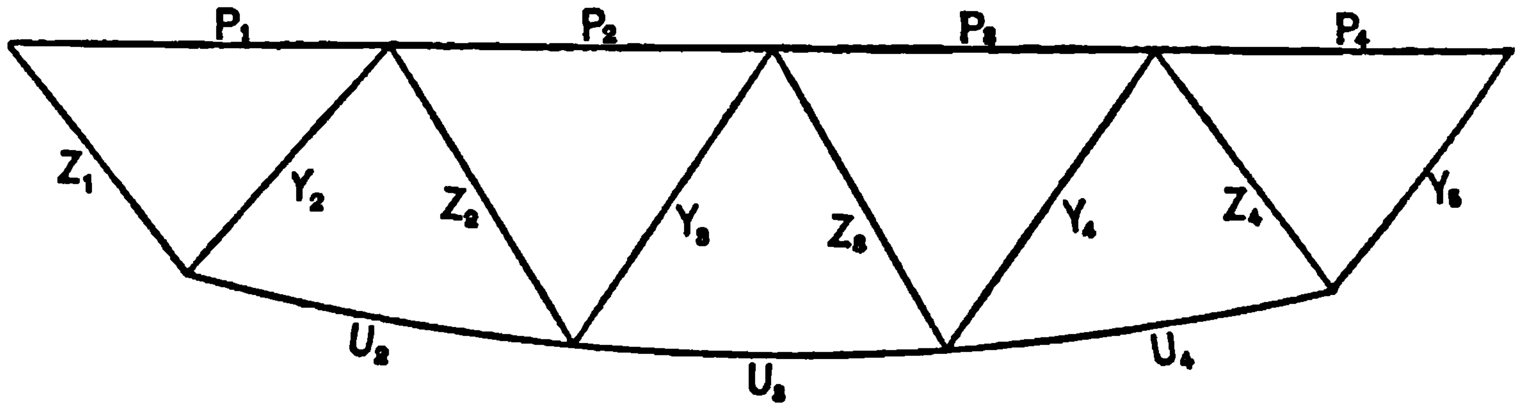


FIG. 42.—INVERTED TRUNCATED BOWSTRING.

FORMULÆ FOR CLASS III. $\alpha = 0$.

Method of Moments.

$$H = M \div h.$$

$$\Delta H = \frac{M_{r+1}}{h_{r+1}} - \frac{M_r}{h_r}.$$

$$P = H_{r+1}.$$

$$U = -H_r \div \cos\beta.$$

$$Y = \Delta_r H \div \cos\phi.$$

$$Z = \Delta_{r+1} H \div \cos\theta.$$

CLASS IV.—BOTH CHORDS HORIZONTAL, WEB MEMBERS INCLINED. $\alpha = 0, \beta = 0$.

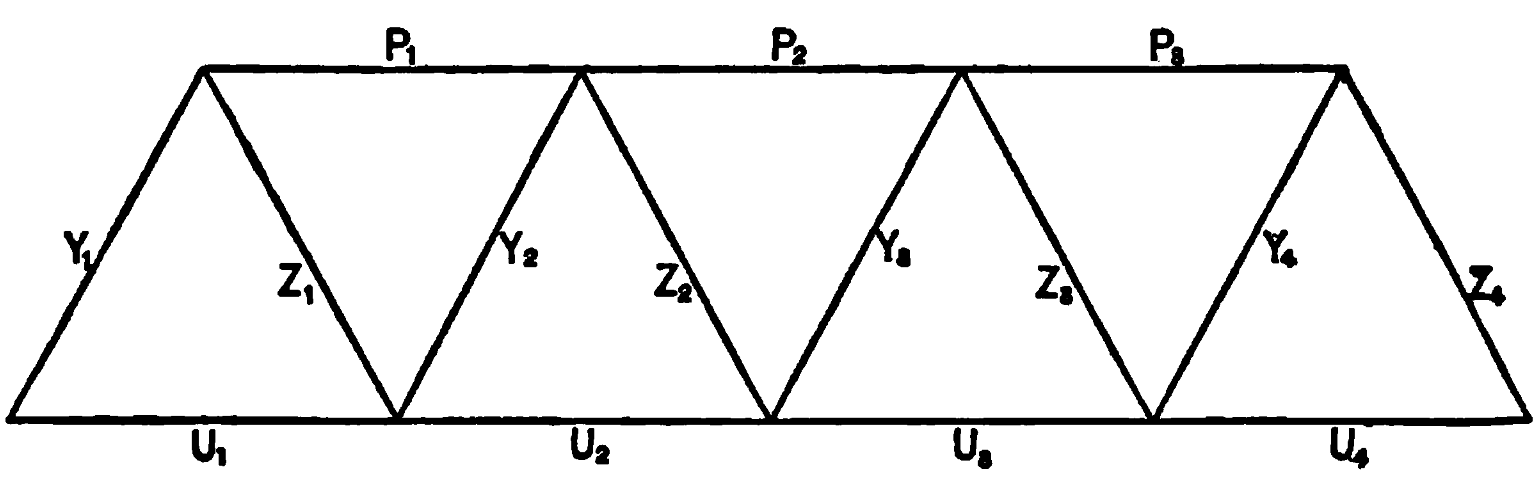


FIG. 43.—THE TRIANGULAR GIRDER. ERECT, OR "THROUGH."

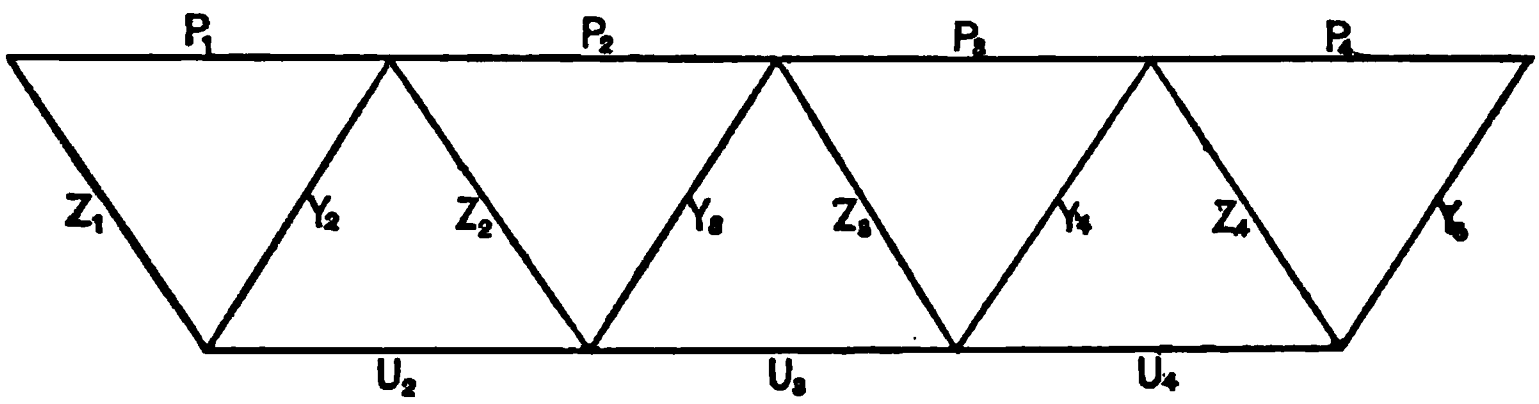


FIG. 44.—TRIANGULAR GIRDER. SUSPENDED, OR "DECK." $\alpha = 0, \beta = 0$.

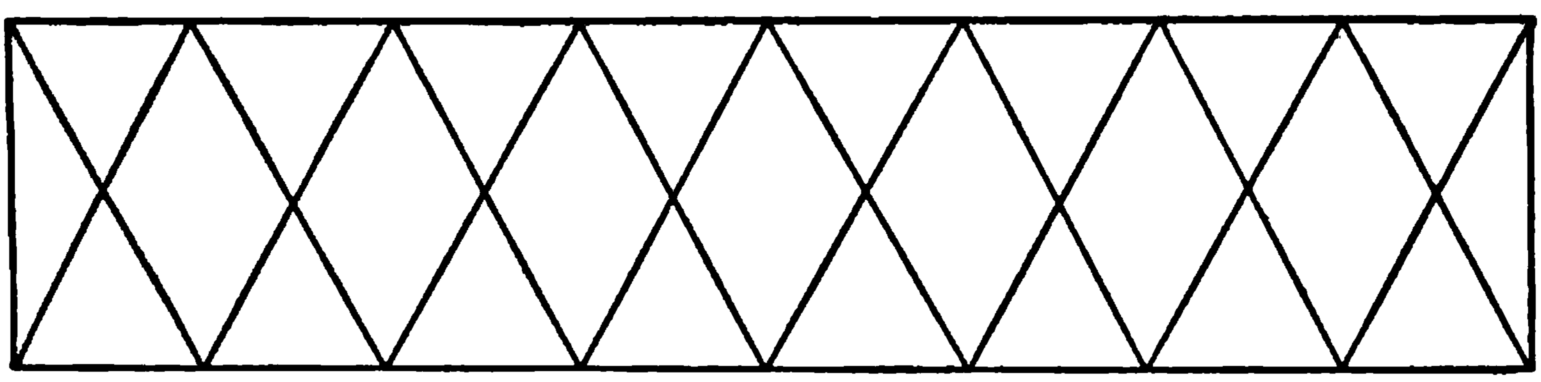


FIG. 45.—DOUBLE TRIANGULAR GIRDER. FIGS. 43 AND 44 COMBINED.

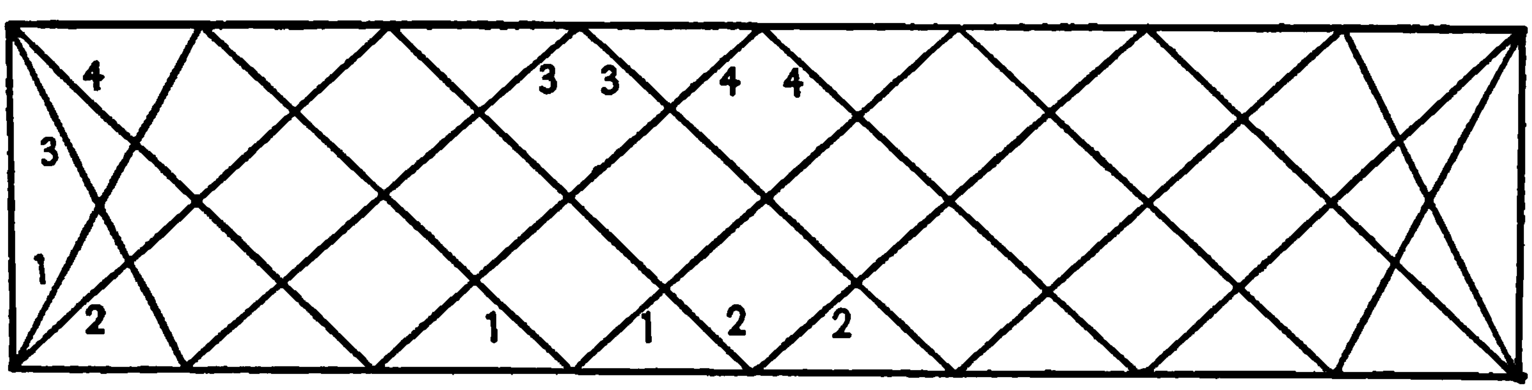


FIG. 46.—QUADRUPLE TRIANGULAR SYSTEM. EACH SYSTEM INDEPENDENT.

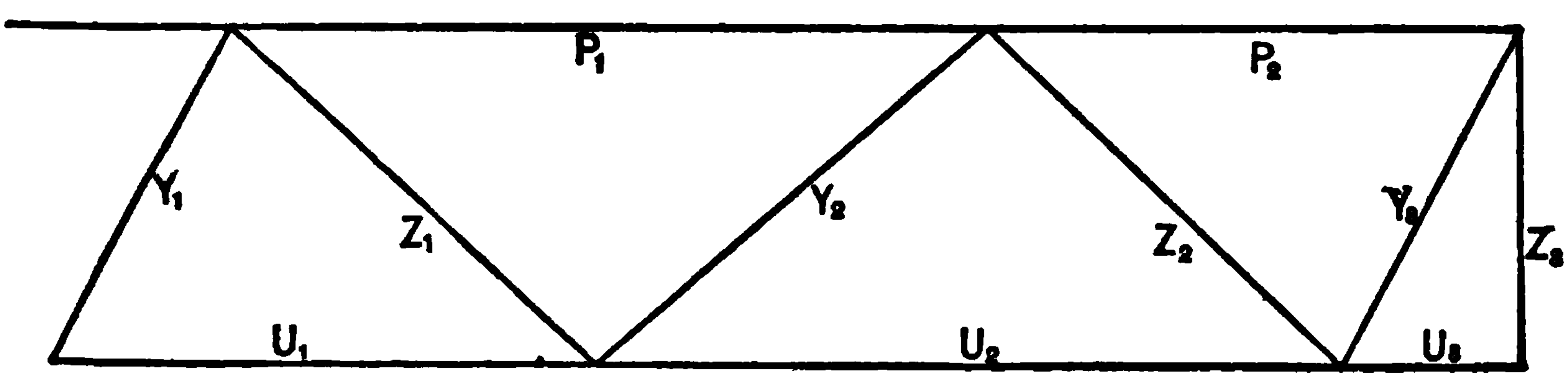


FIG. 46a.—FIRST SYSTEM.

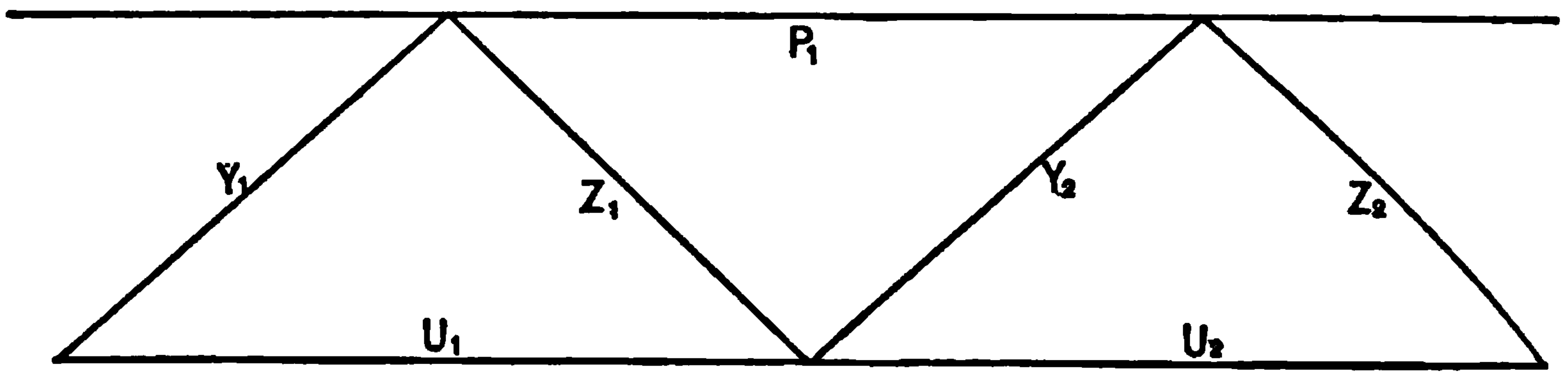


FIG. 46*b*.—SECOND SYSTEM.

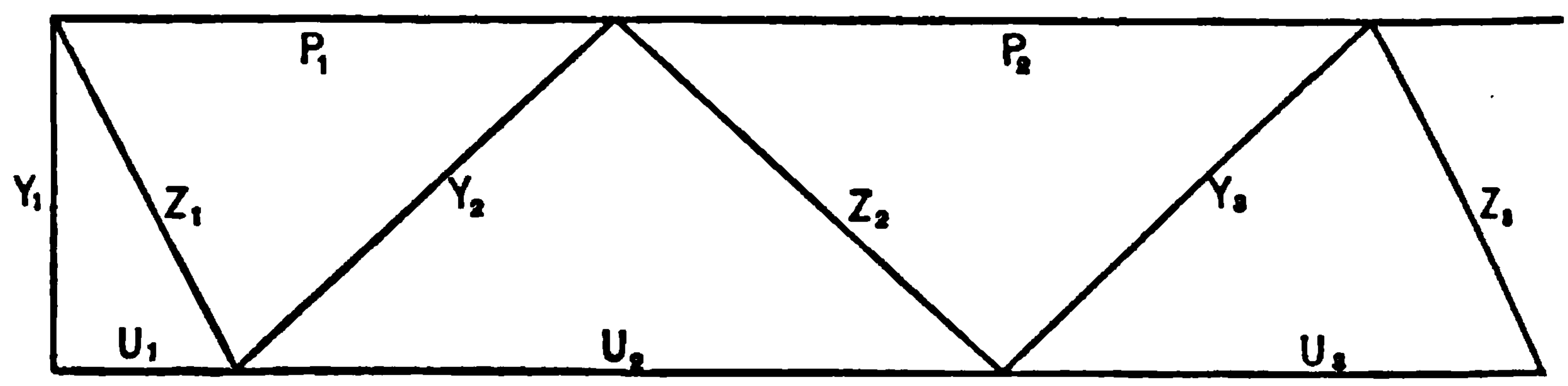


FIG. 46*c*.—THIRD SYSTEM.

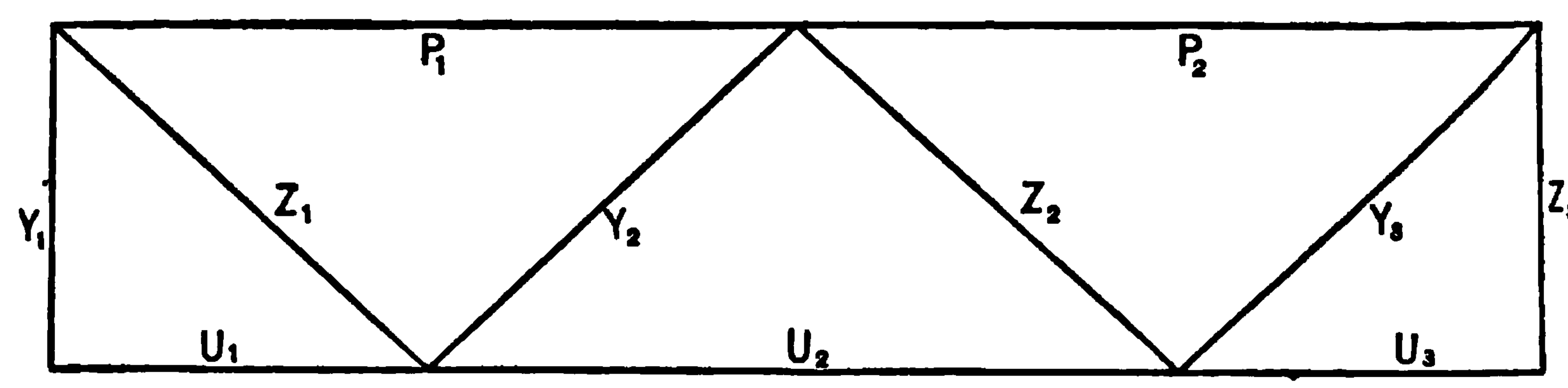


FIG. 46*d*.—FOURTH SYSTEM.

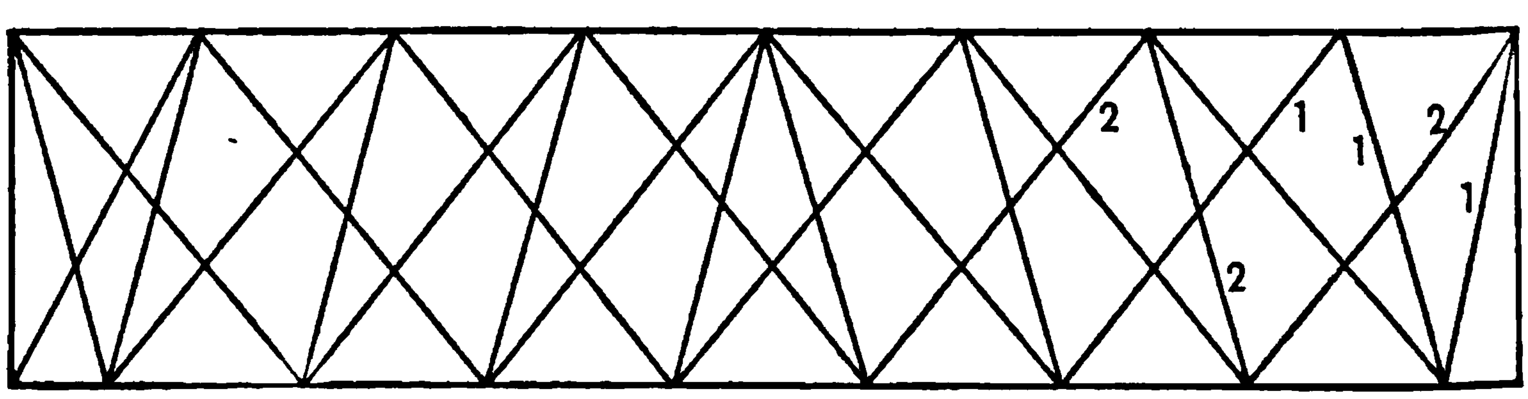
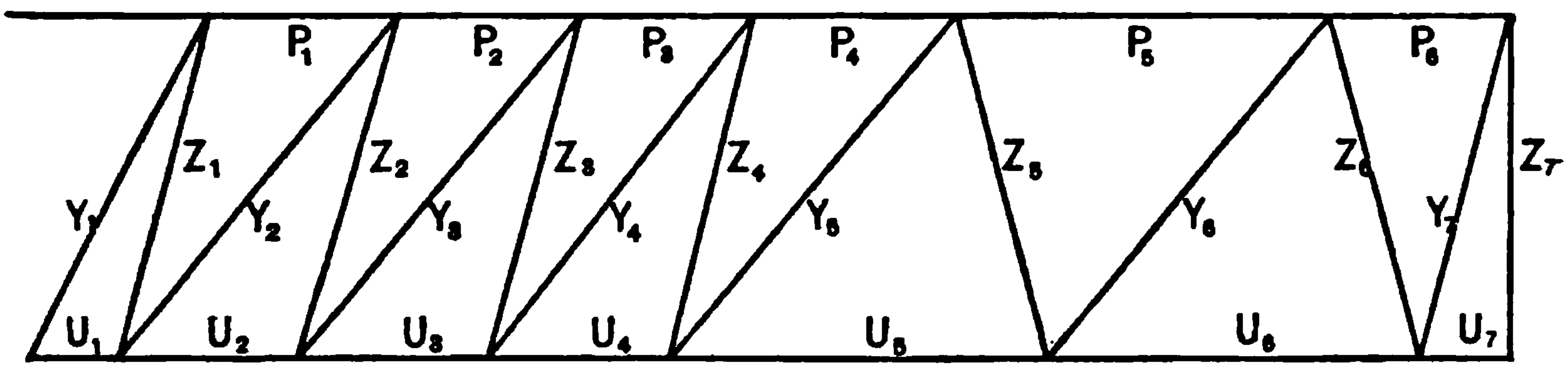
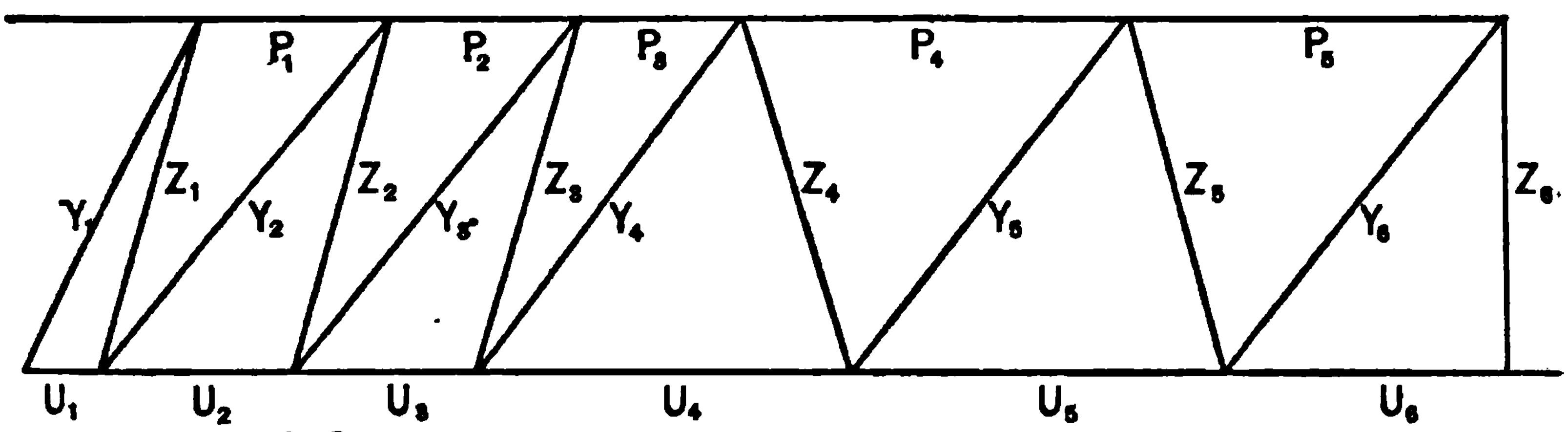


FIG. 47.—THE POST TRUSS. TWO SYSTEMS.



As Counters. As Mains.

FIG. 47*a*.—FIRST SYSTEM.



As Counters. As Mains.

FIG. 47*b*.—SECOND SYSTEM.

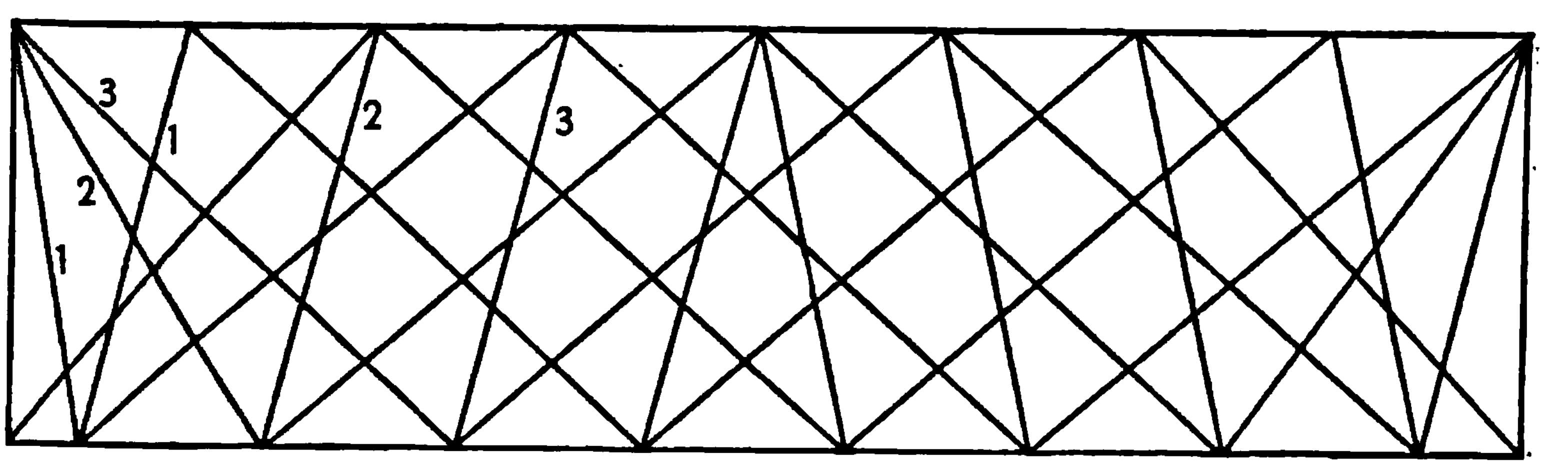


FIG. 47*c*.—POST TRUSS. THREE SYSTEMS.

FORMULÆ FOR CLASS IV. $\alpha = 0, \quad = 0.$

Method of Moments.

$$H = M \div h.$$

$$\Delta H = \Delta M \div h.$$

$$P = H_{r+1}.$$

$$U = -H_r.$$

$$Y = \Delta_r H \div \cos\phi.$$

$$Z = \Delta_{r+1} H \div \cos\theta.$$

CLASS V. — ALTERNATE WEB MEMBERS VERTICAL. BOTH CHORDS INCLINED.

Generally $\left\{ \begin{array}{l} \text{Verticals in compression,} \\ \text{Diagonals in tension.} \end{array} \right\}$ $\theta = 90°$.

Only one set of diagonals shown in figures. These are counters in first half-span, mains in second half-span.

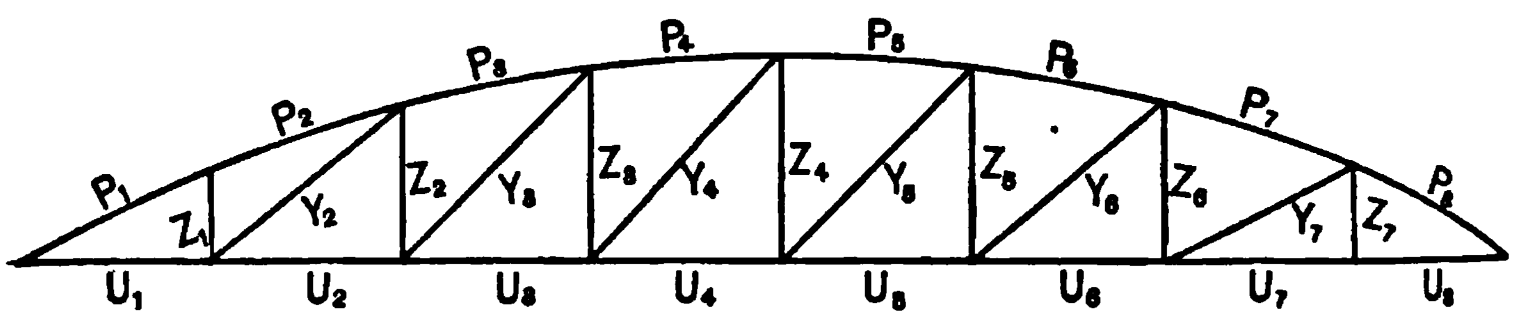


FIG. 48. — THE CRESCENT.

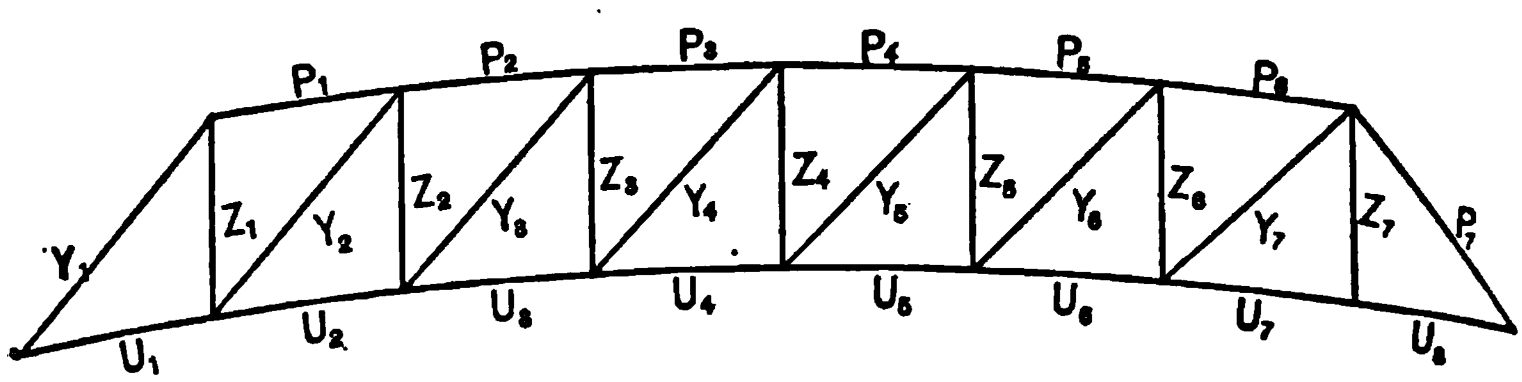


FIG. 49. — TRUNCATED CRESCENT.

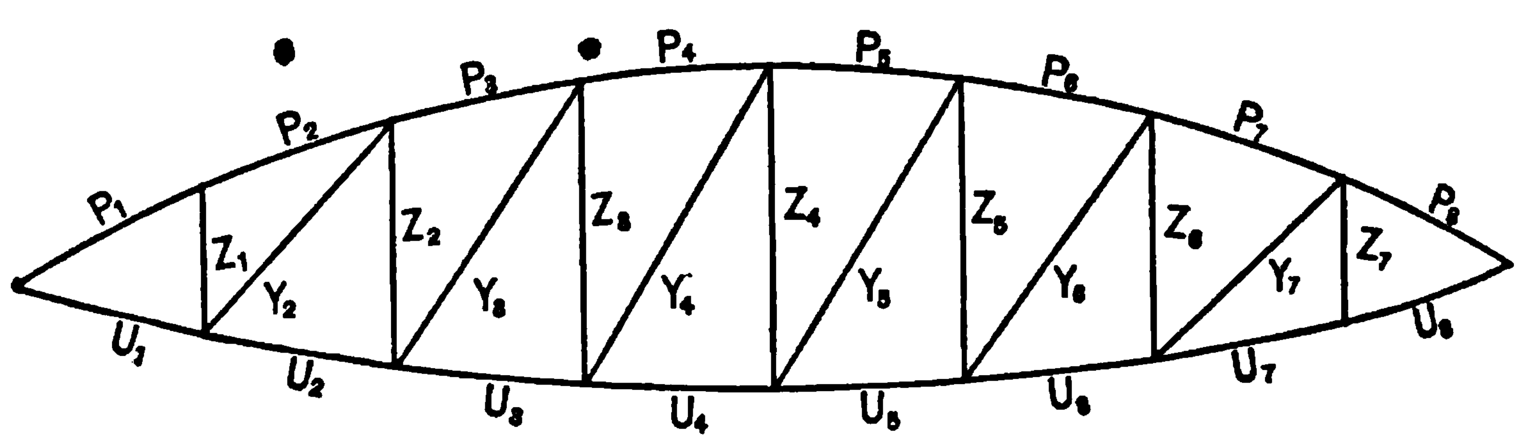


FIG. 50. — DOUBLE BOW, OR BRUNEL GIRDER.

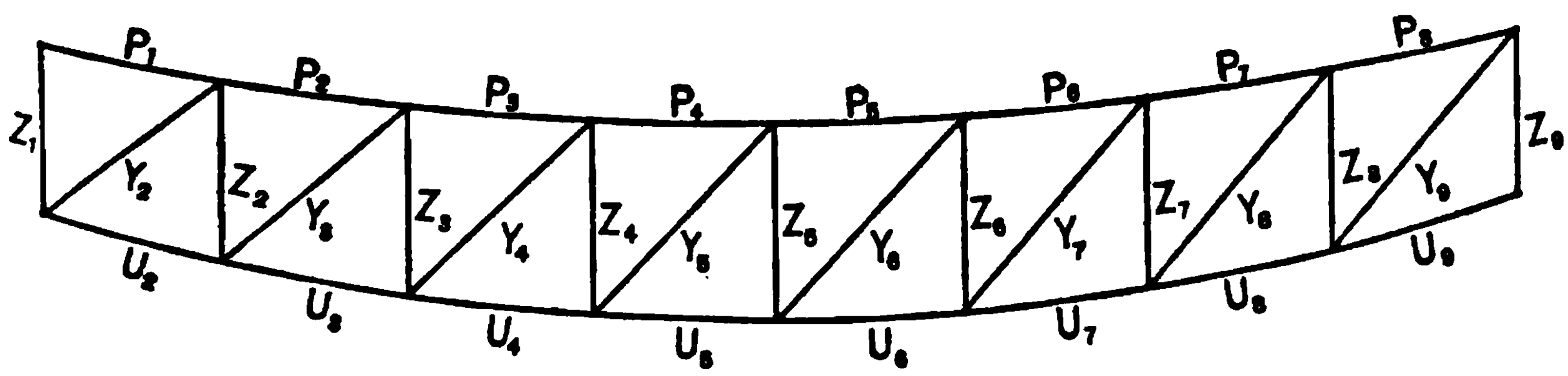


FIG. 51.—TRUNCATED CRESCENT INVERTED.

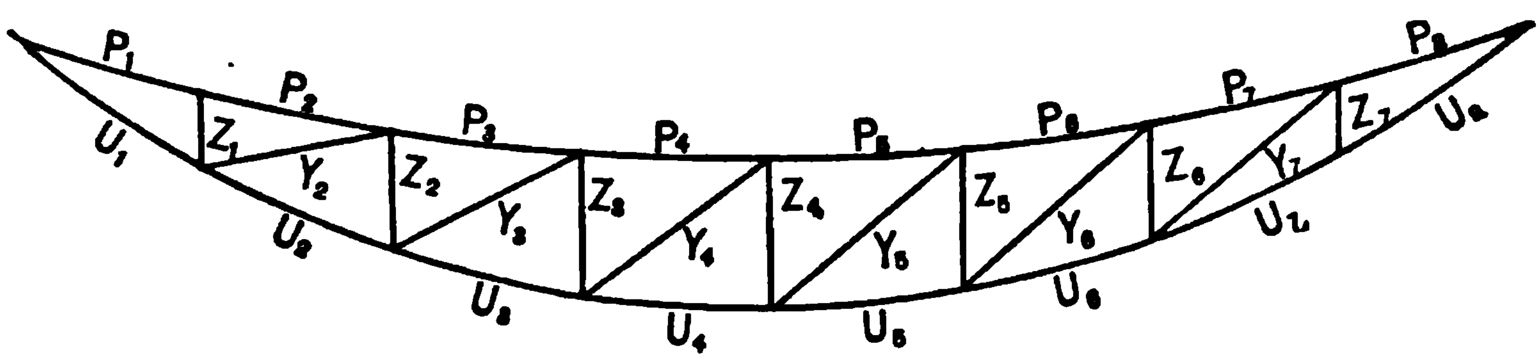


FIG. 52.—CRESCENT SUSPENDED.

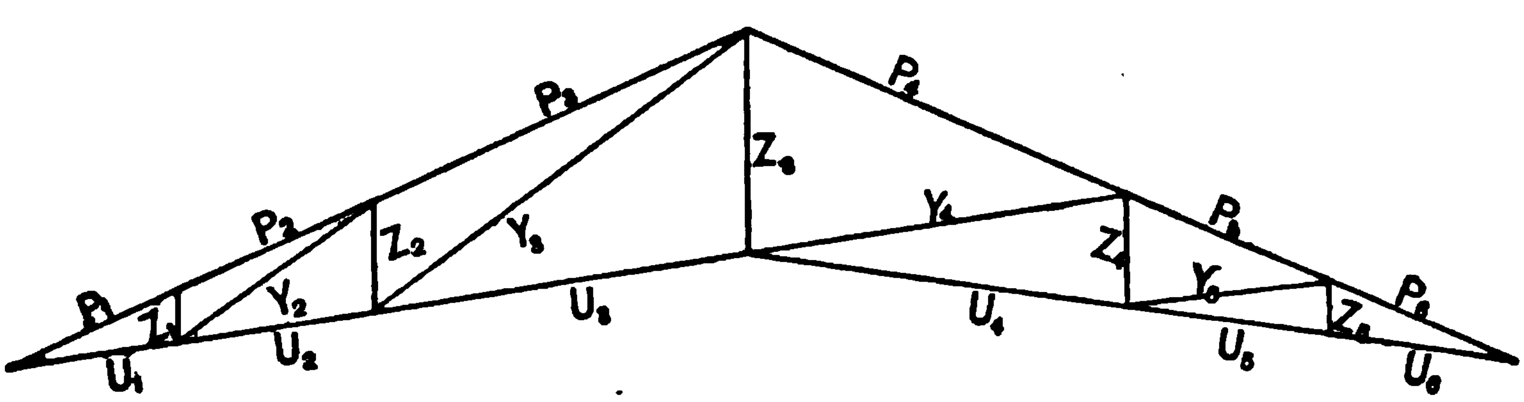


FIG. 53.—ROOF PRINCIPAL.

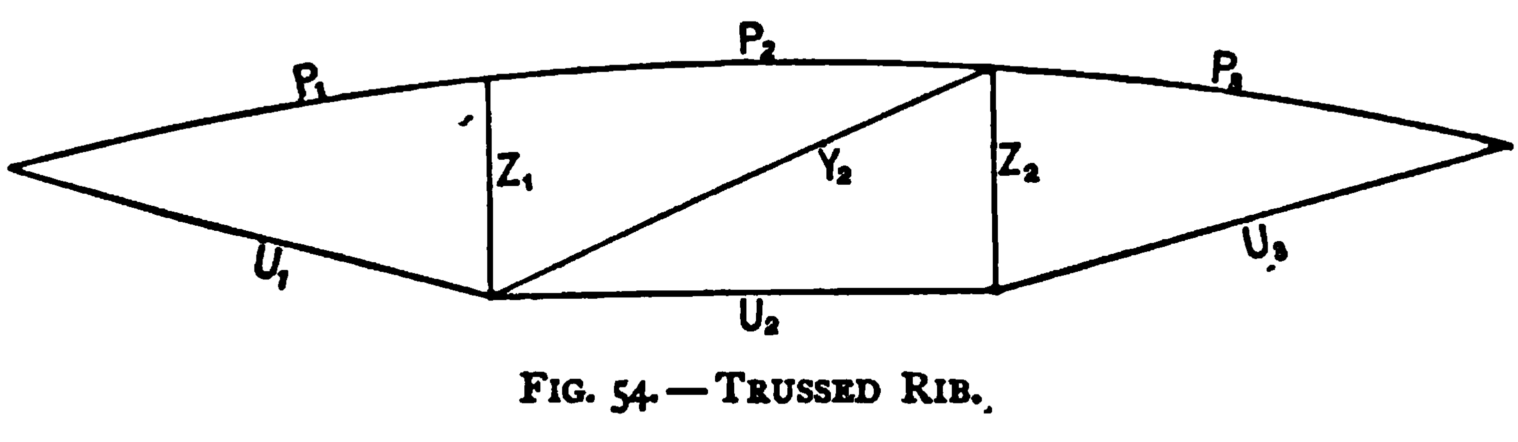


FIG. 54.—TRUSSED RIB.

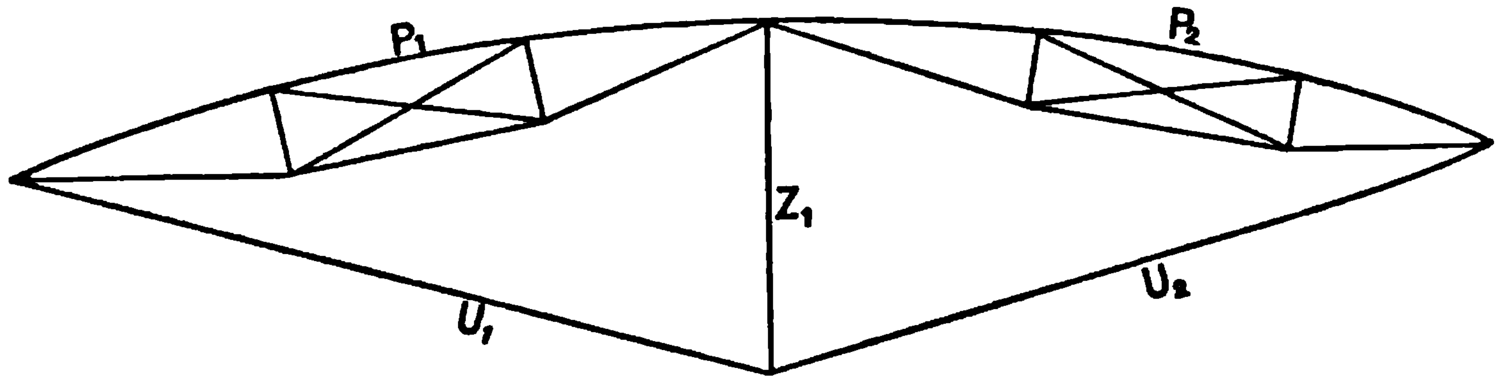


FIG. 55.—DOME PRINCIPAL. PRIMARY SYSTEM. SECONDARY SYSTEM SAME AS IN FIG. 54.

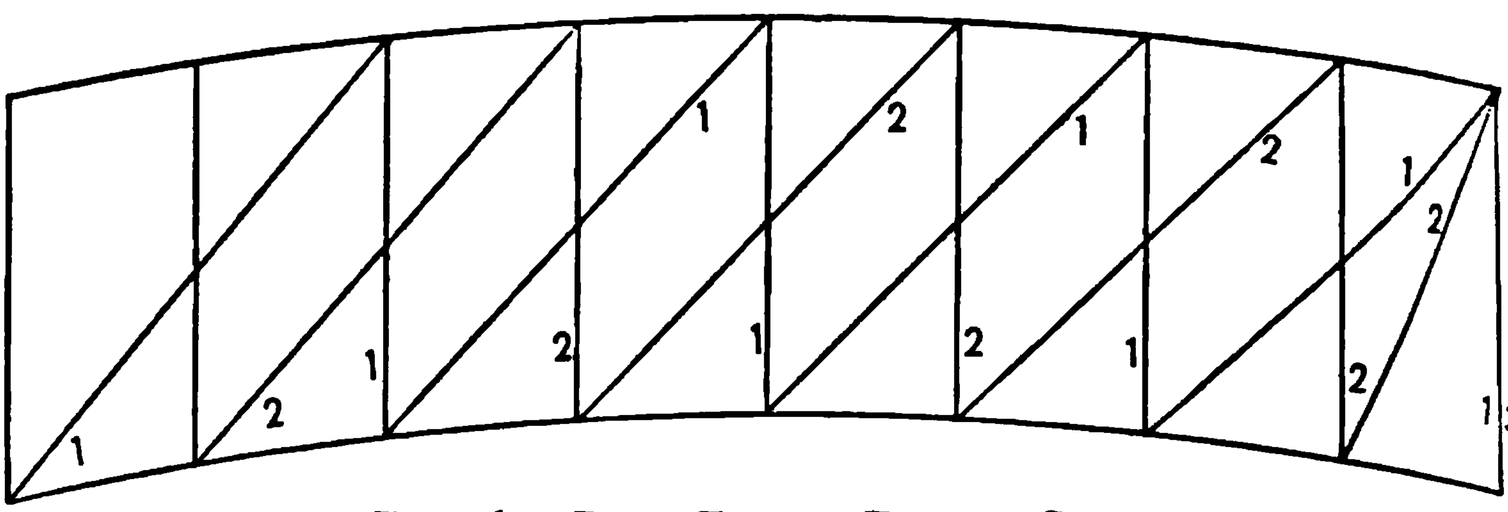


FIG. 56.—BENT TRUSS. DOUBLE SYSTEM.

FORMULÆ FOR CLASS V. $\theta = 90°$.

Method of Moments.

$$H = M \div h.$$

$$\Delta H = \frac{M_{r+1}}{h_{r+1}} - \frac{M_r}{h_r}.$$

$$P = H \div \cos\alpha.$$

$$U_{-1} = -H \div \cos\beta.$$

$$Y = \Delta_r H \div \cos\phi.$$

$$Z = P_{r+1}\sin\alpha_{r+1} - P_r\sin\alpha_r - Y_r\sin\phi_r$$
$$= H_{r+1}\tan\alpha_{r+1} - H_r\tan\alpha_r - \Delta_r H\tan\phi_r.$$

(Load applied at bottom.)

$$Z = U_r\sin\beta_r - U_{r+1}\sin\beta_{r+1} - Y_r\sin\phi_r$$
$$= -H_r\tan\beta_r + H_{r+1}\tan\beta_{r+1} - \Delta_r H\tan\phi_r.$$

(Load applied at top.)

The value of Z in equation

$$Z = \Delta H \div \cos\theta,$$

here becomes indeterminate, since $\Delta H = 0$ for the horizontal projection of the Z member, and $\cos 90° = 0$.

CLASS VI. — BOTH CHORDS INCLINED. ALTERNATE WEB MEMBERS VERTICAL.

$$\text{Generally} \left\{ \begin{array}{l} \text{Verticals in tension,} \\ \text{Diagonals in compression.} \end{array} \right\} \phi = 90°.$$

Only one set of diagonals shown in figures. These are counters in first half-span, mains in second half-span.

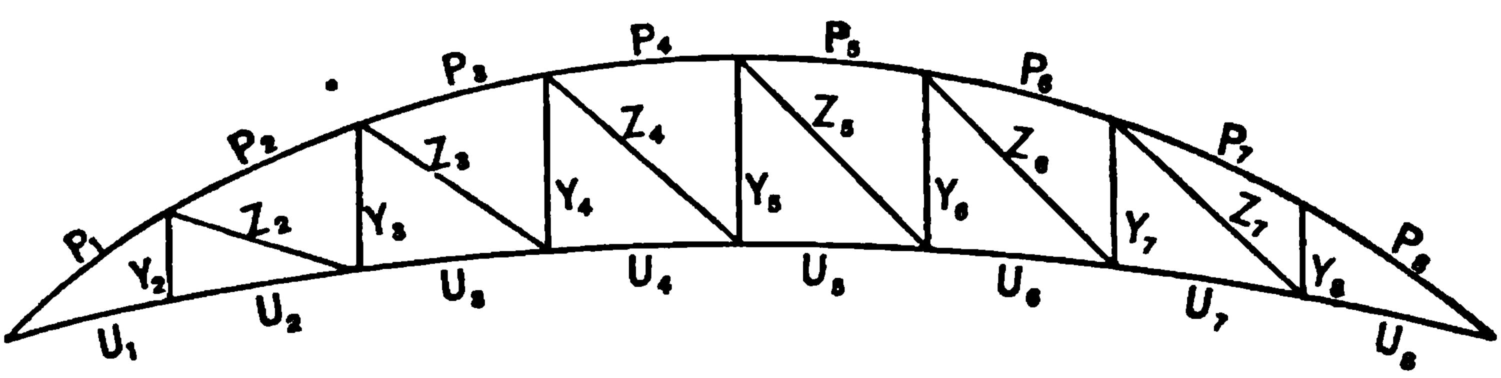


FIG. 57.—THE CRESCENT.

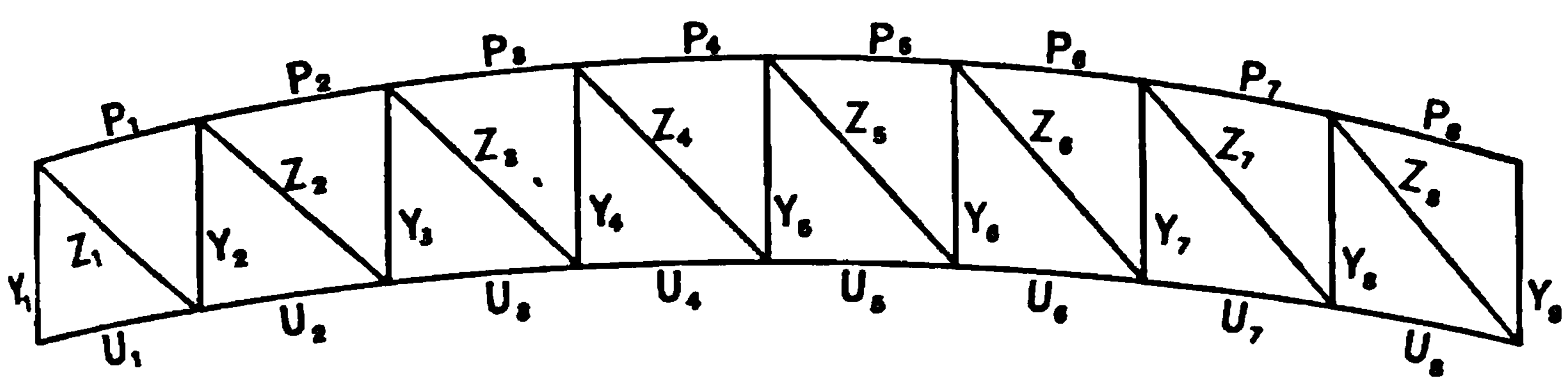


FIG. 58.—TRUNCATED CRESCENT.

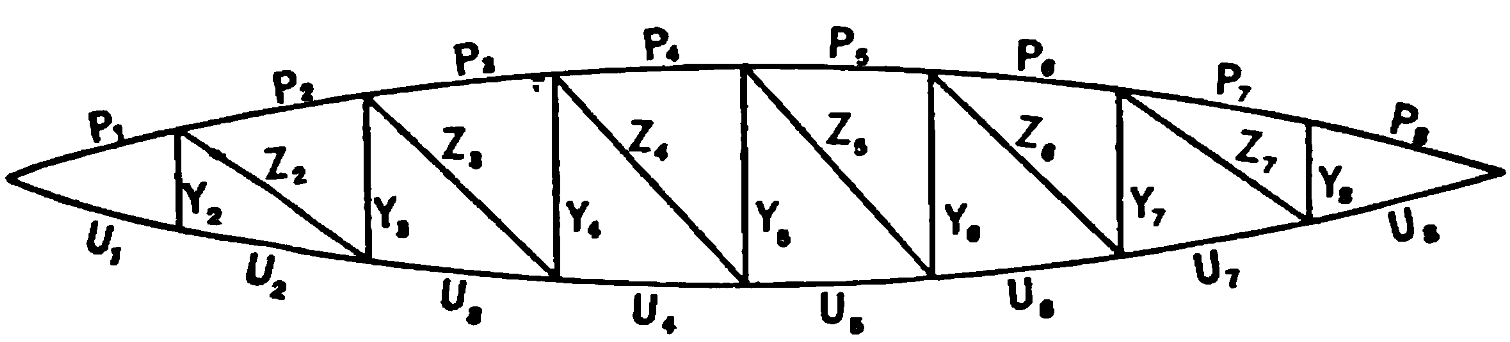


FIG. 59.—THE BRUNEL GIRDER.

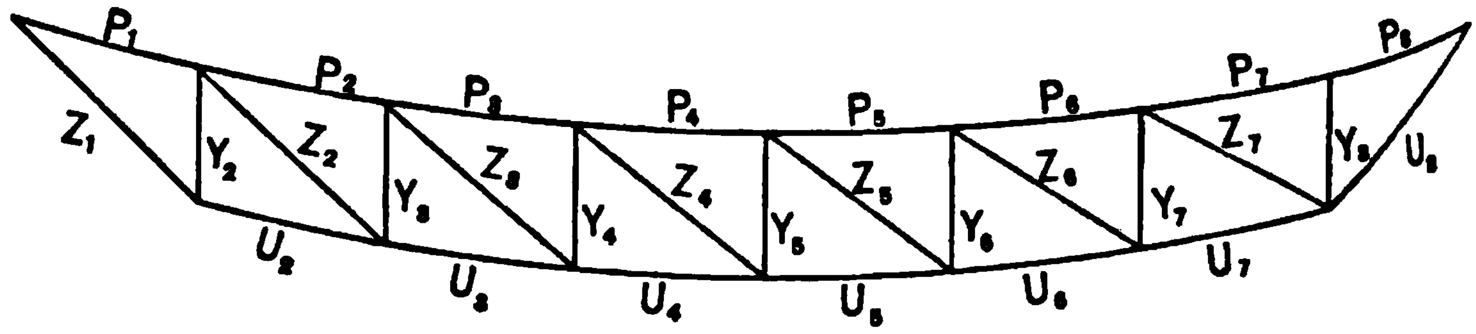


FIG. 60.—TRUNCATED CRESCENT SUSPENDED.

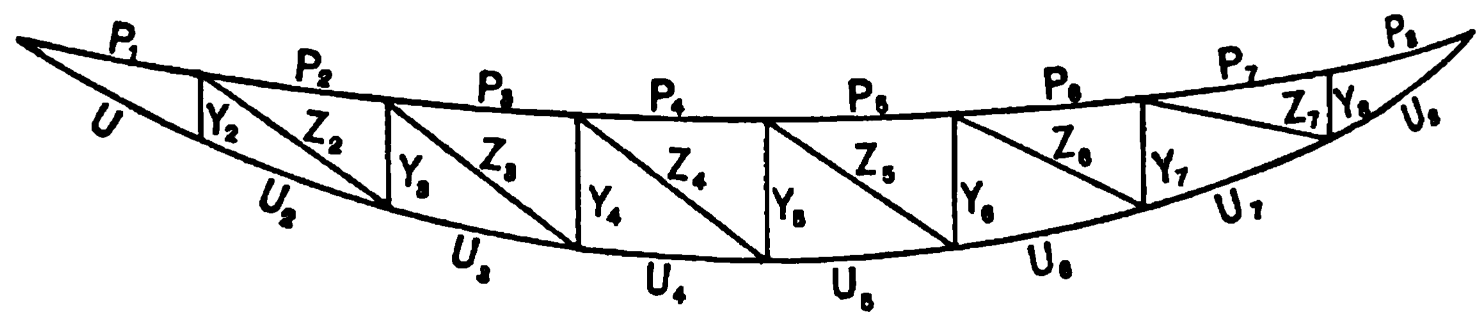


FIG. 61.—SUSPENDED CRESCENT.

FORMULÆ FOR CLASS VI. $\phi = 90°$.

Method of Moments.

$$H = M \div h.$$

$$\Delta H = \frac{M_{r+1}}{h_{r+1}} - \frac{M_r}{h_r}.$$

$$P = H \div \cos\alpha.$$

$$U_{-1} = -H \div \cos\beta.$$

$$\begin{aligned} Y &= P_{r+1}\sin\alpha_{r+1} - P_r\sin\alpha_r - Z_{r+1}\sin\theta_{r+1} \\ &= H_{r+1}\tan\alpha_{r+1} - H_r\tan\alpha_r - \Delta_{r+1}H\tan\theta_{r+1}. \end{aligned}$$

(Load applied at bottom.)

$$\begin{aligned} Y &= U_r\sin\beta_r - U_{r+1}\sin\beta_{r+1} - Z_r\sin\theta_r \\ &= -H_r\tan\beta_r + H_{r+1}\tan\beta_{r+1} - \Delta_r H\tan\theta_r. \end{aligned}$$

(Load applied at top.)

$$Z = \Delta_r H \div \cos\theta.$$

Multiple systems of this class are seldom built, since long struts are not economical.

CLASS VII. — BOTTOM CHORD HORIZONTAL. $\beta = 0$. ALTERNATE WEB MEMBERS VERTICAL. $\theta = 90°$.

In general, $\left\{\begin{array}{l}\text{Verticals in compression,}\\ \text{Diagonals in tension.}\end{array}\right.$

But one set of diagonals shown in figures. These are counters in first half-span, mains in second half-span.

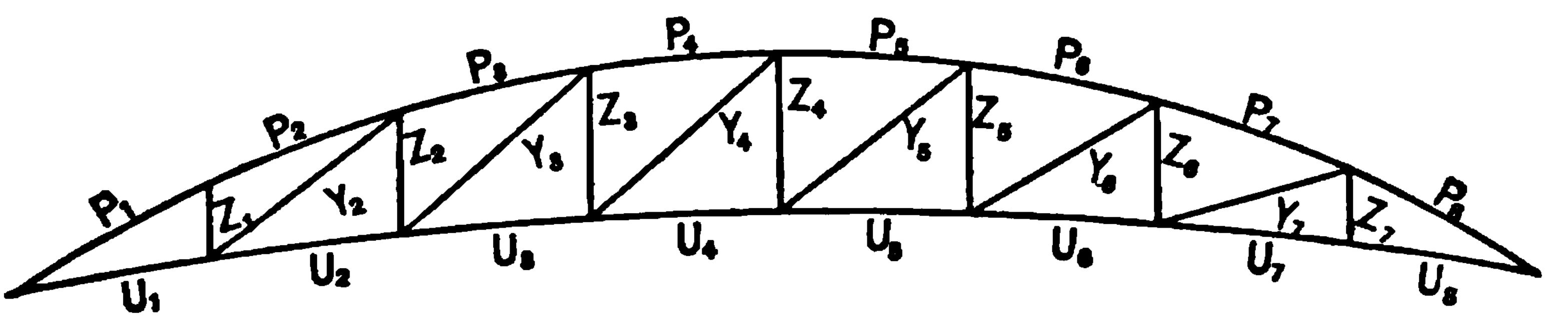


FIG. 62. — THE BOWSTRING.

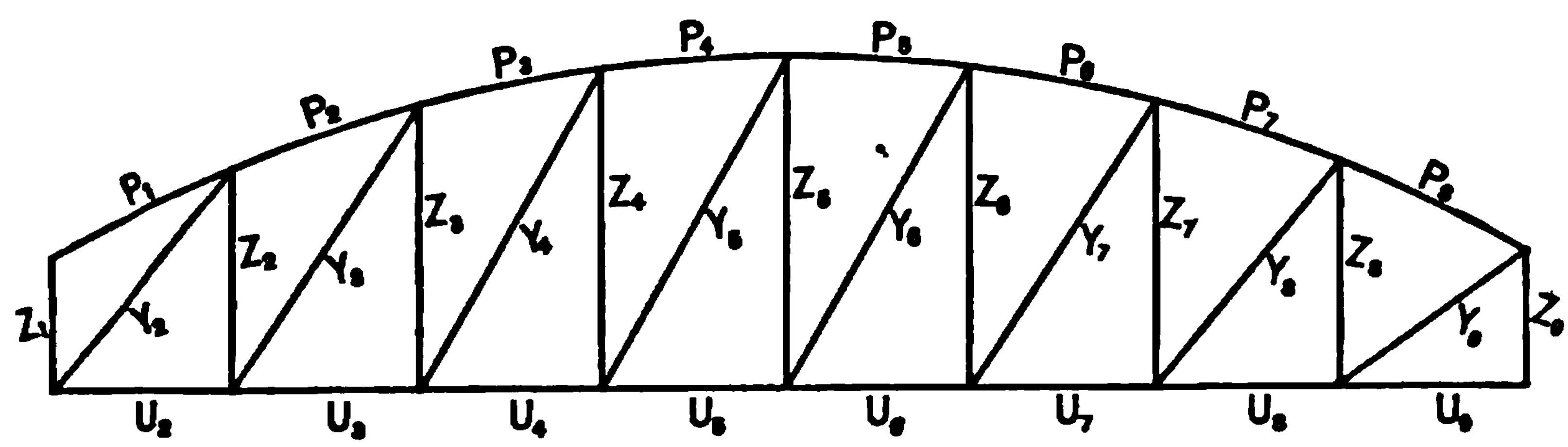


FIG. 63. — TRUNCATED BOWSTRING.

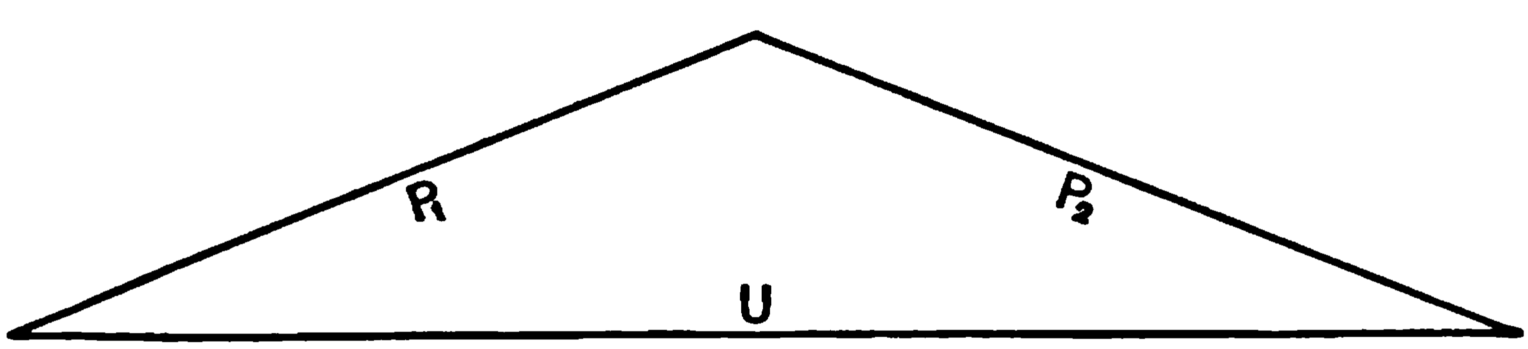


FIG. 64. — RAFTERS AND TIE.

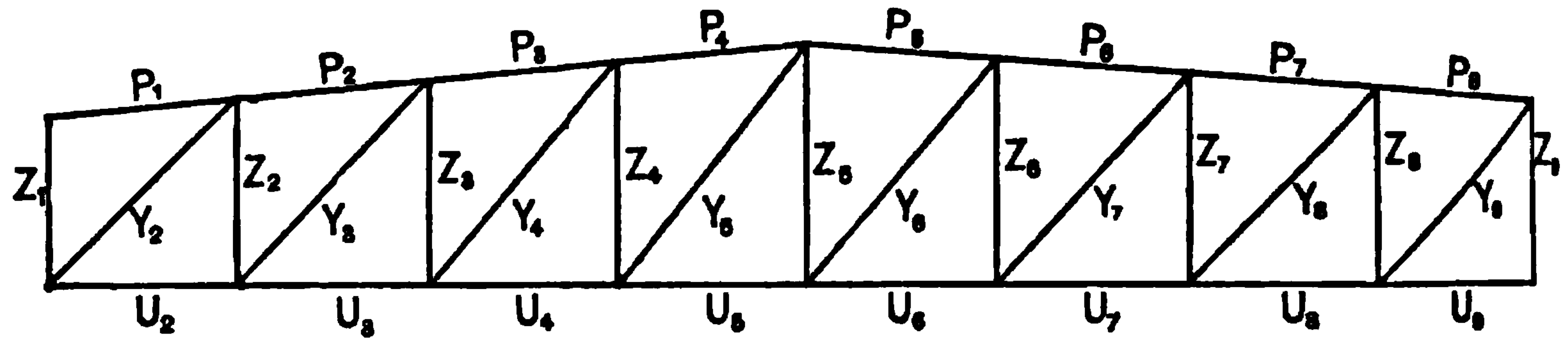


FIG. 65. — TOP CHORD UNIFORMLY SLOPED. $a_1 = a_2 = a_3$.

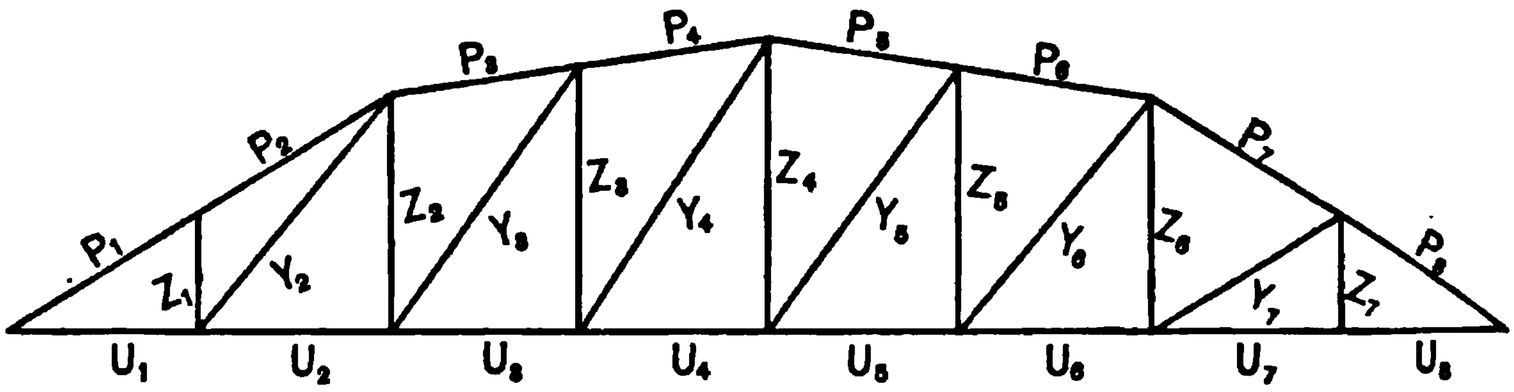


FIG. 66. — POLYGONAL TOP CHORD.

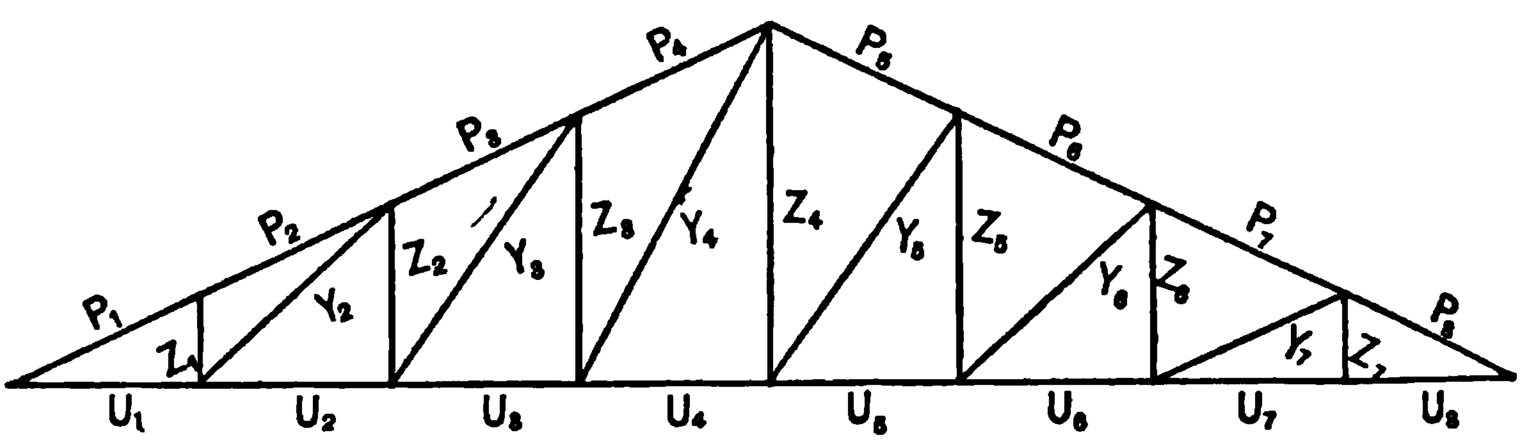


FIG. 67. — ROOF PRINCIPAL. $a_1 = a_2 = a_3$.

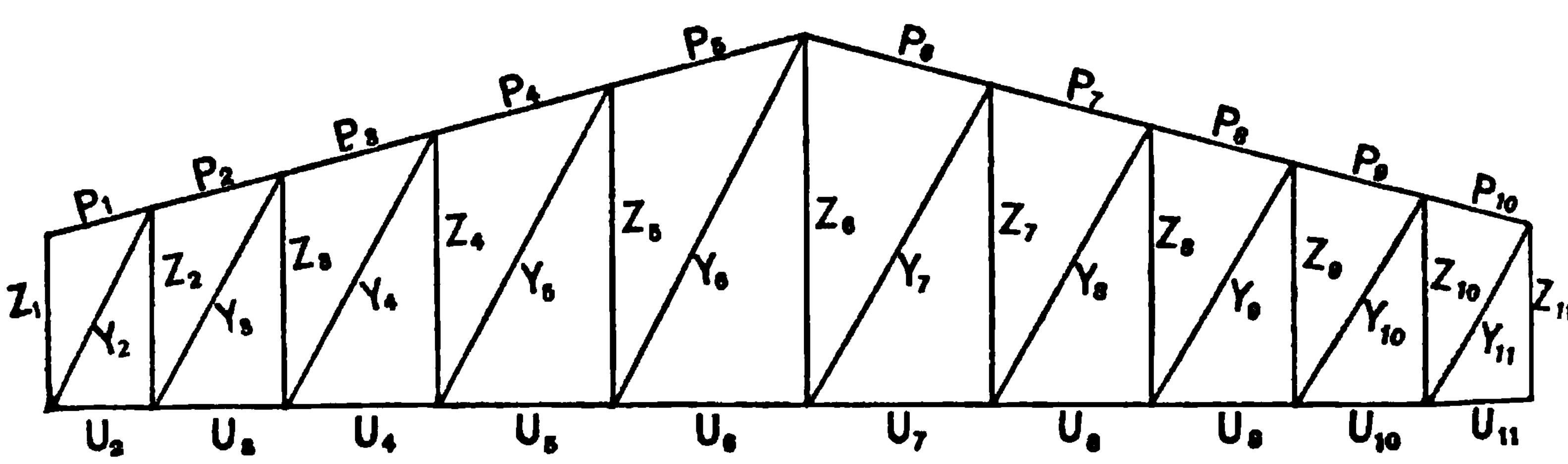


FIG. 68. — PARALLEL BRACING.

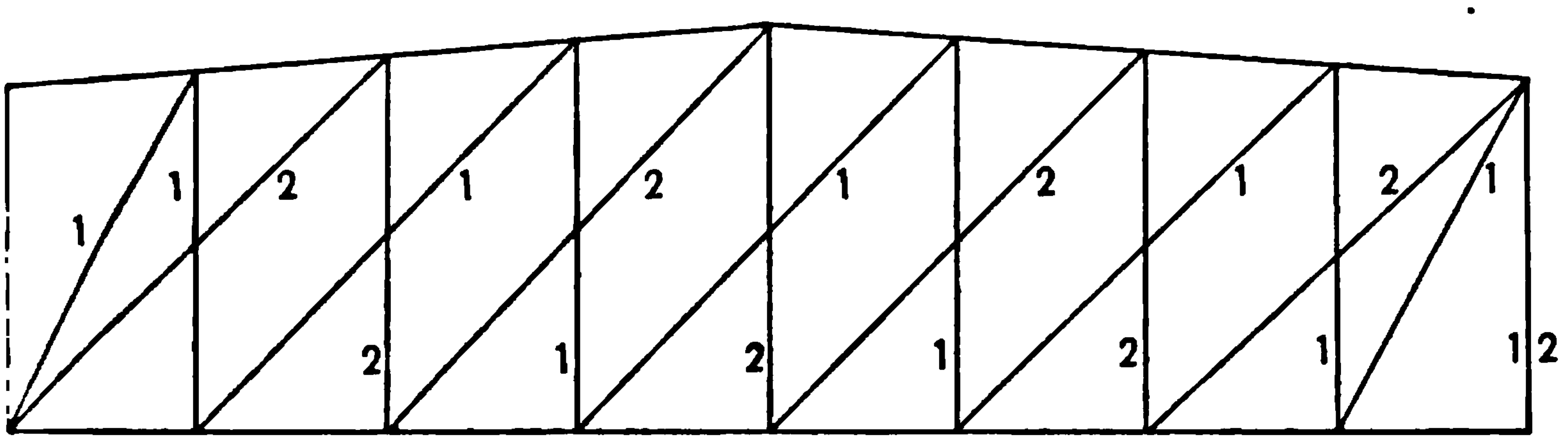


FIG. 69. — UNIFORM SLOPE. DOUBLE SYSTEM.

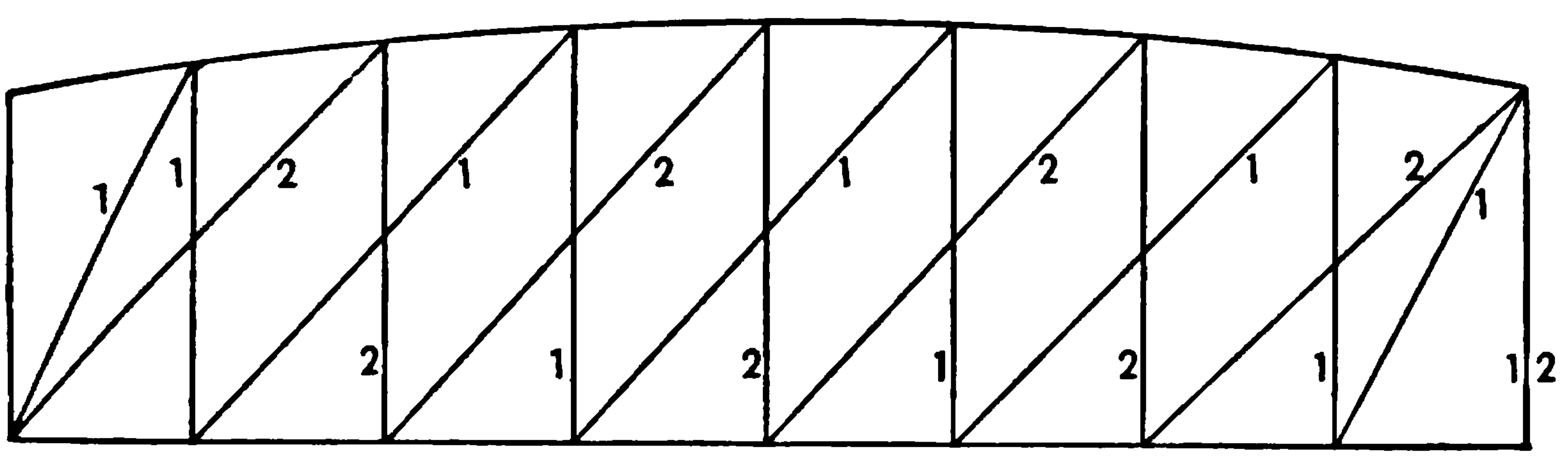


FIG. 70. — CURVED TOP. DOUBLE SYSTEM.

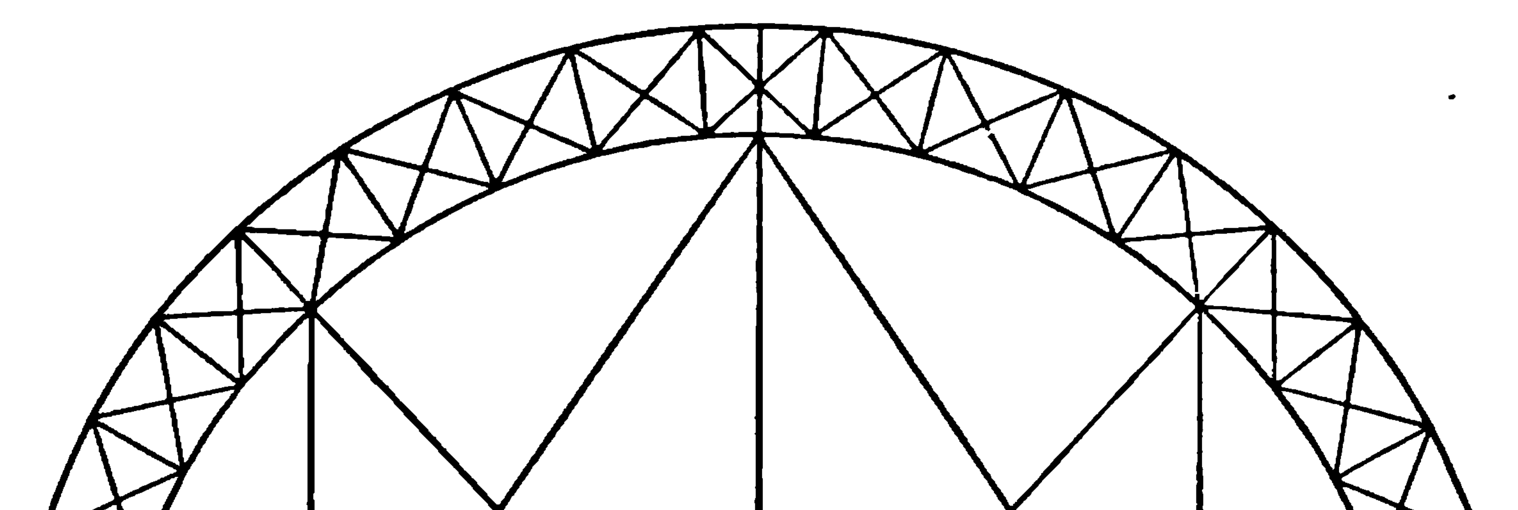


FIG. 71. — ROOF PRINCIPAL. (SEE FIG. 23.)

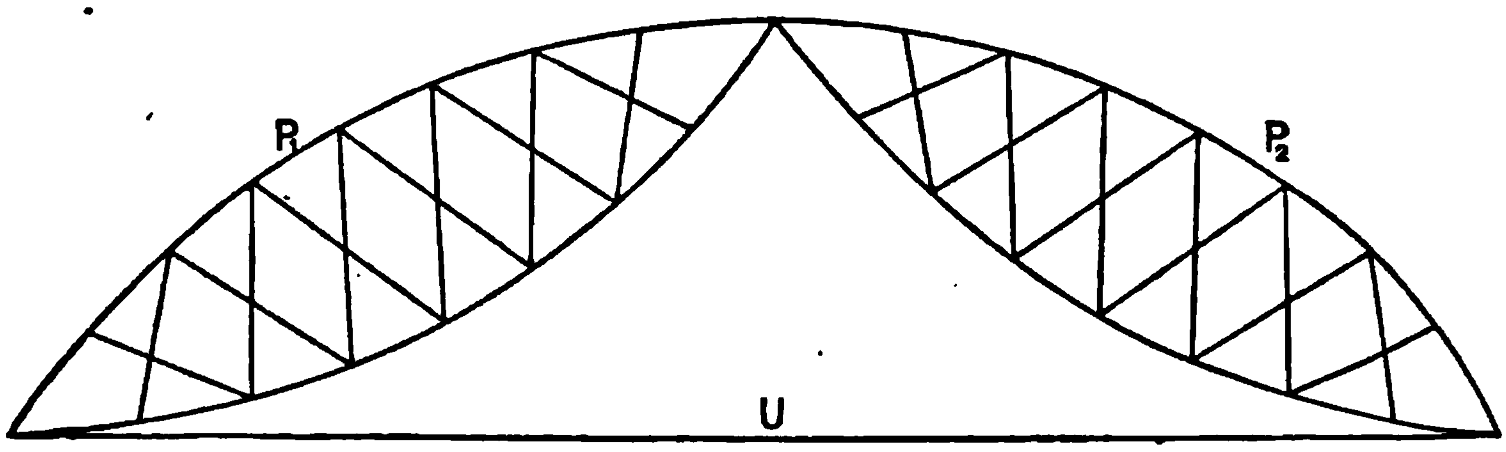


FIG. 71*a*.—THE TWIN FISHES, LONG SPAN. (SEE FIG. 22.)

FORMULÆ FOR CLASS VII. $\beta = 0, \theta = 90°$.

Method of Moments.

$$H = M \div h.$$

$$\Delta H = \frac{M_{r+1}}{h_{r+1}} - \frac{M_r}{h_r}.$$

$$P = H_r \div \cos\alpha.$$

$$U = -H_{r+1}.$$

$$\left.\begin{array}{l} Y = \Delta_r H \div \cos\phi_r. \\ Z = \Delta_{r+1} H \tan\phi_{r+1}. \end{array}\right\} \text{simultaneous.}$$

Foremost end of live load at Z_{r-1} for maximum Y and Z_r.

When it is desired to have the diagonals in each half-span parallel for a given number of panels, as in Fig. 68, the lengths of the panels and the inclination of the diagonals may be found as follows:—

Let l = length of span.
h_0 = height at each end.
h = central height.
m = number of panels in each half-span.
ϕ = inclination to horizon of counters in first half-span, and of mains in second half-span.
α = inclination of top chord.
Δl = the variable panel length.

Then

$$\Delta l = \frac{h_r}{\tan\phi}, \quad \text{and} \quad \Delta h = h_r \frac{\tan\alpha}{\tan\phi}.$$

$$\tan\alpha = \frac{h - h_0}{\frac{1}{2}l}.$$

$$h_{m-1} = h - \Delta h = h - h\frac{\tan\alpha}{\tan\phi} = h\left(1 - \frac{\tan\alpha}{\tan\phi}\right).$$

$$h_{m-2} = h_{m-1}\left(1 - \frac{\tan\alpha}{\tan\phi}\right) = h\left(1 - \frac{\tan\alpha}{\tan\phi}\right)^2.$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$h_0 = h\left(1 - \frac{\tan\alpha}{\tan\phi}\right)^m, \tag{154}$$

$$\therefore \quad \tan\phi = \frac{\tan\alpha}{1 - \left(\frac{h_0}{h}\right)^{\frac{1}{m}}} \tag{155}$$

for the first half-span.

Similarly, for the second half-span

$$h = h_0\left(1 + \frac{\tan\alpha}{\tan\phi}\right)^m, \tag{156}$$

$$\tan\phi = \frac{\tan\alpha}{1 + \left(\frac{h}{h_0}\right)^{\frac{1}{m}}}. \tag{157}$$

Generally,

$$h_r = h_0\left(1 + \frac{\tan\alpha}{\tan\phi}\right)^r. \tag{158}$$

CLASS VIII. — TOP CHORD HORIZONTAL. $\alpha = 0$. STRUTS VERTICAL. $\theta = 90°$.

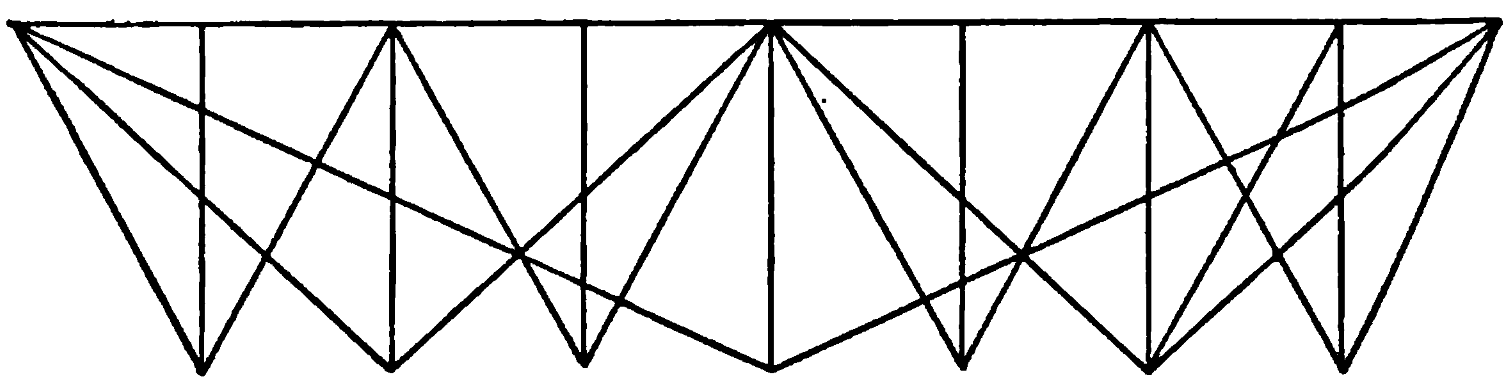


FIG. 72. — THE FINK TRUSS.

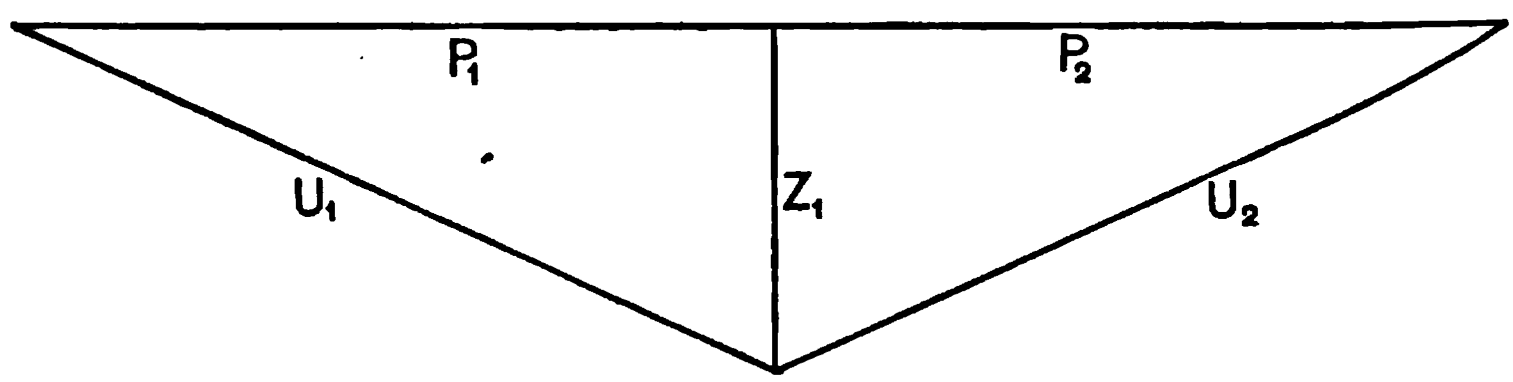


FIG. 72a. — MAIN SUSPENDERS.

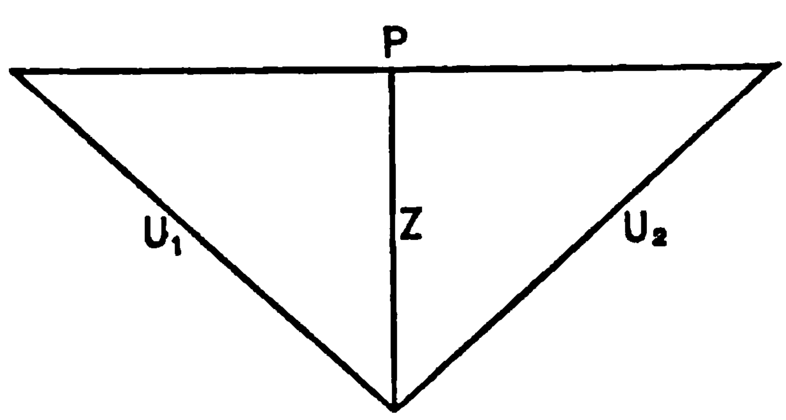
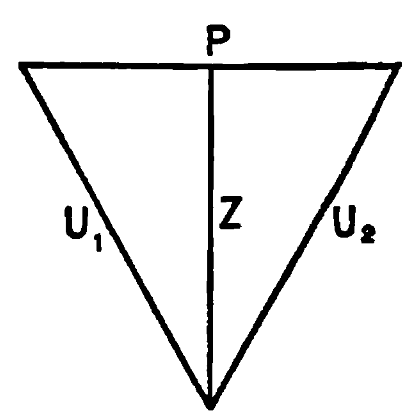


FIG. 72b. — SECONDARIES.



TERTIARIES.

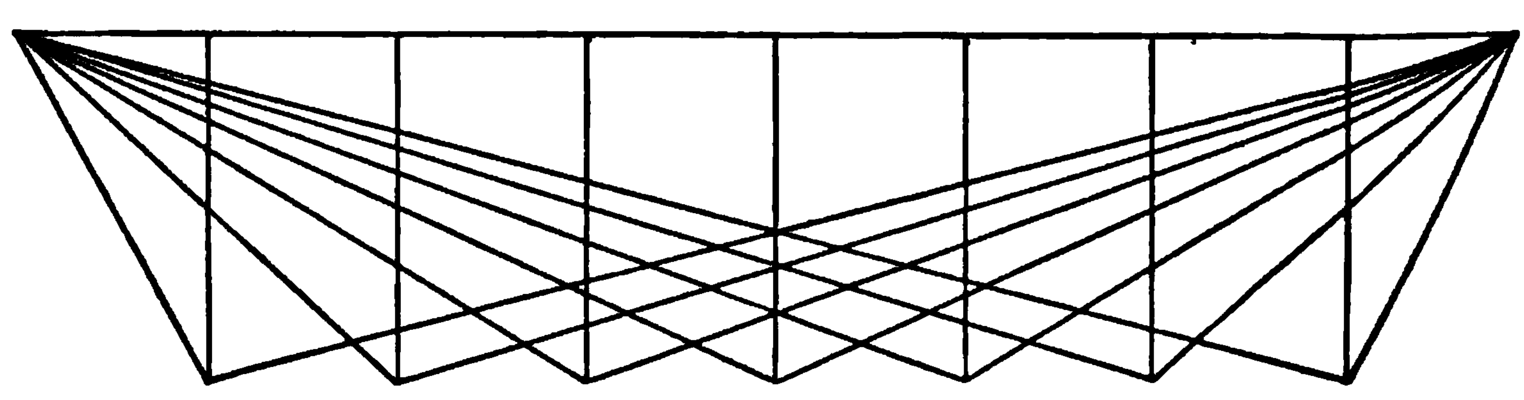


FIG. 73. — THE BOLLMAN TRUSS.

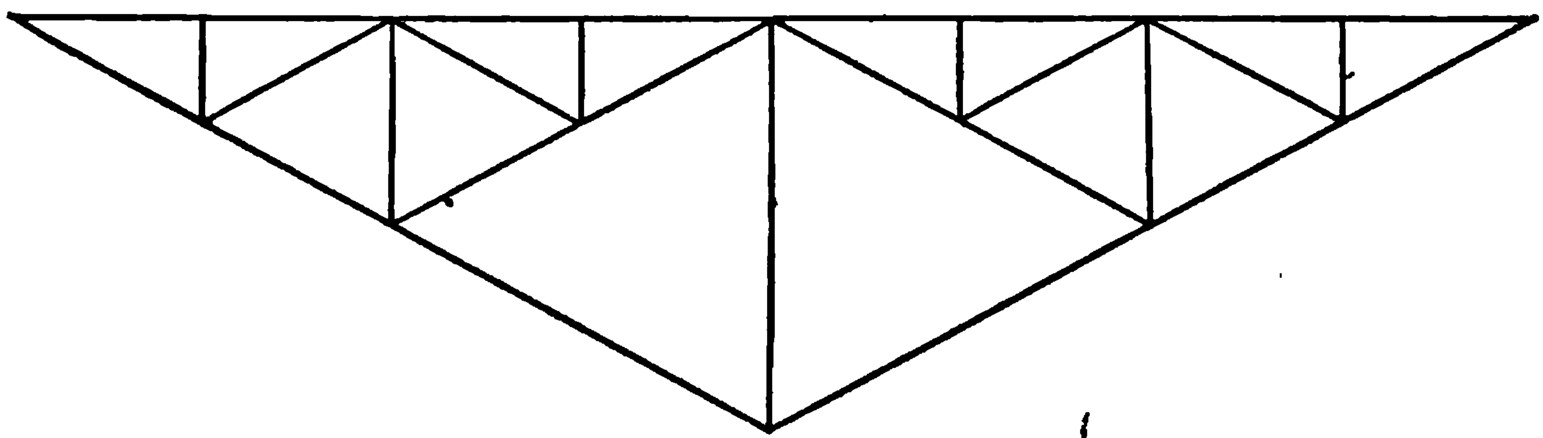


FIG. 74.—TRUSSED RAFTER OF FIG. 34.

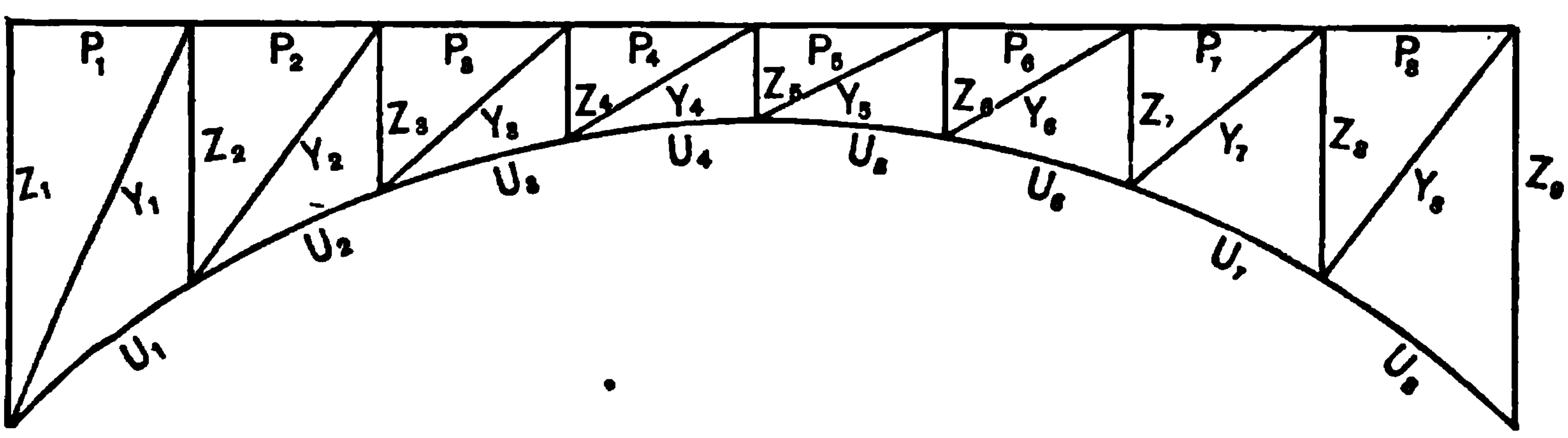


FIG. 75.—SPANDREL FILLED.

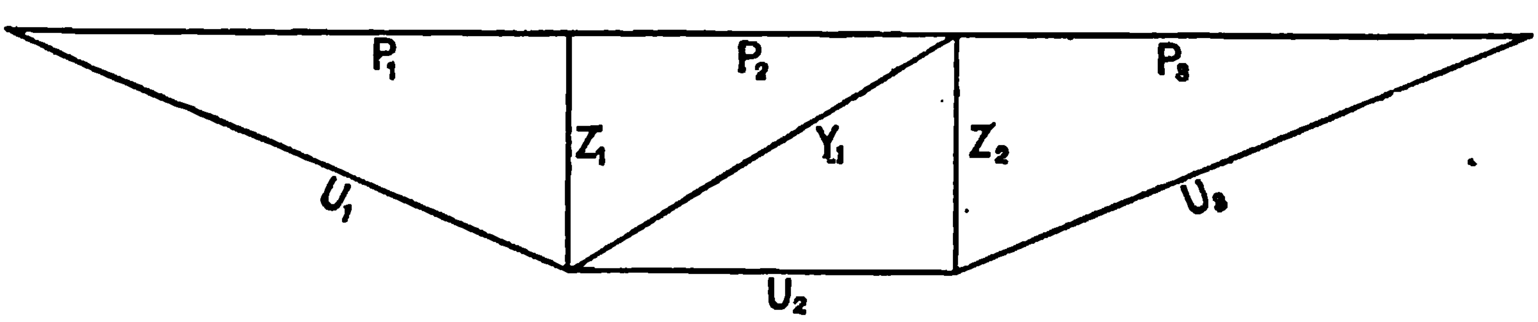


FIG. 76.—TRUSSED BEAM, THREE LINKS.

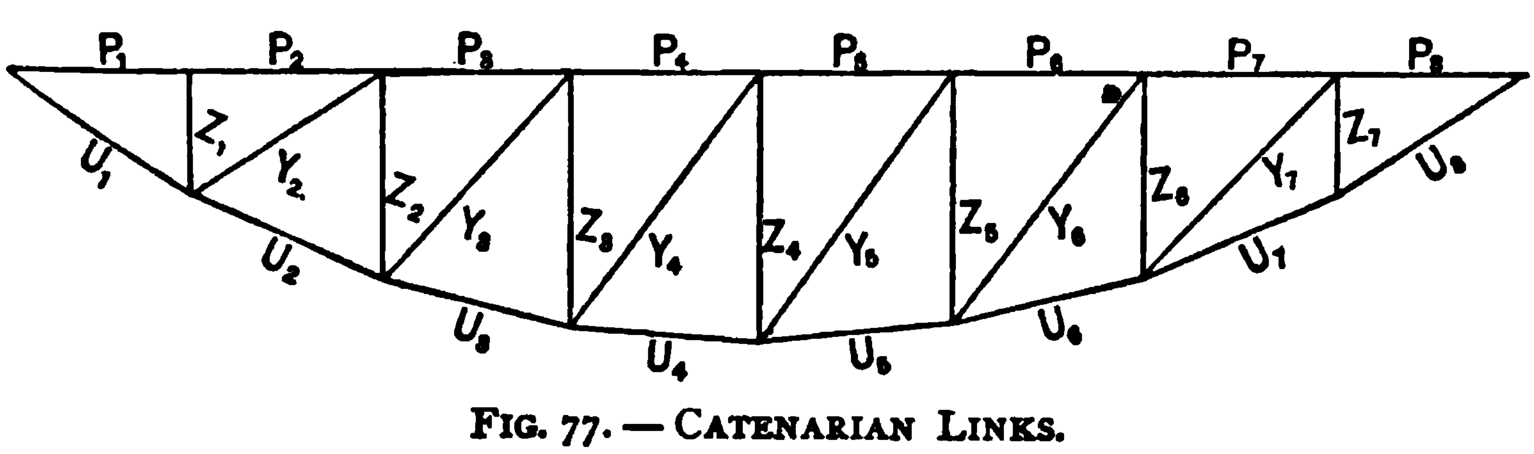


FIG. 77.—CATENARIAN LINKS.

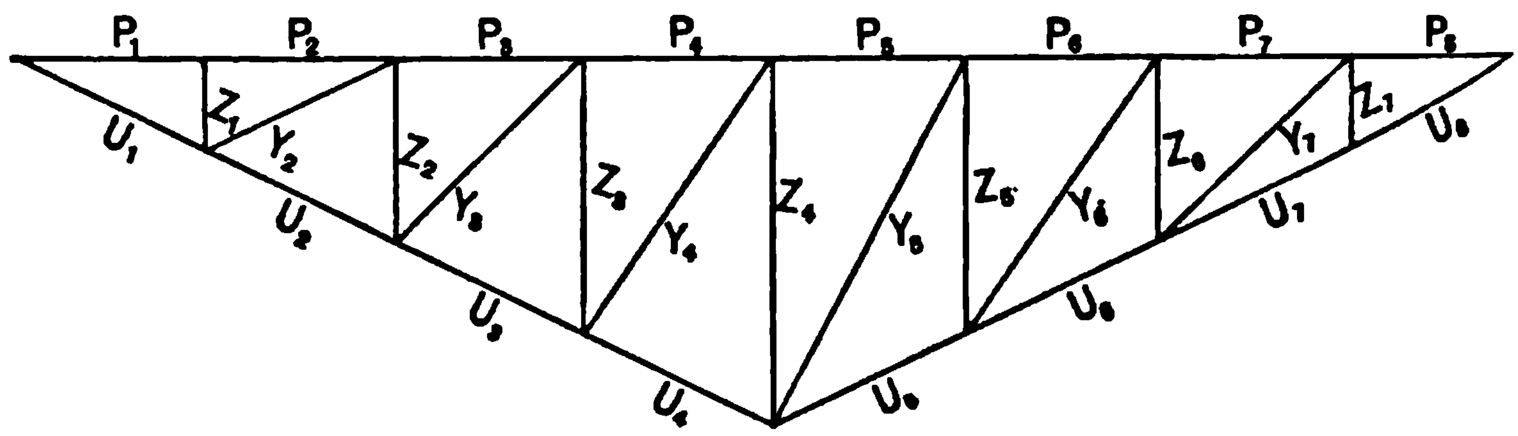


FIG. 78. — TRUSSED BEAM.

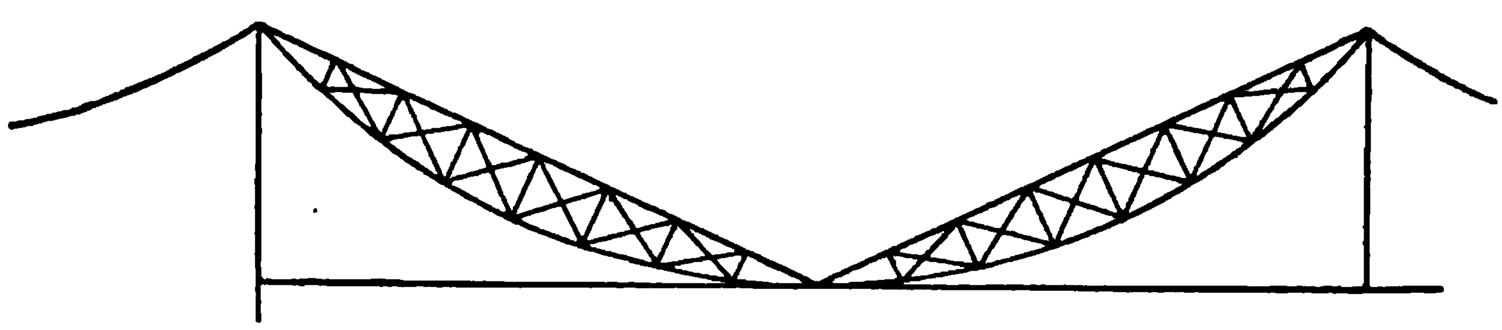


FIG. 79. — THE POINT SUSPENSION, STIFFENED CATENARY.

FORMULÆ FOR CLASS VIII. $\alpha = 0, \theta = 90°$.

Method of Moments.

$$H = M \div h.$$

$$\Delta H = \frac{M_{r+1}}{h_{r+1}} - \frac{M_r}{h_r}.$$

$$P = H_{r+1}.$$

$$U = -H_r \div \cos\beta.$$

$$Y = \Delta H \div \cos\phi.$$

$$Z = \Delta H \tan\phi.$$

When the vertical member has no diagonal attached at its top, then, of course, the strain upon the vertical is, for Class VIII., equal to the load applied at the upper apex.

CLASS IX. — BOTH CHORDS HORIZONTAL. $\alpha = 0$.

Verticals in compression. $\beta = 0$.

Diagonals in tension. $\theta = 90°$.

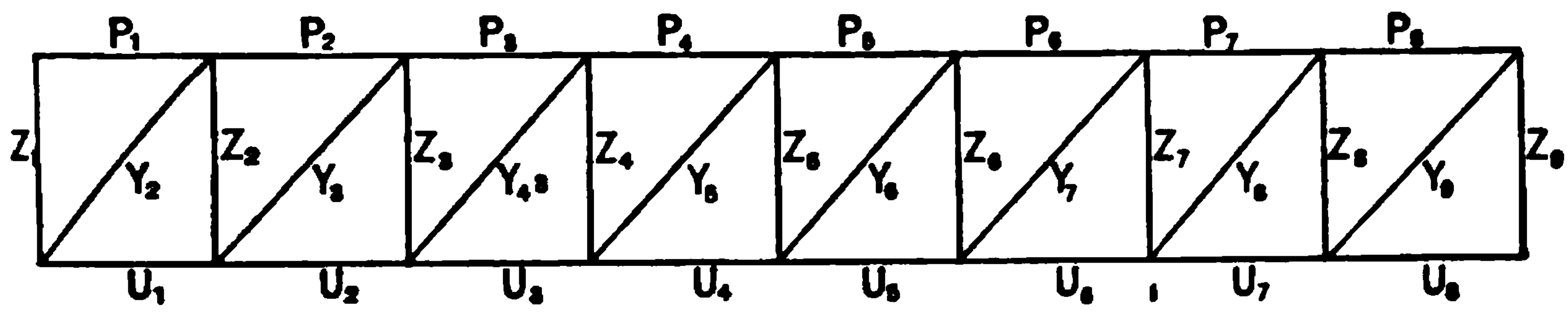


FIG. 80. — THE PRATT TRUSS.

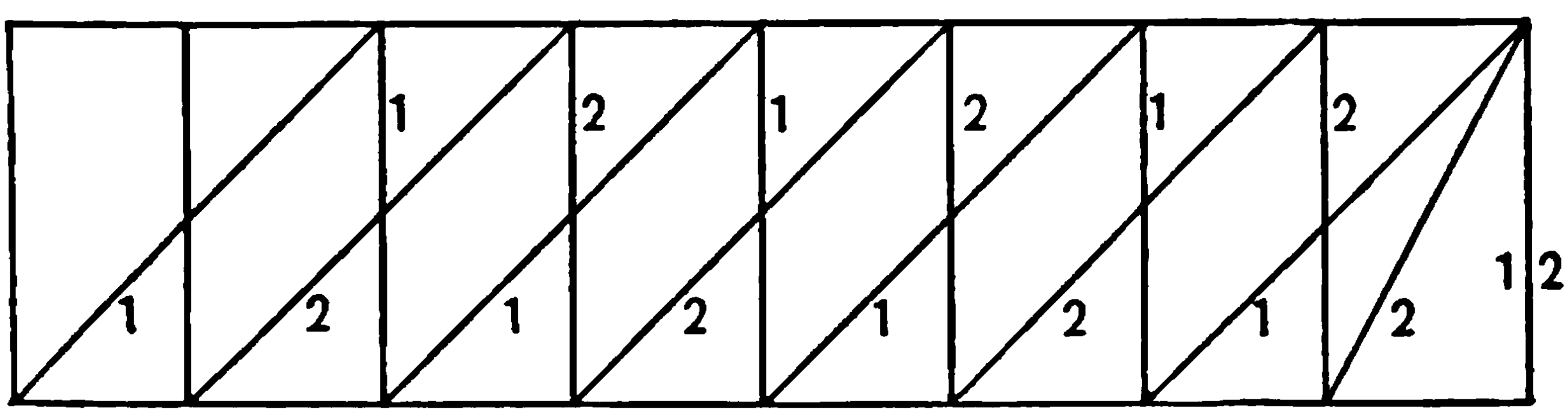


FIG. 80*a*. — THE LINVILLE, OR PRATT OF TWO SYSTEMS.

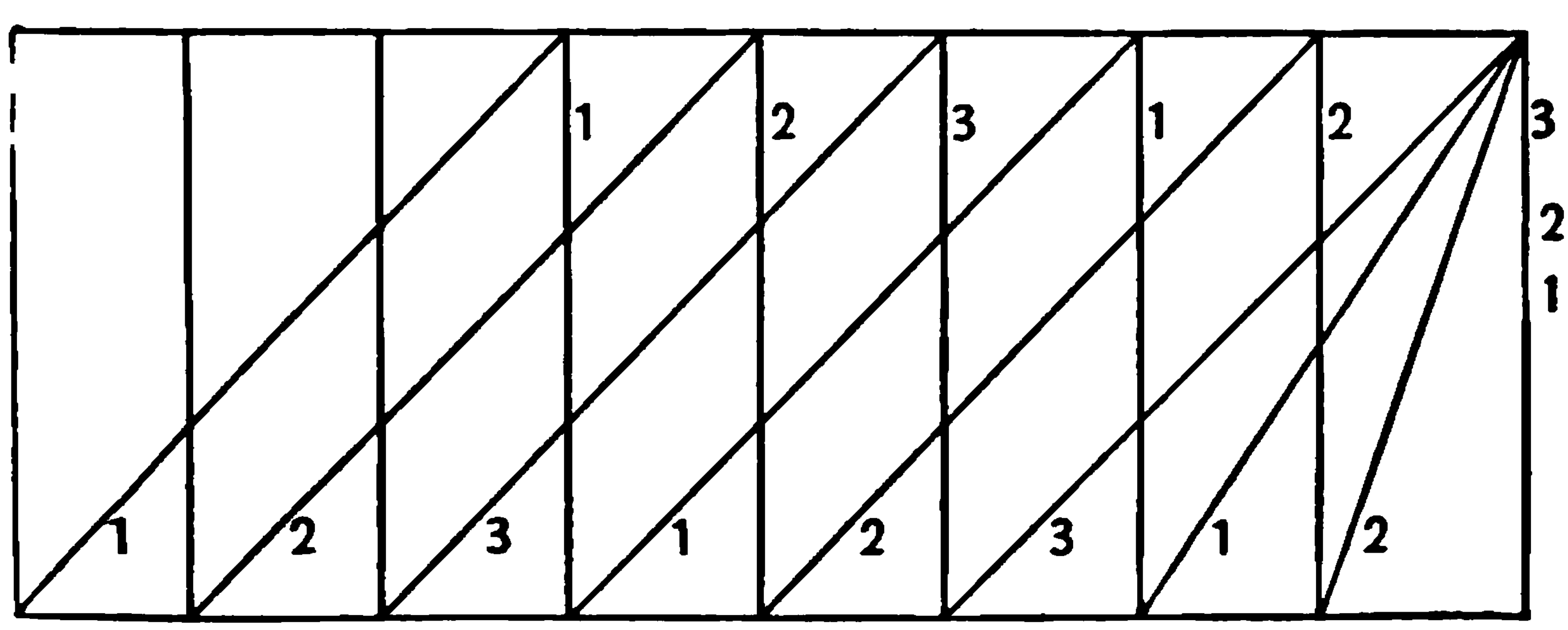
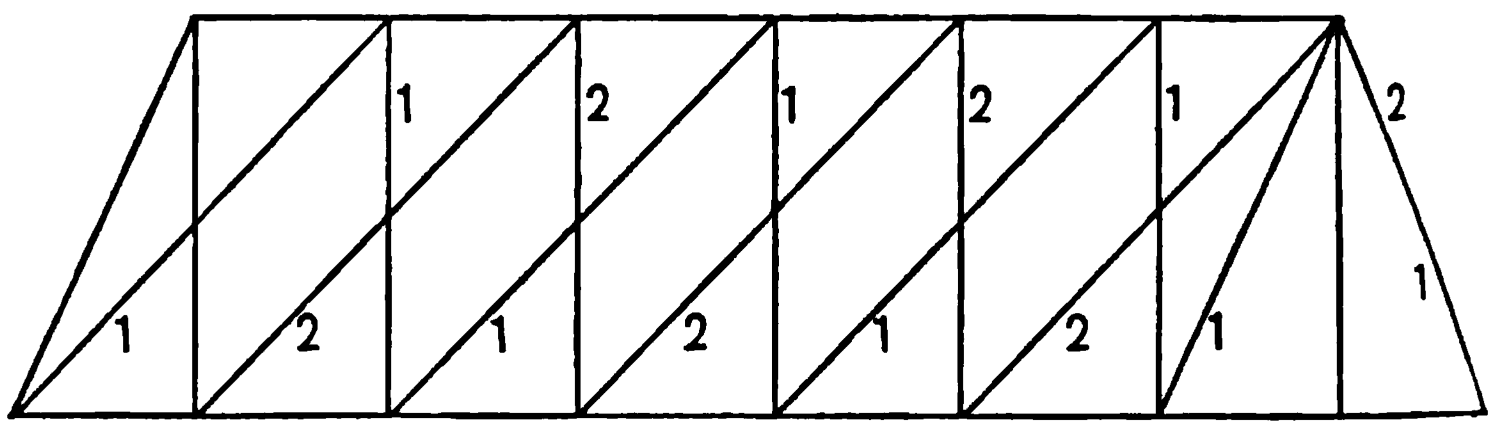


FIG. 80*b*. — PRATT TRUSS OF THREE SYSTEMS.



For end struts, $\tan\theta = 2\tan\phi$.

Fig. 80*c*. — Linville, with Inclined End-Posts.

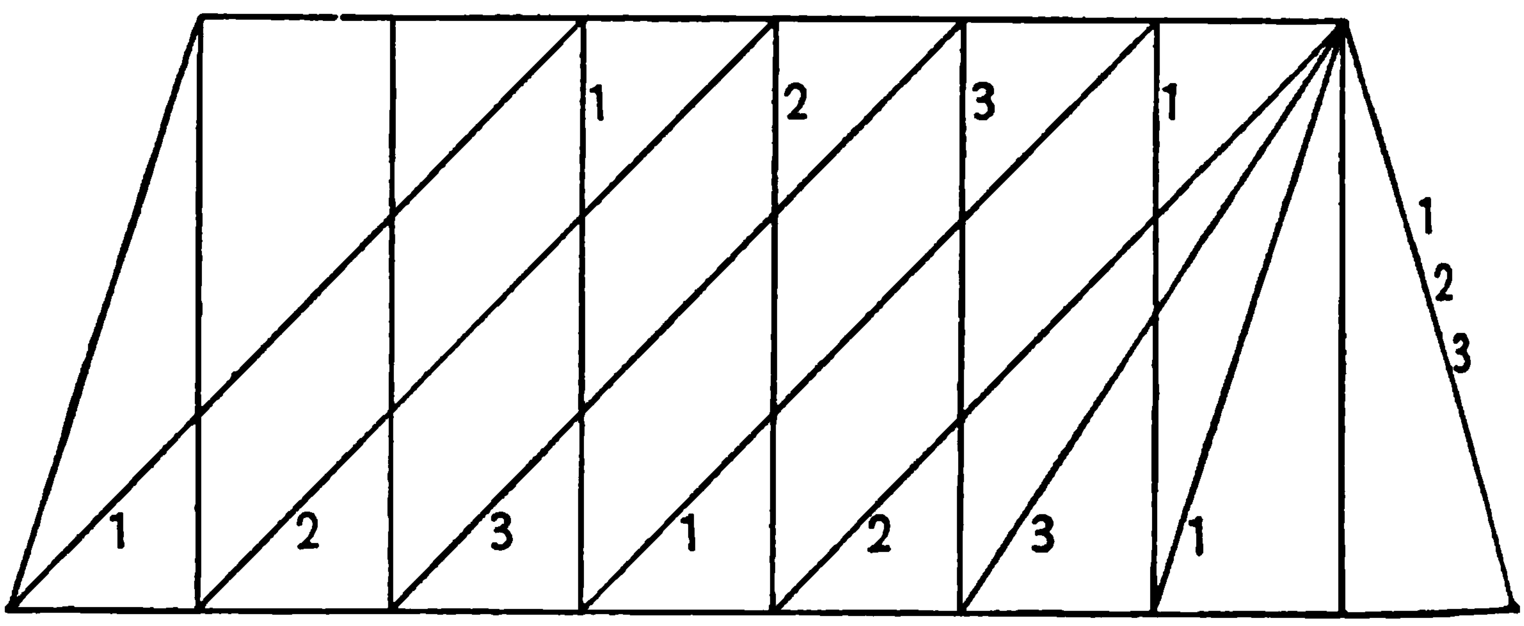


Fig. 80*d*. — Three Systems, Inclined End Posts.

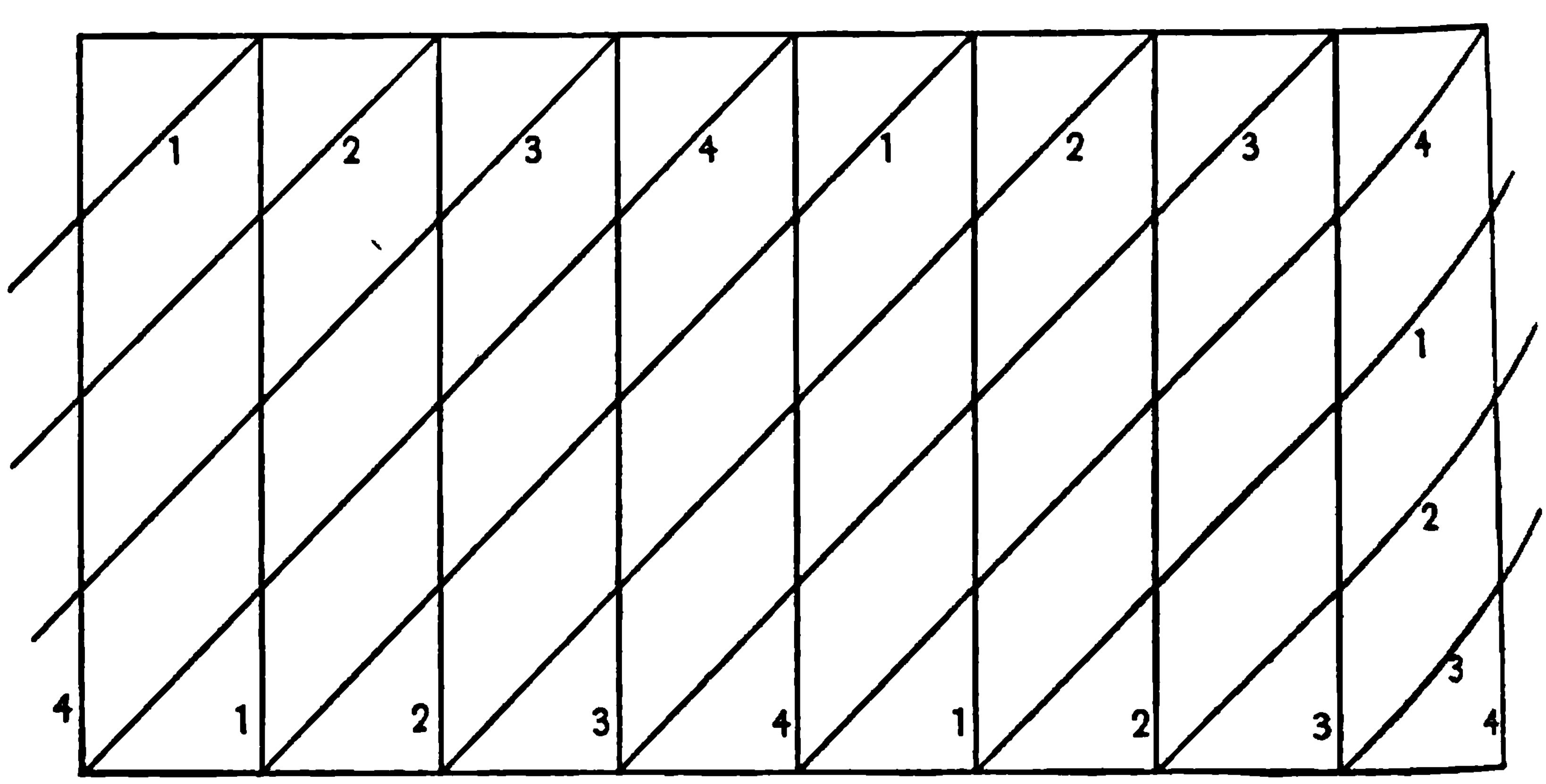


Fig. 80*e*. — Truss Systems of Niagara Bridge.

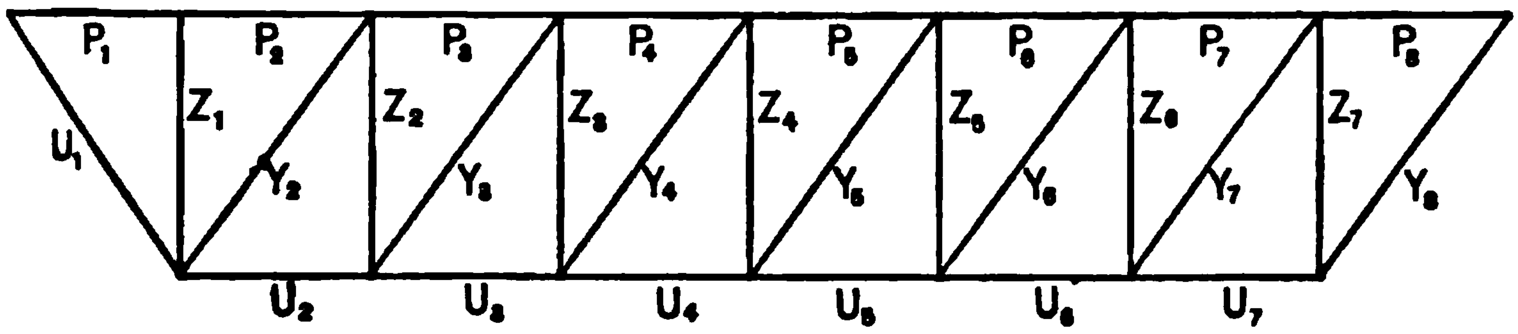


FIG. 80*f*. — PRATT TRUSS SUSPENDED.

FORMULÆ FOR CLASS IX. $\alpha = 0, \beta = 0, \theta = 90°$.

Method of	
Moments.	**Moments and Shearing-Forces.**
$H = M \div h.$	$H = (M_W + M_L) \div h.$
$\Delta H = \Delta M \div h.$	$S = S_W + S_L.$
$P = H_r.$	$P = H_{W+L}.$
$U = -H_{r+1}.$	$U = -H_{+1\,W+L}.$
$Y = \Delta H \div \cos\phi.$	$Y = -S \div \sin\phi.$
$Z = \Delta H \tan\phi.$	$Z = S_{+1}$, load applied at top.
	$Z = S$, load applied at bottom.

CLASS X. — BOTTOM CHORD HORIZONTAL. $\beta = 0$.

Verticals in tension. $\phi = 90°$.
Diagonals in compression.

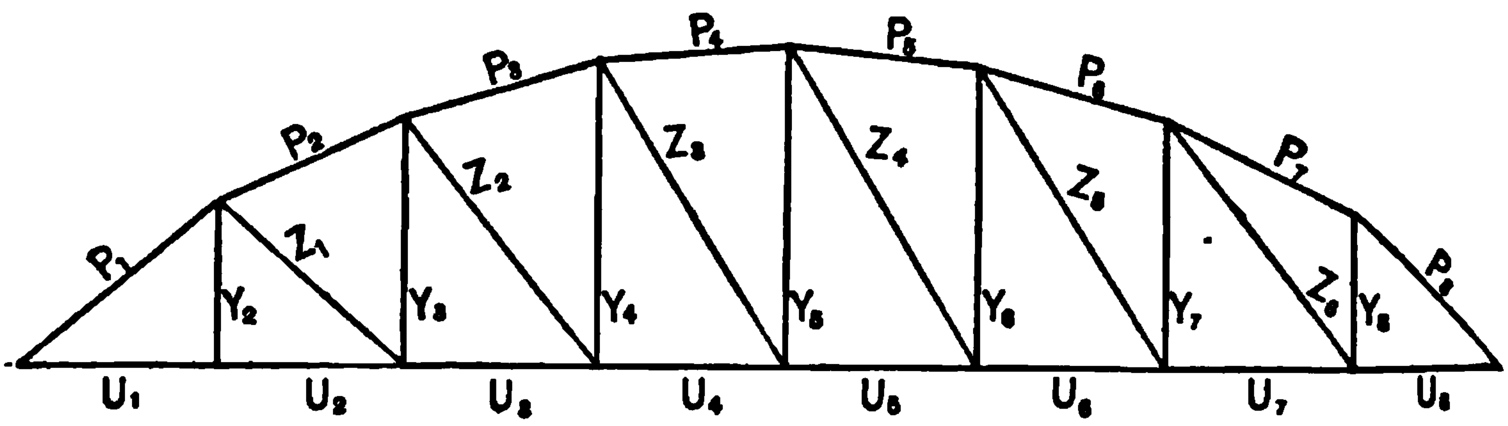


FIG. 81. — THE POLYGONAL BOWSTRING.

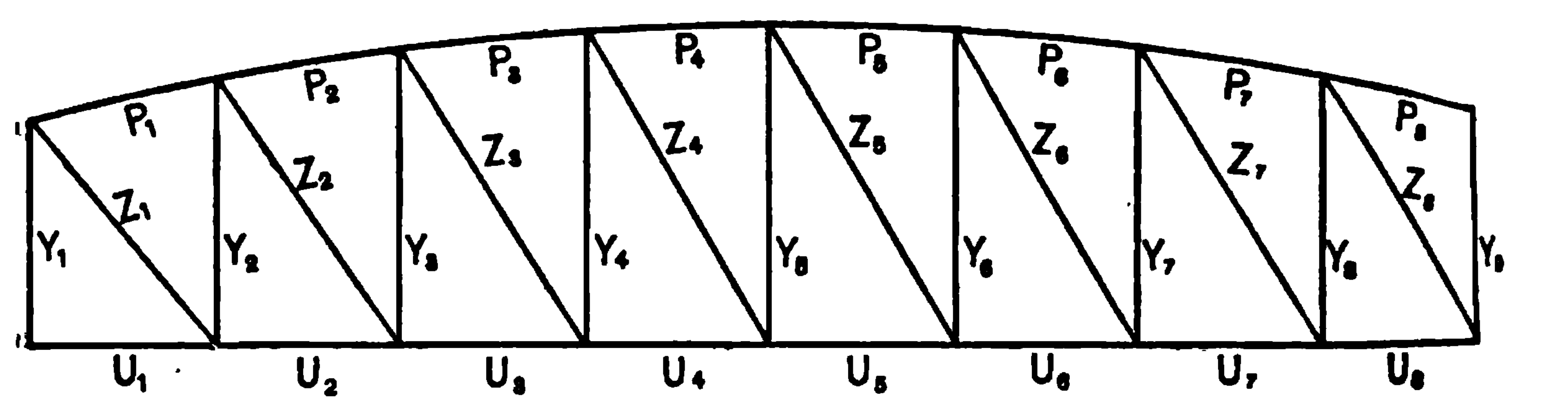


FIG. 82. — THE HOWE TRUSS, WITH CURVED TOP.

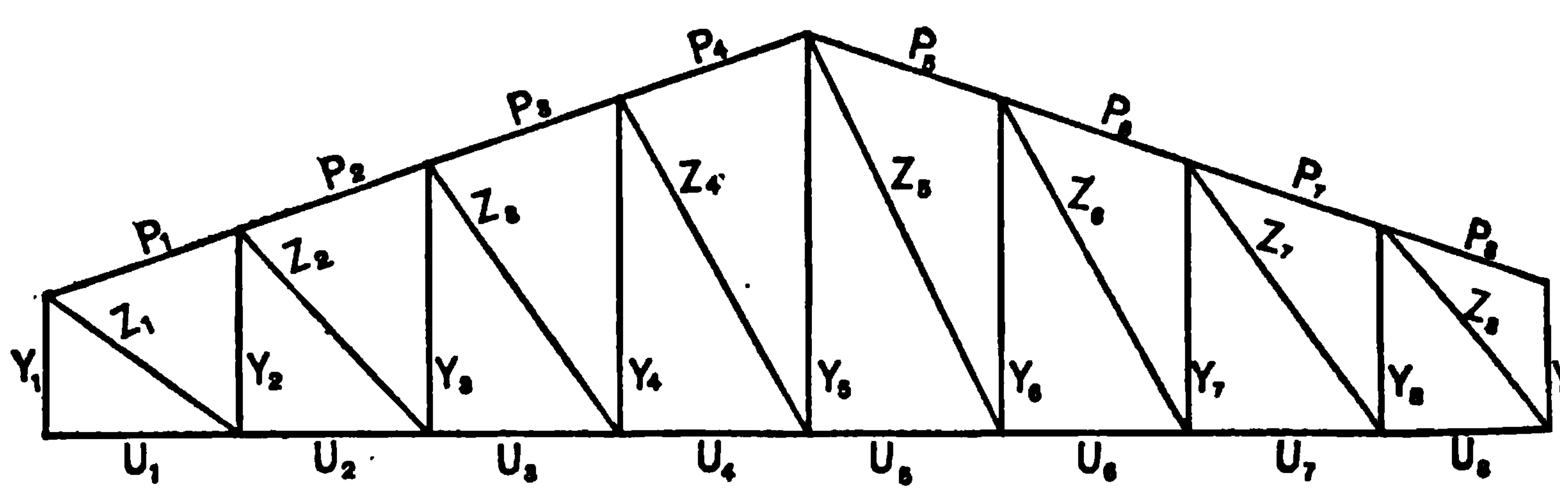


FIG. 83. — HOWE TRUSS, INCLINED TOP CHORD.

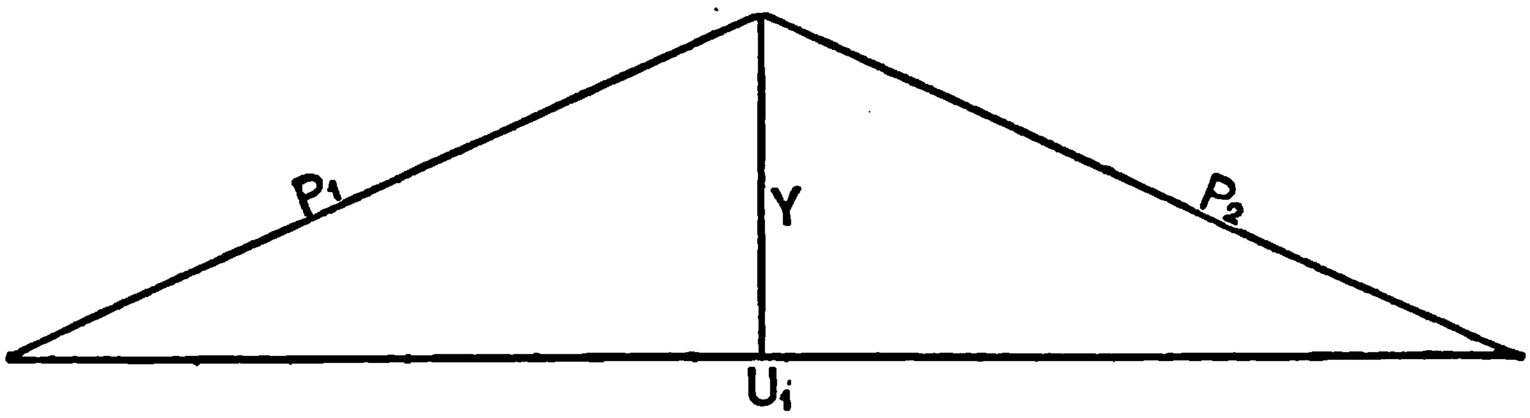


FIG. 84. — RAFTERS, WITH VERTICAL TIE.

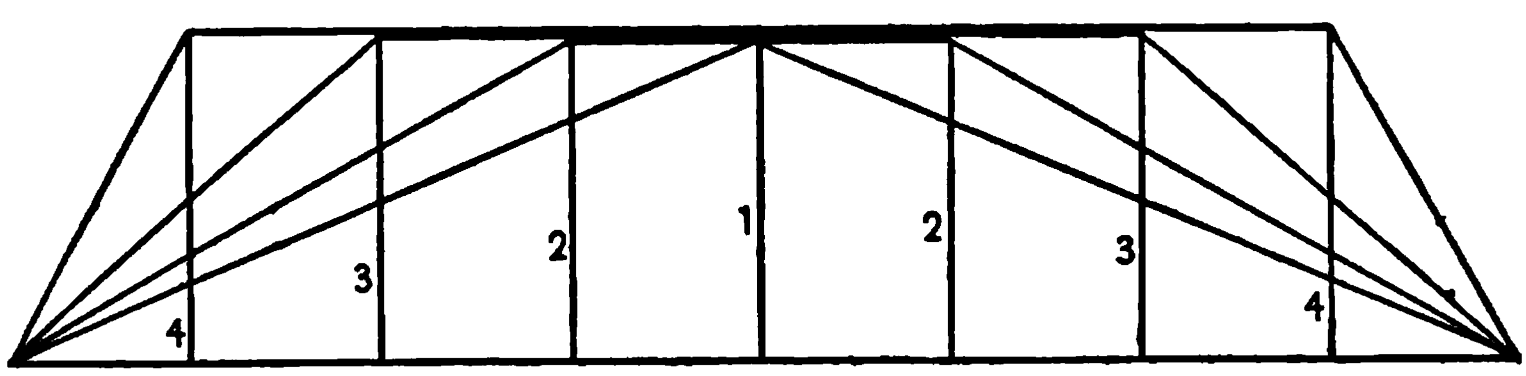


FIG. 84*a*. — SYSTEMS OF THE SCHAFFHÄUSEN TRUSS.

FORMULÆ FOR CLASS X. $\beta = 0, \phi = 90°$.

Method of Moments.

$$H = M \div h.$$

$$\Delta H = \frac{M_{r+1}}{h_{r+1}} - \frac{M_r}{h_r}.$$

$$P = H_{r+1} \div \cos\alpha.$$

$$U = -H_r.$$

$$Y = \Delta H \tan\theta.$$

$$Z = \Delta H \div \cos\theta.$$

CLASS XI. — TOP CHORD HORIZONTAL. $\alpha = 0$. STRUTS INCLINED. TIES VERTICAL. $\phi = 90°$.

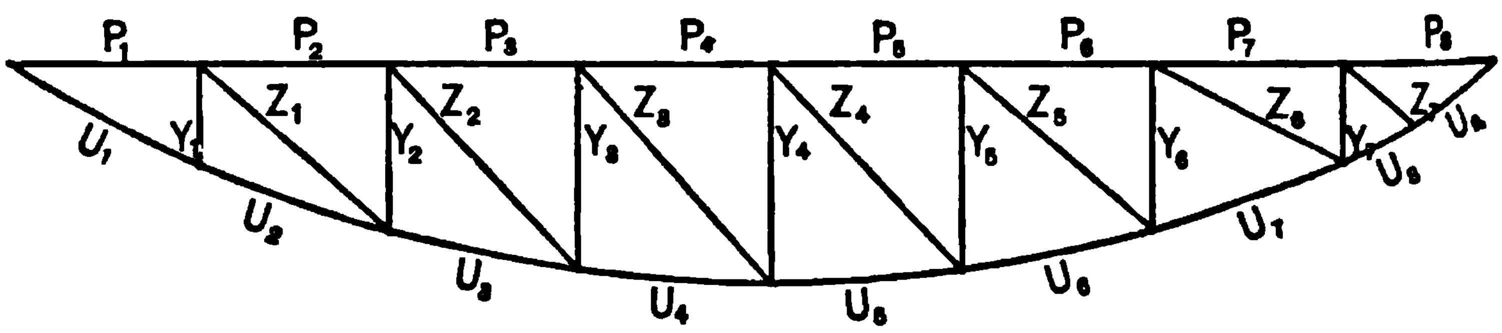


FIG. 85. — SUSPENDED BOW.

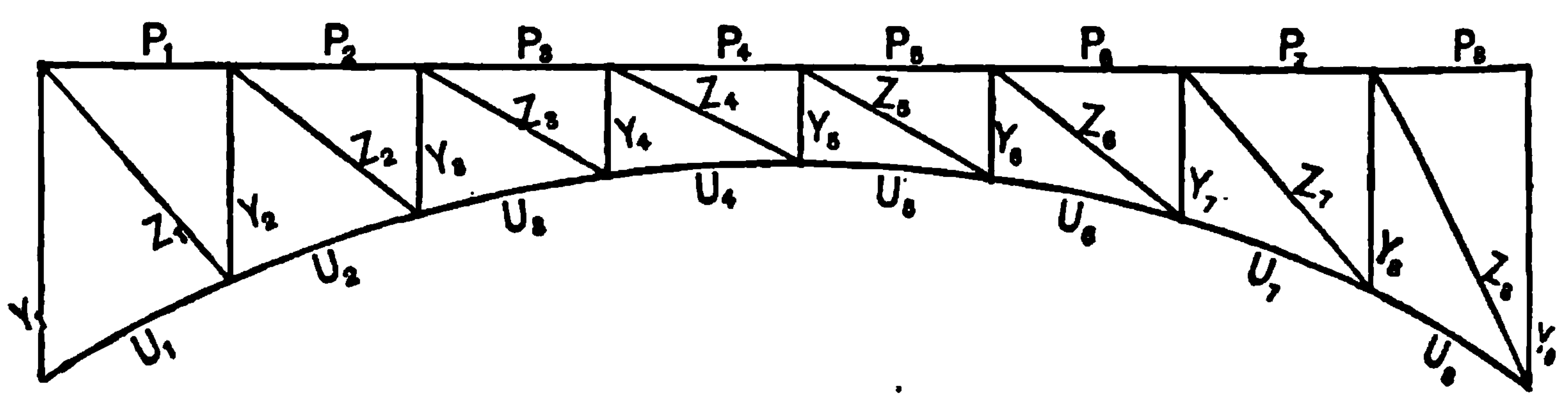


FIG. 86. — FILLED SPANDRELS.

FORMULÆ FOR CLASS XI. $\alpha = 0, \phi = 90°$.

Method of Moments.

$$H = M \div h.$$

$$\Delta H = \frac{M_{r+1}}{h_{r+1}} - \frac{M_r}{h_r}.$$

$$P = H_{r+1}.$$

$$U = -H_r \div \cos\beta_r.$$

$$Y = \Delta H \tan\theta.$$

$$Z = \Delta H \div \cos\theta.$$

CLASS XII. — BOTH CHORDS HORIZONTAL. $\alpha = 0, \beta = 0.$
STRUTS INCLINED. TIES VERTICAL. $\phi = 90°.$

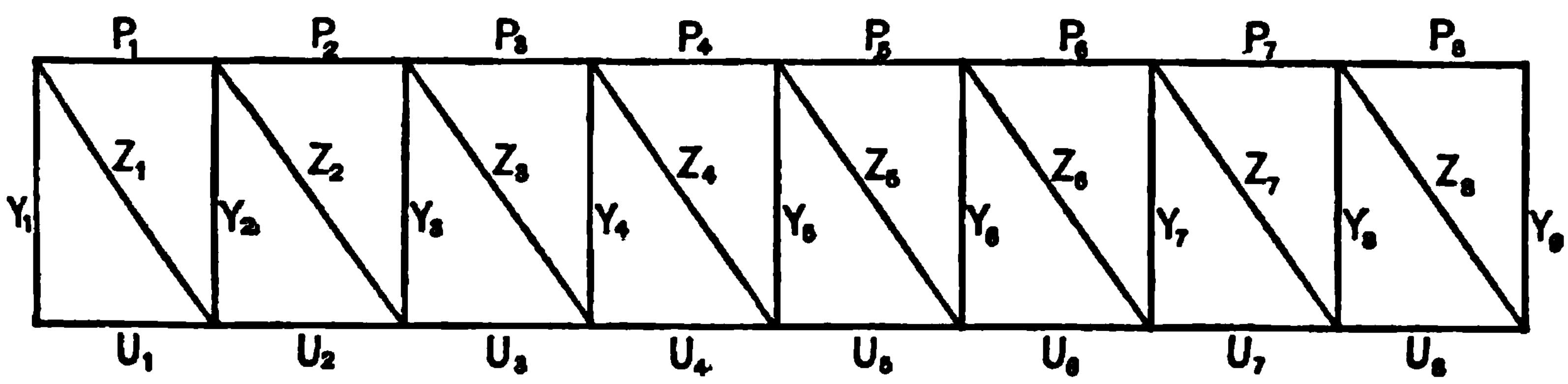


FIG. 87. — THE HOWE TRUSS.

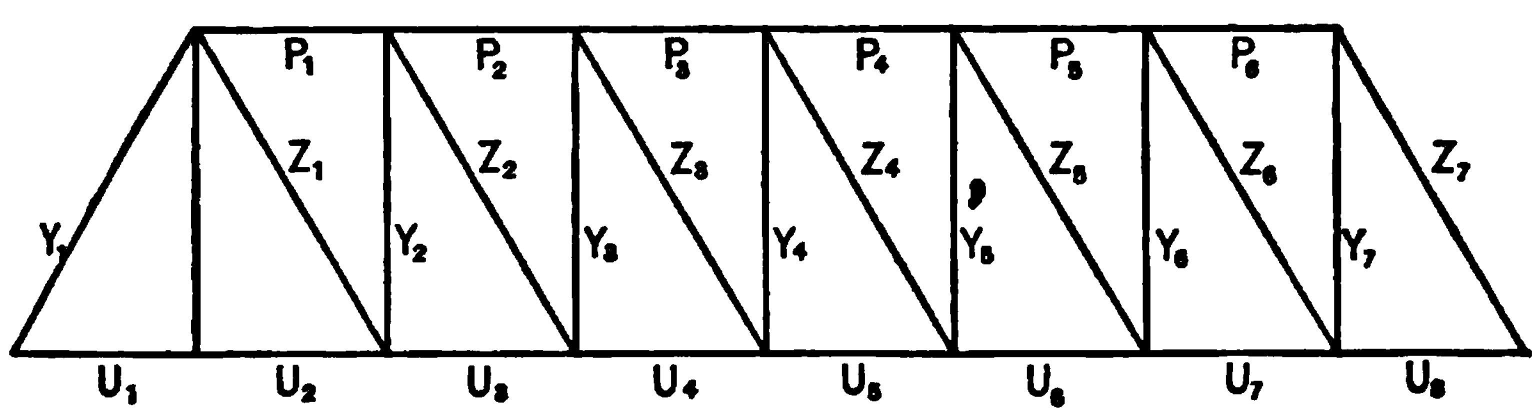


FIG. 88. — HOWE TRUSS, INCLINED END POSTS.

FORMULÆ FOR CLASS XII. $\alpha = 0, \beta = 0, \phi = 90°.$

Method of	
Moments.	Moments and Shearing-Forces.
$H = M \div h.$	$H = (M_W + M_L) \div h.$
$\Delta H = \Delta M \div h.$	$S = S_W + S_L.$
$P = H_{r+1}.$	$P = H_{+1W+L}.$
$U = -H_r.$	$U = -H_{W+L}.$
$Y = \Delta H \tan\theta.$	$Y = -S_{+1}.$
$Z = \Delta H \div \cos\theta.$	$Z = S \div \sin\theta.$

CHAPTER V.

MOMENTS OF RESISTANCE OF THE INTERNAL FORCES OF A BEAM OR GIRDER HAVING A CONTINUOUS WEB.

SECTION I.

General Formula found and applied to Particular Cross-Sections of Beams with Continuous Web.

50. The mode of estimating the moment of resistance offered by the cohesion of the particles of the material composing a beam, we now proceed to illustrate.

Let $ABCD$, Fig 89, be the vertical longitudinal central section of a beam of any cross-section whatever, under the influence of given applied forces or pressures.

It is required to find the moment of resistance offered by the material of the

FIG. 89.

beam at any normal section, OTQ.

Let NS be the intersection of the neutral surface of the beam with the plane of the paper. The neutral surface of a beam coincides with the position of that longitudinal lamina which, for a given strain, is neither compressed nor elongated.

All fibres not in the neutral surface are assumed to be increased or diminished in length by a quantity in direct proportion to their distance from the neutral surface, and also in direct proportion to the intensity of the force acting on the fibres.

Let f = the force acting on a unit of area of any given normal section, at right angles to the section, and at the unit's distance from the neutral surface, either above or below.

dz = an element of the thickness of the beam.

dy = an element of its depth.

y = the distance of the fibre whose area is $dz\,dy$ from the neutral surface.

Then the pressure upon the area $dz\,dy$ is

$$fy\,dy\,dz,$$

and the elementary moment due this stress is

$$dM = fy^2\,dy\,dz;$$

whence the total moment for the cross-section is

$$R = M = f\iint y^2\,dy\,dz, \tag{159}$$

which expression is to be integrated between limits depending on the form of the given cross-section whose centre of gravity may be taken as the origin of co-ordinates.

51. We will now apply (159) to the determination of the moment of resistance, R, for various cross-sections occurring in practice.

Let the beam have a rectangular cross-section of the breadth b, and the height h, as in Fig. 90.

h

FIG. 90.

Then equation (159) becomes

$$R = f\int_{-\frac{1}{2}b}^{+\frac{1}{2}b}\int_{-\frac{1}{2}h}^{+\frac{1}{2}h} y^2\,dy\,dx = fb\int_{-\frac{1}{2}h}^{+\frac{1}{2}h} y^2\,dy,$$

$$\therefore\quad R = \tfrac{1}{12}fbh^3 = \tfrac{1}{6}Bbh^2, \qquad (160)$$

where f is the unit strain at the unit's distance from the neutral surface, and $B = \frac{1}{2}hf =$ the unit strain at the distance $\frac{1}{2}h$ from the neutral surface, or at the upper and lower surfaces of the beam, since the neutral surface is here assumed to be in its centre. The quantity B is the unit strain which, at the instant of rupture, would be developed at the upper and lower surfaces of a beam having its neutral surface midway between those outer surfaces. B is called the *modulus of rupture*, or the ultimate unit resistance of the material to cross-breaking.

A table giving values of B is inserted in article 60.

52. Beam of Hollow [illegible] Section or [illegible] of Equal Flanges, Fig. 91.

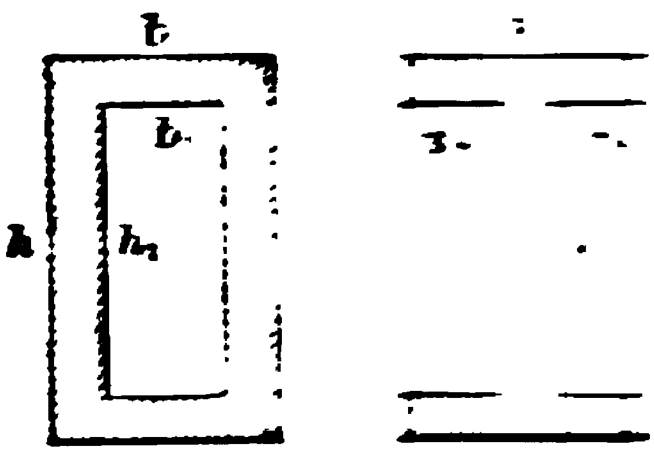


FIG. 91.

Let h = height of beam.
h_1 = height of cavity or web.
b = breadth of beam or flange.
b_1 = breadth of cavity.

Then, as in article 51, we shall have for the whole area $b \times h$,

$$R_1 = \tfrac{1}{12} fbh [illegible]$$

and for the area of the cavity [illegible],

$$R_2 = \tfrac{1}{12} f [illegible]$$

Whence, for net area of cross-section,

$$R = \tfrac{1}{12} f(bh^3 - [illegible] \qquad (161)$$

where $B = \frac{1}{2}hf$ = unit strain at the upper and lower surfaces of the beam.

If the beam is square and hollow [illegible] $h = b$, and $h_1 = b_1$, we have, from equation 161,

$$R = [illegible] \qquad (162)$$

53. Beam composed of Two Vertical Plates and Two Horizontal Channels.

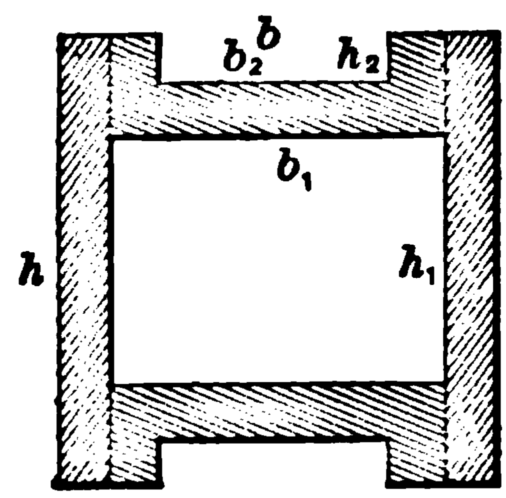


FIG. 92.

Let the two plates and the two channels, Fig. 92, have equal cross-sections.

b = entire breadth of beam.
h = entire height of beam.
b_1 = width of channel.
b_2 = width of its web.
h_1 = distance between channels.
h_2 = depth of channel cavity.

The neutral axis (that is, the line of intersection of the neutral surface with the normal section) is here central.

Whence, as in article 51, we have, for the area $(b - b_2) \times h$,

$$R_1 = \tfrac{1}{12} f (b - b_2) h^3;$$

for the area $b_2 \times (h - 2h_2)$,

$$R_2 = \tfrac{1}{12} f b_2 (h - 2h_2)^3;$$

for the area $b_1 \times h_1$,

$$R_3 = \tfrac{1}{12} f b_1 h_1^3.$$

Whence the total moment of resistance,

$$R = \frac{B}{6h}[(b - b_2)h^3 + b_2(h - 2h_2)^3 - b_1h_1^3], \quad (163)$$

where $B = \frac{1}{2}hf =$ unit strain in extreme top and bottom fibres.

54. **Beam composed of Two Vertical I-Beams and Two Equal Horizontal Plates.**

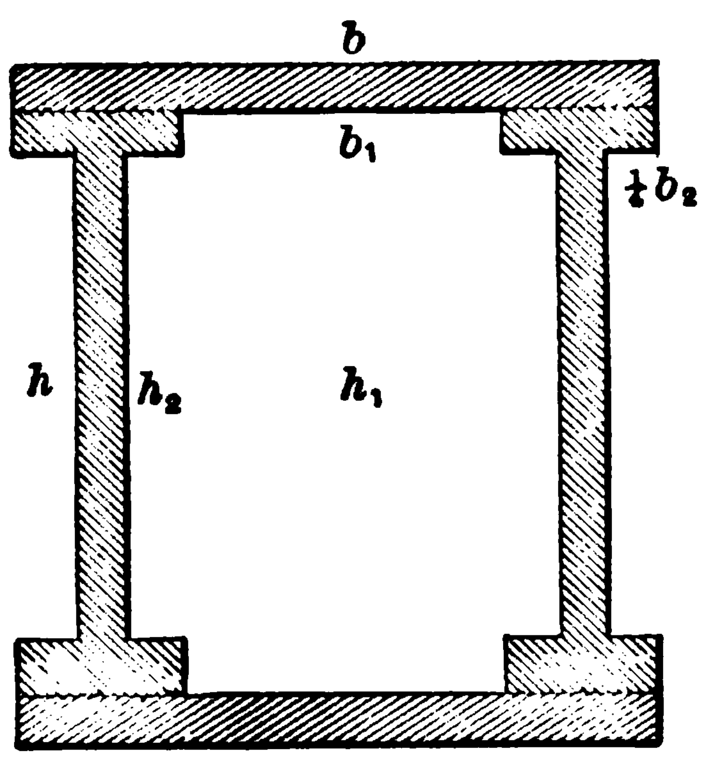


FIG. 93.

In Fig. 93, let h = height of beam.
h_1 = height of I-beams.
h_2 = height of their webs.
b = width of plates.
b_1 = width between beams.
b_2 = width of cavities of beams.

Then, proceeding as in the last article, we find

$$R = \frac{B}{6h}(bh^3 - b_1h_1^3 - b_2h_2^3). \quad (164)$$

55. Beam composed of Two Vertical Channels and Two Horizontal Plates.

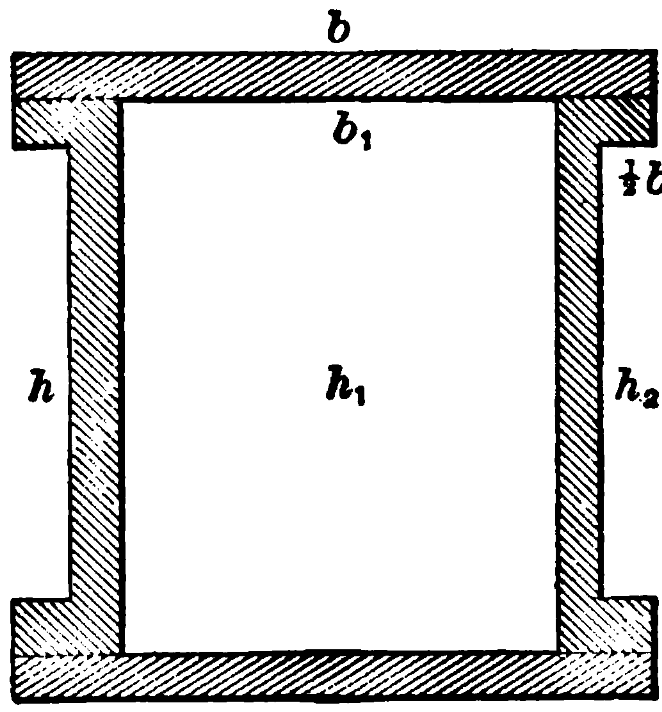


FIG. 94.

In Fig. 94, let h = height of beam.
h_1 = height of channels.
h_2 = height of their webs.
b = width of plates.
b_1 = width between channels.
b_2 = width of cavities.

Then we have

$$R = \frac{B}{6h}(bh^3 - b_1h_1^3 - b_2h_2^3). \qquad (165)$$

56. Beam with but One Flange.

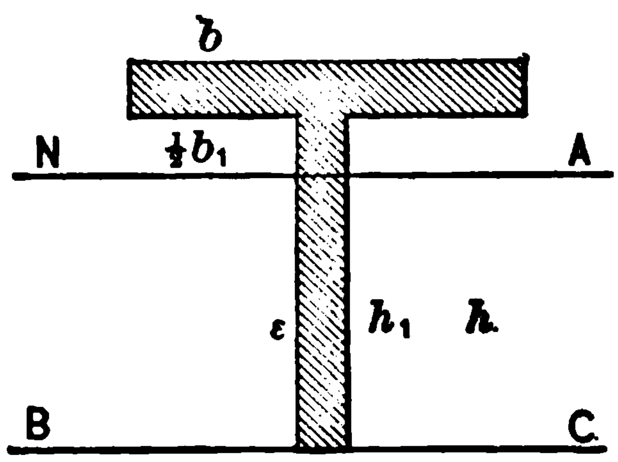


FIG. 95.

Let the cross-section, Fig. 95, have the form of the letter T.

b = width of flange.
h = whole height of beam.
$b - b_1$ = thickness of web.
h_1 = height of web.
s = distance of neutral axis NA from any line, BC, parallel to NA, in the plane of the given cross-section.

1st, To find s, and determine the position of the neutral axis.

Take the moment of the area of the section about BC as an axis, and divide this moment by the area. The quotient will be the value of s.

Moment of flange about BC

$$= b(h - h_1) \times \tfrac{1}{2}(h + h_1) = \tfrac{1}{2}b(h^2 - h_1^2).$$

Moment of web about $BC = h_1(b - b_1) \times \tfrac{1}{2}h_1 = \tfrac{1}{2}h_1^2(b - b_1)$.

Total moment of area about BC is

$$\tfrac{1}{2}b(h^2 - h_1^2) + \tfrac{1}{2}h_1^2(b - b_1) = \tfrac{1}{2}(bh^2 - b_1h_1^2).$$

Total area $= bh - b_1h_1$; therefore

$$s = \frac{bh^2 - b_1h_1^2}{2(bh - b_1h_1)}.$$

2d, By means of (159) we find,—
For area $b \times (h - s)$,

$$R_1 = fb\int_0^{h-s} y^2dy = \tfrac{1}{3}fb(h - s)^3;$$

for area $(b - b_1) \times s$,

$$R_2 = f(b - b_1)\int_0^{s} y^2dy = \tfrac{1}{3}f(b - b_1)s^3;$$

for area $b_1 \times (h_1 - \varepsilon)$,

$$R_3 = fb_1 \int_0^{h_1-\varepsilon} y^2 dy = \tfrac{1}{3} fb_1(h_1 - \varepsilon)^3.$$

Whence the moment of resistance due to the net cross-section is

$$R = \frac{B}{3\varepsilon}[b(h - \varepsilon)^3 + (b - b_1)\varepsilon^3 - b_1(h_1 - \varepsilon)^3], \quad (166)$$

where $B = \varepsilon f =$ unit strain at the extreme edge of the beam.

In a similar manner may all other beams be treated whose cross-sections are composed of rectangles having two sides parallel to the neutral axis.

57. Solid or Hollow Beam of Square Cross-Section and Diagonal Vertical.

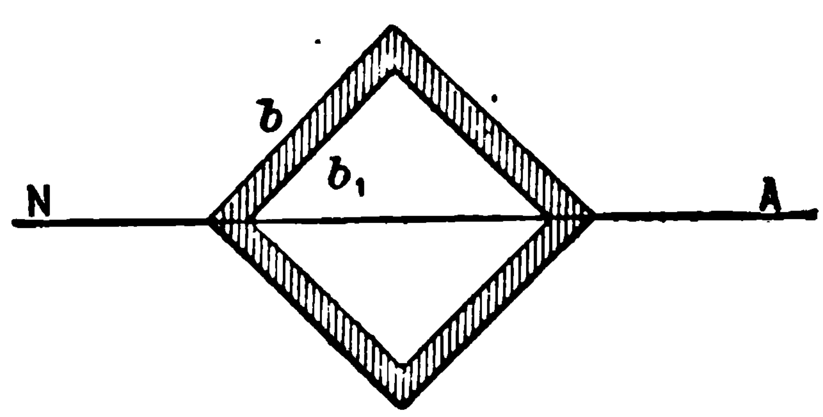


FIG. 96.

Let b, Fig. 96, be a side of the beam's cross-section, and b_1 a side of the square concavity whose centre coincides with the beam's centre. Then the diagonals are $b\sqrt{2}$ and $b_1\sqrt{2}$; and (159) becomes, since $z = \frac{1}{2}b\sqrt{2} - y$,—

For solid beam,

$$R = 2f\int_0^{\frac{1}{2}b\sqrt{2}} 2(\tfrac{1}{2}b\sqrt{2} - y)y^2 dy,$$

$$\therefore \; R = \tfrac{1}{12} fb^4 = \frac{\sqrt{2}}{12}Bb^3, \quad (167)$$

where $B = \frac{1}{2}fb\sqrt{2} =$ intensity of stress at extreme upper or lower edge of the beam whose diagonal is vertical.

If in (160) we make $h = b$, then

$$R = \tfrac{1}{12}fb^4 = \tfrac{1}{6}Bb^3,$$

where $B = \frac{1}{2}fb$ = intensity of stress at upper or lower surface of a square beam whose side is vertical.

Hence, although the identity of the middle members of equations (160) and (167) shows that the total moment of resistance, R, is the same for a given solid square beam whether its side or its diagonal be vertical, yet the extreme fibres for these two positions of the beam are strained in the ratio of their distances from the neutral axes; that is, in the ratio of 1 to $\sqrt{2}$.

If, therefore, B expresses the ultimate strength of the material, when in equation (167) it is equal to $\frac{1}{2}fb\sqrt{2}$, we may evidently give to B the same extreme value in equation (160), and thus make the beam $\sqrt{2}$ times stronger when its side is vertical than when its diagonal is vertical.

Again, for the vacant square whose side is b_1, since $z = \frac{1}{2}b_1\sqrt{2} - y$, we have

$$R = 2f\int_0^{\frac{1}{2}b_1\sqrt{2}} 2(\tfrac{1}{2}b_1\sqrt{2} - y)y^2dy,$$

$$\therefore \quad R = \tfrac{1}{12}fb_1^4.$$

And therefore, for a hollow square beam with diagonal vertical, the moment of resistance is

$$R = \frac{\sqrt{2}}{12}B\frac{b^4 - b_1^4}{b}, \qquad (168)$$

where $B = \frac{1}{2}fb\sqrt{2}$ = unit strain at extreme edge of beam when the diagonal is vertical.

58. Solid or Hollow Beam of Circular Cross-Section.

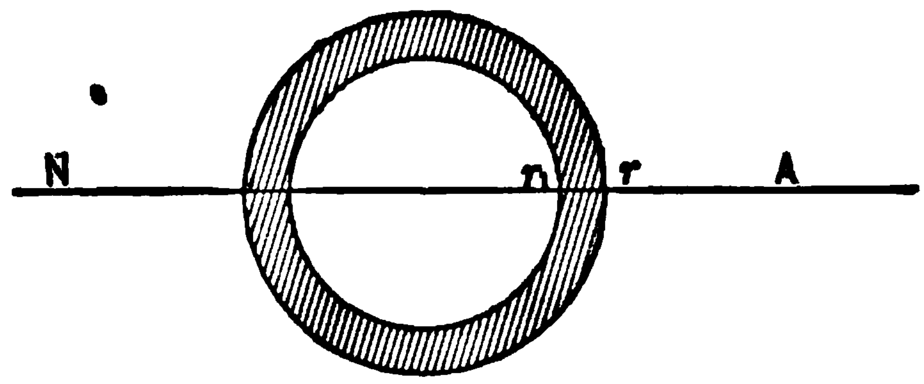


FIG. 97.

Let r = radius of the outer circle, Fig. 97, and
r_1 = radius of the inner circle.
The equation of the outer circle is

$$r^2 = y^2 + z^2,$$

$$\therefore \quad z = (r^2 - y^2)^{\frac{1}{2}};$$

and equation (159) becomes

$$R = 2f\int_{-r}^{+r}(r^2 - y^2)^{\frac{1}{2}}y^2dy,$$

$$\therefore \quad R = f\pi\frac{r^4}{4} = 0.7854Br^3, \qquad (169)$$

which is the moment of resistance for a solid beam of circular cross-section with the radius r, and where $B = fr$ = the unit strain on the highest and lowest fibres.

If the beam is hollow, the inner and outer circles being concentric, we manifestly have

$$R = 0.7854B\frac{r^4 - r_1^4}{r}, \qquad (170)$$

where B = unit strain on highest and lowest fibres.

59. Beam of Elliptical Cross-Section, Solid or Hollow; Longer Axis vertical; Axes of Outer and Inner Ellipses coincident.

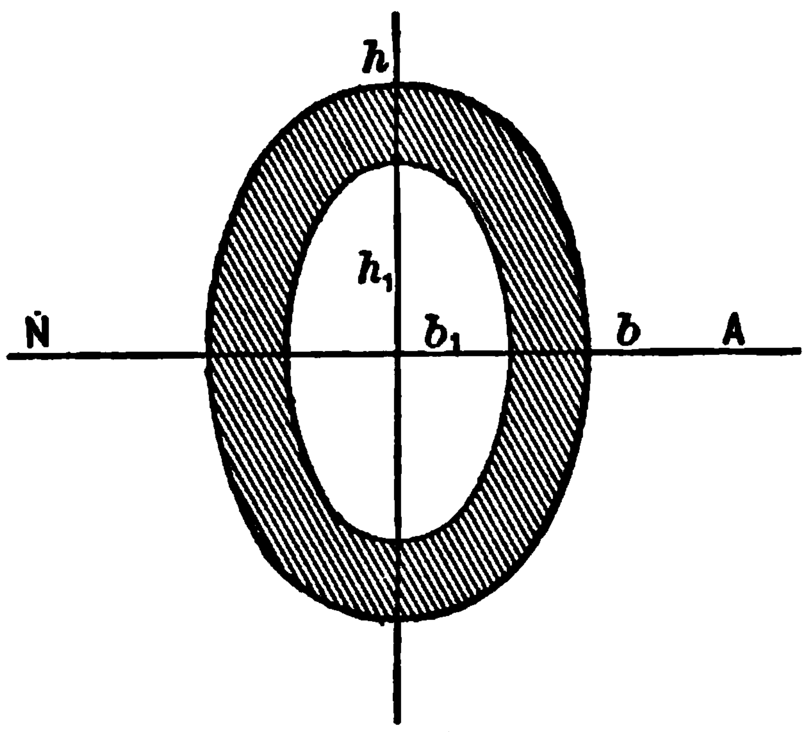


FIG. 98.

Let h, Fig. 98, be the length of the semi-transverse axis of the outer ellipse, and h_1 that of the inner ellipse; b = length of semi-conjugate axis of outer ellipse, and b_1 = the same for the inner ellipse. The equation of the outer ellipse is

$$\frac{z^2}{b^2} + \frac{y^2}{h^2} = 1,$$

$$\therefore \quad z = \frac{b}{h}(h^2 - y^2)^{\frac{1}{2}}.$$

Whence (159) becomes

$$R = \frac{2bf}{h}\int_{-h}^{+h}(h^2 - y^2)^{\frac{1}{2}}y^2dy,$$

$$\therefore \quad R = \frac{\pi}{4}fbh^3 = 0.7854Bbh^2, \qquad (171)$$

which is the moment of resistance for the solid elliptical beam, where $B = hf$ = unit strain on highest and lowest fibres.

Similarly, for the area of the inner ellipse,

$$R = \frac{\pi}{4} f b_1 h_1^3;$$

and therefore, for the hollow elliptical beam, the moment of resistance is

$$R = 0.7854 B \frac{bh^3 - b_1 h_1^3}{h}, \qquad (172)$$

where $B = hf$ = unit strain on highest and lowest fibres.

60. These illustrations may suffice for girders of continuous web.

We close this section with a table giving the limiting value of B, in pounds avoirdupois to the square inch, for the ordinary materials used in beams; that is, the values of B in this table are values which cannot be exceeded in the equations of this section, and represent the ultimate resistance of the material to cross-breaking.

It should, however, be borne in mind, that B may not represent the actual unit strain which the material is capable of resisting, either in tension or compression; but that it, in general, has some mean value between the ultimate resistance of material to crushing, and the ultimate resistance of the same material to tearing by direct pull.

Continuing the suppositions made in article 50, we may find the relation existing among these three ultimate unit strains in rectangular beams as follows: —

Let us take

h = depth of beam.
b = breadth of beam.
l = length of beam.
x = distance of the neutral surface from the *compressed* side of the beam.

C = ultimate resistance of the material to crushing by direct thrust, in pounds, per square inch.

T = ultimate resistance of the material to extension, in pounds, per square inch.

B = the unit strain which, at the instant of rupture, would be developed at the upper and lower surfaces of a beam having its neutral surface midway between these outer surfaces; that is, B = the modulus of rupture, also in pounds per square inch.

Then, using (159), we find for the compressed part of any cross-section, moment of internal forces,

$$R = \tfrac{1}{3}Cbx^2; \tag{173}$$

and for the extended part of the same cross-section,

$$R = \tfrac{1}{3}Tb(h - x)^2. \tag{174}$$

But, if the neutral axis bisected the given cross-section, we should have moment of internal forces on either side of this axis,

$$R = \tfrac{1}{3}Bb(\tfrac{1}{2}h)^2. \tag{175}$$

Now, since experimental researches show, that, for many materials used in construction, these three expressions are nearly equal to one another, we have approximately

$$\tfrac{1}{3}Cbx^2 = \tfrac{1}{3}Tb(h - x)^2 = \tfrac{1}{3}Bb(\tfrac{1}{2}h)^2; \tag{176}$$

whence

$$x = \tfrac{1}{2}h\sqrt{\frac{B}{C}} = h\frac{\sqrt{T} - \frac{1}{2}\sqrt{B}}{\sqrt{T}} = h\frac{\sqrt{T}}{\sqrt{C} + \sqrt{T}}, \tag{177}$$

$$C = \frac{B}{\left(2 - \sqrt{\frac{B}{T}}\right)^2}, \tag{178}$$

$$T = \frac{B}{\left(2 - \sqrt{\frac{B}{C}}\right)^2}, \tag{179}$$

$$B = \frac{4C}{\left(1 + \sqrt{\frac{C}{T}}\right)^2} = \frac{4T}{\left(1 + \sqrt{\frac{T}{C}}\right)^2}. \tag{180}$$

When, therefore, any two of the three quantities, C, T, and B, are given, the third may be found, and also the position of the neutral surface.

It is probable, that after the elastic limit of the material is passed, and rupture is about to take place, the expressions in (176) do not represent the actual moments, but are similar functions of C, T, and B, and are *proportional* to the *forces* then developed. For within the elastic limits the forces are

$$\tfrac{1}{2}C_1bx = \tfrac{1}{2}T_1b(h - x) = \tfrac{1}{2}B_1b(\tfrac{1}{2}h), \tag{181}$$

C_1, T_1, and B_1 being the limited unit strains at the surfaces. But when the strain on the extreme fibres passes the elastic limit, and the fibres expand or contract more rapidly than the strain increases, then an increment is given to all the previous inner strains proportional to their distances from the neutral surface, which is equivalent to introducing a factor of the form kx into the expressions (181), whereby, at the instant of rupture, they become

$$\tfrac{1}{2}Cbkx^2 = \tfrac{1}{2}Tbk(h - x)^2 = \tfrac{1}{2}Bbk(\tfrac{1}{2}h)^2, \tag{182}$$

which is identical with (176).

Experiments indicate, that in the case of cast-iron, owing to the superior hardness of the outer over the inner portions of the metal, an increment should be given to B in (180) equal to one-ninth of itself.

TABLE II.

MATERIAL.	Authority.	Ultimate Resistance, in Lbs., per Square Inch, to			Modulus of Elasticity, E.
		Compression, C.	Tension, T.	Cross-Breaking, B.	
Cast-Iron.					
Means of 9 irons	Stoney.	105945 H.	16720 H.	37695 H.F.	-
Means of 16 irons	"	86284 H.	15298 H.	-	12000000
Bars not > 1 inch wide	"	-	-	45696 C.	-
Bars 3 inches wide	"	-	-	30240 C.	-
Bars, small round	"	-	-	26880 C.	-
Circular tubes	"	-	-	38304 C.	-
Square tubes	"	-	-	45965 C.	-
Average	Rankine.	112000	16500	38250	17000000
Salisbury, No. 2	Thurston.	87429	20500	45760	11450754
Salisbury, No. 4	"	127323	34407	67035	15968254
Wrought-Iron.					
Bars, new	Stoney.	-	-	51341 C.	-
Bars, previously strained	"	-	-	74995 C.	-
Bars, new round	"	-	-	30240 C.	-
Boiler tubes, welded	"	-	-	70291 C.	-
Circular tubes, riveted	"	-	-	43814 C.	-
Rolled I-beams, about	"	-	-	61824	-
T-iron, flange up about	"	-	-	53760	-
T-iron, flange down about	"	-	-	51475	-
Average	"	40320	57555 K.	52567 C.	24000000
Bars and bolts	Rankine.	36000	60000	-	-
Bars and bolts	"	40000	70000	-	29000000
Plates	"	-	51000	-	-
Plates, double-riveted	"	-	35700	-	-
Plates, single-riveted	"	-	28600	-	-
Hoops, best-best	"	-	64000	-	-
Wire	"	-	70000	-	-
Wire	"	-	100000	-	25300000
Wire ropes	"	-	90000	-	15000000
Plate beams	"	-	-	42000	-
Mean of 113 tests	Lovett.	-	50915	-	-
Mean of 27 tests	"	-	-	-	27311111
Steel.					
Bessemer, hammered	Stoney.	225568 F.	83391 F.	128083 K.	31000000
Bessemer, rolled	"	-	71658 K.	115181 K.	-

TABLE II. — *Continued.*

MATERIAL.	Authority.	Ultimate Resistance, in Lbs., per Square Inch, to			Modulus of Elasticity, E.
		Compression, C.	Tension, T.	Cross-Breaking, B.	
Steel (continued).					
Crucible, hammered	Stoney.	-	85546 K.	147840 K.	-
Crucible, rolled	"	-	68589 K.	118272 K.	-
Cast, not hardened	"	198944 Wd.	-	-	-
Cast, low temper	"	354544 Wd.	-	-	-
Cast, mean temper	"	391985 Wd.	-	-	-
Cast, high temper	"	372598 Wd.	-	-	-
Bars	Rankine.	-	100000	-	29000000
Bars	"	-	130000	-	42000000
Plates, average	"	-	80000	-	-
Average	-	-	-	127344	-
Wood.[1]					
Alder	Stoney.	6831 H.	13900 Mu.	-	-
Ash	"	9363 H.	16700 Bv.	12156 B.	-
Ash	Rankine.	9000	17000 B.	13000	1600000
Beech	"	11500	9360	10500	1350000
Beech	Stoney.	9363 H.	11500 B.	9336 B.	-
Beech	"	-	17300 Mu.	-	-
Birch, American	"	11663 H.	-	12366 B.D.	1645000
Birch, English	"	6402 H.	15000 Bv.	11568 B.	-
Box	"	-	20000 B.	14670 T.	-
Box	Rankine.	10300	20000	-	-
Cedar, American white . . .	Stoney.	-	-	4596 D.	-
Cedar of Lebanon	"	5863 H.	11400 Bv.	8958 D.	-
Cedar of Lebanon	Rankine.	5860	11400	7400	486000
Chestnut, Spanish	Stoney.	-	13300 Ro.	-	-
Chestnut	"	-	10500 Bv.	-	-
Chestnut, horse	-	-	12100 Bv.	-	-
Chestnut	Rankine.	-	11500	10660	1140000
Chestnut	Haswell.	5350	-	-	-
Cypress	Stoney.	-	6000 Mu.	-	-
Deal, Christiana	"	-	12900 Bv.	9372 B.	-
Deal, red	"	6586 H.	-	-	-
Deal, white	"	7293 H.	-	-	-
Elm	Rankine.	10300	14000	7850	1020000
Elm	Stoney.	10331 H.	14400 Bv.	-	-
Elm, English	"	-	-	4692 B.D.	-
Elm, Canada Rock	"	-	-	11820 D.N.	-

TABLE II. — *Continued.*

Material.	Authority.	Ultimate Resistance, in Lbs., per Square Inch, to			Modulus of Elasticity, E.
		Compression, C.	Tension, T.	Cross-Breaking, B.	
Wood (continued).					
Fir, spruce	Stoney.	6819 H.	-	8076 M.	-
Fir, spruce, American black .	"	-	-	6216 D.	-
Fir, Mar forest	"	-	12000 B.	7392 B.	-
Fir, red pine	Rankine.	5375	12000	7100	1460000
Fir, red pine	"	6200	14000	9540	1900000
Fir, yellow pine, American .	"	5400	-	-	-
Fir, spruce	"	-	12400	9900	1400000
Fir, spruce	"	-	-	12300	1800000
Fir, larch	"	5570	9000	5000	900000
Fir, larch	"	-	10000	10000	1360000
Hemlock	Stoney.	-	-	6852 D.	-
Hickory, American	"	-	-	12774 D.N.	-
Hickory, bitter-nut	"	-	-	8790 D.	-
Larch	"	5568 H.	10220 Ro.	8010 B.D.	-
Larch	"	-	8900 Bv.	-	-
Larch, American	"	-	-	5466 D.	-
Lignum-vitæ	"	-	11800 Bv.	12078 N.	-
Lignum-vitæ	Rankine.	9900	11800	12000	-
Locust	"	-	16000	11200	-
Locust	Stoney.	-	20100 Mu.	20580 B.	-
Locust	Haswell.	9113	-	-	-
Mahogany	Rankine.	-	8000	7600	1255000
Mahogany	"	8200	21800	11500	-
Mahogany	Stoney.	8198 H.	8000 B.	-	-
Mahogany	"	-	16500 Bv.	10314 M.N.	-
Maple	"	-	17400 Bv.	10164 D.	-
Maple	Rankine.	-	10600	-	-
Maple	Haswell.	8150	-	-	-
Oak, European	Rankine.	7700	10000	8700	1200000
Oak, European	"	10000	19800	13600	1750000
Oak, American red	"	6000	10250	10600	2150000
Oak, English	Stoney.	10058 H.	10000 B.	10164 B.D.	-
Oak, English	"	-	19800 Bv.	-	-
Oak, French	"	-	13950 Ro.	8898 M.	-
Oak, Quebec	"	5982 H.	-	-	-
Oak, American red	"	-	-	10122 D.N.	-
Oak, American white . . .	"	-	-	10458 B.D.	-
Pine, American red	"	7518 H.	-	9162 B.D.	-
Pine, American pitch . . .	"	6790 H.	7650 Mu.	10362 B.D.	-
Pine, American white . . .	"	-	-	7374 D.N.	-

TABLE II.—*Continued.*

Material.	Authority.	Ultimate Resistance, in Lbs., per Square Inch, to			Modulus of Elasticity, E.
		Compression, C.	Tension, T.	Cross-Breaking, B.	
Wood (continued).					
Pine, American yellow . . .	Stoney.	5445 H.	-	7110 B.D.	-
Pine, Norway	"	-	14300 Bv.	-	-
Pine, Norway	"	-	7287 Bv.	-	-
Sycamore	Rankine.	-	13000	9600	1040000
Sycamore	Stoney.	7082 H.	13000 Bv.	-	-
Teak	"	12101 H.	15000 B.	12648 B.M.	-
Teak, Indian	Rankine.	12000	15000	12000	2400000
Teak, Indian	"	-	-	19000	-
Teak, African	"	-	-	14980	-
Walnut	Stoney.	7227 H.	8130 Mu.	-	-
Walnut	"	-	7800 Bv.	-	-
Willow	"	6128 H.	14000 Bv.	-	-
Willow	Rankine.	-	-	6600	-
Stone.					
Granite	Stoney.	3173 Wi.	-	456 Wi.	-
Granite	"	13440 Wi.	-	2442 Wi.	-
Granite	Rankine.	5500	-	-	-
Granite	"	11000	-	-	-
Limestone	"	4000	-	-	-
Limestone	"	4500	-	-	-
Limestone	Stoney.	3050 F.	-	1698 Wi.	-
Limestone	"	18043 Wi.	-	2484 Wi.	-
Limestone	Haswell.	-	670	-	-
Limestone	"	-	2800	-	-
Marble	Stoney.	3216 Re.	551 H.	-	-
Marble	"	20160 Wi.	722 Bu.	-	-
Marble	Rankine.	5500	-	-	-
Marble, white	-	-	-	1252 H.	-
Marble, black	-	-	-	2697 H.	-
Marble, black	Moseley.	-	-	2664	-
Sandstone	Stoney.	2185	1054 Bu.	2010 Re.	-
Sandstone	"	7884	1261 Bu.	5142 Re.	-
Sandstone	Rankine.	5500	-	2360	-
Sandstone	"	2200	-	1100	-
Slate	"	-	9600	5000	13000000
Slate	"	-	12800	-	16000000
Slate	Stoney.	17344	-	7370	-

TABLE II. — *Concluded.*

MATERIAL.	Authority.	Ultimate Resistance, in Lbs., per Square Inch, to Compression, C.	Tension, T.	Cross-Breaking, B.	Modulus of Elasticity, E.
Bricks, etc.					
Pale red	Stoney.	562 Re.	-	-	-
Red	"	808 Re.	-	-	-
Fire	"	1717 Re.	-	-	-
Gault clay	"	2240 Gr.	-	-	-
Ordinary	Rankine.	-	280	-	-
Ordinary	"	-	300	-	-
Lime mortar, average	Stoney.	618 Ro.	51	-	-
Portland cement	"	5984 Gr.	358 Gr.	-	-
Plaster of Paris	"	-	71 Ro.	-	-
ROMAN CEMENT: —					
2 years	"	-	546 Gr.	-	-
3 years	"	-	604 Gr.	-	-
4 years	"	-	632 Gr.	-	-
5 years	"	-	627 Gr.	-	-
6 years	"	-	666 Gr.	-	-
7 years	"	-	709 Gr.	-	-

The value of B, the modulus of rupture in Table II., is that due to a rectangular cross-section, unless otherwise specified.

The works from which this Table is made up are the following well-known authorities: —

1st, "A Manual of Civil Engineering," by William John Macquorn Rankine.

2d, "The Theory of Strains in Girders and Similar Structures," by Bindon B. Stoney.

3d, "The Mechanical Principles of Engineering and Architecture," by Henry Moseley.

4th, "Engineers' and Mechanics' Pocket-Book," by Charles H. Haswell.

5th, "Report on the Progress of Work, etc., of the Cincinnati Southern Railway," by Thomas D. Lovett.

6th, "Report on Tests of Salisbury Cast-Iron," in "Railroad Gazette" of Nov. 30, 1877, by Robert H. Thurston.

Names of the experimenters cited are thus abbreviated: viz., H., Hodgkinson; F., Fairbairn; Bv., Bevan; Bu., Buchanan; B., Barlow; D., Denison; N., Nelson; M., Moore; K., Kirkaldy; Ro., Rondelet; Re., Rennie; C., Clark; Gr., Grant; Wi., Wilkinson; Wd., Wade; Mu., Musschenbroek; T., Trickett.

SECTION 2.

Moment of Inertia and Radius of Gyration of a Given Cross-Section.

61. In equation (159) the factor

$$\iint y^2\,dy\,dz = I \tag{183}$$

is called the *moment of inertia* of the surface of the cross-section, relatively to the axis of z, the factor being analogous to the real *moment of inertia* of a material plate whose thickness is unity.

The moment of inertia divided by the area of the section gives the *square* of the *radius of gyration*, which we will call r^2.

We then have, if S is that area,

$$\text{Square of radius of gyration} = r^2 = \frac{I}{S} = \frac{\iint y^2\,dy\,dz}{\iint dy\,dz}. \tag{184}$$

62. From the moments of resistance already found, equations (160) to (171), and from similar applications of (183), we derive values of I and of r^2 as given below in Table III., where the axes of gyration are assumed to pass through the centre of gravity of the cross-section.

TABLE III.

FORM OF CROSS-SECTION.		Moment of Inertia of Section, I.	Square of Radius of Gyration, r^2
1. Rectangle (Fig. 90).			
About least axis b	Max.	$\frac{1}{12}bh^3$	$\frac{1}{12}h^2$.
About greater axis h	Min.	$\frac{1}{12}b^3h$	$\frac{1}{12}b^2$.
2. Square.			
About b or h		$\frac{1}{12}h^4$	$\frac{1}{12}h^2$.
3. Hollow Rectangle (Fig. 91).			
About least axis b	Max.	$\frac{1}{12}(bh^3 - b_1h_1^3)$	$\dfrac{bh^3 - b_1h_1^3}{12(bh - b_1h_1)}$.
About greater axis h	Min.	$\frac{1}{12}(b^3h - b_1^3h_1)$	$\dfrac{b^3h - b_1^3h_1}{12(bh - b_1h_1)}$.
4. Hollow Square.			
About b or h		$\frac{1}{12}(h^4 - h_1^4)$	$\frac{1}{12}(h^2 + h_1^2)$.
5. I-Section (Fig. 91).			
About vertical axis h		$\frac{1}{12}b^2A$	$\dfrac{b^2A}{12(A + B)}$.
About horizontal axis b. . . .		$\frac{1}{12}(bh^3 - b_1h_1^3)$	$\dfrac{bh^3 - b_1h_1^3}{12(A + B)}$.
A = area of flanges. B = area of web.			
6. Plates and Channels (Fig. 92).			
About axis b, normal to plates .		$\left\{\begin{array}{l}\frac{1}{12}(b - b_2)h^3 \\ +\frac{1}{12}b_2(h - 2h_2)^3 \\ -\frac{1}{12}b_1h_1^3\end{array}\right.$	$I \div S$.
About axis b, parallel to plates (Fig. 94)		$\left\{\begin{array}{l}\frac{1}{12}(bh^3 - b_1h_1^3 \\ \qquad - b_2h_2^3)\end{array}\right.$	$I \div S$.
7. Plates and I-Beams (Fig. 93).			
About axis b, parallel to plates .		$\left\{\begin{array}{l}\frac{1}{12}(bh^3 - b_1h_1^3 \\ \qquad - b_2h_2^3)\end{array}\right.$	$I \div S$.
About axis h, normal to plates .		$\left\{\begin{array}{l}\frac{1}{6}(h - h_2)b^3 \\ +\frac{1}{12}(b - \frac{1}{2}b_2)^3h_2 \\ -\frac{1}{12}(b_1 + \frac{1}{2}b_2)^3h_2 \\ -\frac{1}{6}(h_1 - h_2)b_1^3\end{array}\right.$	$I \div S$.

TABLE III. — *Concluded.*

FORM OF CROSS-SECTION.		Moment of Inertia of Section, I.	Square of Radius of Gyration, r^2.
8. T-Section, erect (Fig. 95).			
About axis b, parallel to flange $e = \dfrac{bh^2 - b_1h_1^2}{2(bh - b_1h_1)}$ = height of neutral axis.		$\frac{1}{3}b(h - e)^3 + \frac{1}{3}(b - b_1)e^3 - \frac{1}{3}b_1(h_1 - e)^3$	$I \div S$.
About axis h, normal to flange .		$\frac{1}{12}(h - h_1)b^3 + \frac{1}{12}h_1(b - b_1)^3$	$I \div S$.
9. Angle Iron; equal ribs b, thickness $= t$	Min.	$\frac{1}{12}tb^3$	$\frac{1}{24}b^2$.
Of unequal ribs h and b . . .	Min.	$\dfrac{b^2h^2(b + h)t}{12(b^2 + h^2)}$	$\dfrac{b^2h^2}{12(b^2 + h^2)}$.
10. Channel Iron; h = depth of flanges + $\frac{1}{2}$ thickness of web, A = area of flanges, B = area of web	Min.	$h^2\left(\dfrac{A}{12} + \dfrac{AB}{4S}\right)$	$h^2\left(\dfrac{A}{12S} + \dfrac{AB}{4S^2}\right)$
11. Star Iron, or cross of equal arms h	Min.	$\frac{1}{24}Sh^2$	$\frac{1}{24}h^2$.
12. Ellipse (Fig. 98).			
About minor axis $2b$	Max.	$\frac{1}{4}\pi bh^3$	$\frac{1}{4}h^2$.
About major axis $2h$	Min.	$\frac{1}{4}\pi b^3h$	$\frac{1}{4}b^2$.
13. Circle; radius h (Fig. 97) . . .		$\frac{1}{4}\pi h^4$	$\frac{1}{4}h^2$.
14. Hollow Ellipse (Fig. 98).			
About minor axis $2b$	Max.	$\dfrac{\pi}{4}(bh^3 - b_1h_1^3)$	$\frac{1}{4}\dfrac{bh^3 - b_1h_1^3}{bh - b_1h_1}$.
About major axis $2h$	Min.	$\dfrac{\pi}{4}(b^3h - b_1^3h_1)$	$\frac{1}{4}\dfrac{b^3h - b_1^3h_1}{bh - b_1h_1}$.
15. Hollow Circle (Fig. 97).			
Radii, h, h_1		$\dfrac{\pi}{4}(h^4 - h_1^4)$	$\frac{1}{4}(h^2 + h_1^2)$.

CHAPTER VI.

DEFLECTION, END MOMENTS, AND POINTS OF CONTRARY FLEXURE FOUND. — CAMBER.

SECTION I.

Deflection of the Semi-Beam having a Uniform Cross-Section.

63. **Equation of the Elastic Curve as applied to a Beam or Pillar.** — Let N, Fig. 89, be the origin, and x and y the current co-ordinates, of the neutral line NTS of any beam or column under a given load; $TU =$ a unit of the length of the beam; $VT = VU =$ the radius of curvature at any point $=$

$$\rho = -\frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = -\frac{1}{\frac{d^2y}{dx^2}} \qquad (185)$$

when the deflection of the beam or pillar is small compared with its length. (See Differential Calculus.) $PP_1 =$ increment of unit on extended side due to flexure; $RR_1 =$ decrement of unit on compressed side due to flexure; $\alpha =$ the angle included between the tangent to the curve at any point and the axis of x, so that $\tan \alpha = \frac{dy}{dx}$.

Suppose the unit strain required to extend a unit of length by the space PP_1 to be T_{11}, and that required to compress a

unit of length by the space RR_1 to be C_1, and that required to extend or compress a unit of length by its own length, TU, to be E = the modulus of transverse elasticity; and put $C_1 + T_1 = 2B_1$ = total unit strain at surfaces.

We then have, if h is the depth of the beam, and if the displacing forces E and $2B_1$ are proportional to the displacements they cause,

$$\frac{P_1P}{PU} = \frac{R_1R}{RU} = \frac{TU}{UV},$$

$$\therefore \quad \frac{2B_1}{h} = \frac{E}{\rho} = -E\frac{d^2y}{dx^2}. \qquad (186)$$

Multiplying (186) by I = moment of inertia of cross-section, we find

$$\frac{2B_1I}{h} = -EI\frac{d^2y}{dx^2} = M_x = R, \qquad (187)$$

which is the moment of resistance of the cross-section, since $B_1 \div \frac{1}{2}h = f$, and $I = \iint y^2\,dy\,dz$ of equation (159).

If, therefore, we put $-EI\dfrac{d^2y}{dx^2}$ equal to the moment at the section due the external forces acting on a beam or pillar of uniform cross-section, and perform two successive integrations, we shall have an equation in which y is the deflection of the neutral line at the distance x from the origin of co-ordinates.

64. Deflection of the Semi-Beam under One Weight.—Let L, Fig. 8, the point where the neutral line of the semi-beam meets the wall, be the origin of co-ordinates, and call x positive to the left, and y positive downward, in accordance with the notation in article 14.

Take semi-beam of length l, with concentrated load W at distance a' from fixed end.

From equation (18),

$$M_x = -W(d' - x) = -EI\frac{d^2y}{dx^2}$$

by (187),

$$\therefore\ EI\frac{d^2y}{dx^2} = W(d' - x).$$

Integrating, with the condition that $\frac{dy}{dx} = 0$ when $x = 0$, we have

$$EI\frac{dy}{dx} = W\left(d'x - \frac{x^2}{2}\right).$$

Integrating again, with the condition that $y = 0$ when $x = 0$,

$$\therefore\ EIy = W\left(d'\frac{x^2}{2} - \frac{x^3}{6}\right).$$

Deflection at any point, x, is

$$y = \frac{W}{EI}(\tfrac{1}{2}d'x^2 - \tfrac{1}{6}x^3). \qquad (188)$$

And when $x = d' = l$, we have

$$\text{Maximum deflection } D = \frac{Wl^3}{3EI}, \qquad (189)$$

where the proper values of E and I are to be taken from Tables II. and III., according to the material used, and the form of cross-section.

65. Semi-Girder, Length l, Uniform Load w per Unit.—From equations (23) and (187),

$$EI\frac{d^2y}{dx^2} = \tfrac{1}{2}w(l - x)^2.$$

When $\frac{dy}{dx} = 0$, $x = 0$;

$$\therefore \quad EI\frac{dy}{dx} = \tfrac{1}{2}wl^2x - \tfrac{1}{2}wlx^2 + \tfrac{1}{6}wx^3.$$

When $y = 0$, $x = 0$;

$$\therefore \quad EIy = \tfrac{1}{4}wl^2x^2 - \tfrac{1}{6}wlx^3 + \tfrac{1}{24}wx^4.$$

Deflection anywhere,

$$y = \frac{w}{24EI}(6l^2x^2 - 4lx^3 + x^4). \tag{190}$$

Maximum deflection,

$$D = \frac{wl^4}{8EI} \text{ when } x = l. \tag{191}$$

66. Semi-Girder, Partial Uniform Load, w', on each Unit of Length, b, at Distance a from Fixed End. Fig. 8.— When $x = a$, or $x < a$, equations (29) and (187) give

$$EI\frac{d^2y}{dx^2} = w'b(\tfrac{1}{2}b + a - x).$$

$\frac{dy}{dx} = 0$ when $x = 0$,

$$\therefore \quad EI\frac{dy}{dx} = w'b\left\{(\tfrac{1}{2}b + a)x - \frac{x^2}{2}\right\}.$$

$y = 0$ when $x = 0$,

$$\therefore \quad EIy = w'b\left\{(\tfrac{1}{2}b + a)\frac{x^2}{2} - \frac{x^3}{6}\right\}.$$

Deflection x not $> a$,

$$y = \frac{w'b}{EI}\left\{(\tfrac{1}{2}b + a)\frac{x^2}{2} - \frac{x^3}{6}\right\}. \qquad (192)$$

Again, $\dfrac{dy}{dx} = \tan\alpha$ when $x = a$,

$$\therefore\ EI\left(\frac{dy}{dx} - \tan\alpha\right) = w'b\left\{(\tfrac{1}{2}b + a)(x - a) - \frac{x^2 - a^2}{2}\right\}.$$

$y = 0$ when $x = 0$,

$$\therefore\ EI(y - x\tan\alpha) = w'b\left\{(\tfrac{1}{2}b + a)\left(\frac{x^2}{2} - ax\right) - \left(\frac{x^3}{6} - \frac{a^2x}{2}\right)\right\}.$$

Let $y = y_1$ when $x = a$,

$$\therefore\ EIa\tan\alpha - EIy_1 = \tfrac{1}{4}w'a^2b^2 + \tfrac{1}{3}w'a^3b.$$

From (192),

$$EIy_1 = \tfrac{1}{4}w'a^2b^2 + \tfrac{1}{3}w'a^3b,$$

$$\therefore\ EI\tan\alpha = \tfrac{1}{2}w'ab(b + a).$$

Also, when x is not $< a$ nor $> (a + b)$, we have, from (26) and (187),

$$EI\frac{d^2y}{dx^2} = \tfrac{1}{2}w'(a + b - x)^2 = \tfrac{1}{2}w'(a + b)^2 - w'(a + b)x + \tfrac{1}{2}w'x^2.$$

$\dfrac{dy}{dx} = \tan\alpha$ when $x = a$,

$$\therefore\ EI\left(\frac{dy}{dx} - \tan\alpha\right)$$
$$= w'\left\{(a + b)^2\frac{x - a}{2} - (a + b)\frac{x^2 - a^2}{2} + \frac{x^3 - a^3}{6}\right\}.$$

$y = y_1$ when $x = a$,

$$\therefore\ EI(y - y_1) - (x - a)\tan\alpha EI$$
$$= \frac{w'}{6}\Big\{3(a + b)^2\Big[\frac{x^2 - a^2}{2} - a(x - a)\Big]$$
$$- 3(a + b)\Big[\frac{x^3 - a^3}{3} - a^2(x - a)\Big]$$
$$+ \frac{x^4 - a^4}{4} - a^3(x - a)\Big\}.$$

Whence, after eliminating y_1 and $\tan\alpha$, we find the deflection

$$y = \frac{w'}{24EI}[x^4 - a^4 - 4(a + b)(x^3 - a^3)$$
$$+ 6(a + b)^2(x^2 - a^2) - 4a^3(x - a)$$
$$+ 6a^2b^2 + 8a^3b]. \qquad (193)$$

And if $x = a + b$, we have

$$\text{Maximum deflection } D = \frac{w'}{24EI}[3(a + b)^4 - 3a^4 - 4a^3b]. \qquad (194)$$

67. If it be required to find the total deflection of a semi-beam at its free extremity when it supports partial uniform or concentrated loads not reaching that extremity, we may proceed as follows: —

Find the deflection at the free end due the beam's own weight, lw; then find the deflection, D, and the inclination, α, of the beam at any point bearing a concentrated load, W, or at the outer end of any partial uniform load, bw'; using $\tan\alpha$, compute the end deflection due W or bw' by the formula

$$\begin{cases} D_1 = D_{w'} + (l - a - b)\tan\alpha \text{ for } bw', \\ D_2 = D_W + (l - a')\tan\alpha \text{ for } W. \end{cases}$$

These deflections added to that due the beam's own weight, will give the total deflection at the free end of the semi-beam.

EXAMPLE. — Suppose the semi-beam, Fig. 8, to be of oak, weighing 54 pounds to the cubic foot, and to have a rectangular cross-section 1 foot deep and $\frac{1}{2}$ foot wide, so that weight of beam per foot of length $= w = 27$ pounds; length $= 15$ feet, $b = 4$ feet, loaded with $w' = 100$ pounds per foot, beginning $a = 5$ feet from the fixed end of beam; $W = 100$ pounds, placed $a' = 11$ feet from fixed end; $E = 2{,}000{,}000$ pounds per square inch $= 288{,}000{,}000$ pounds per square foot.

From Table III., $I = \frac{1}{12}bh^2 = \frac{1}{12} \times \frac{1}{2} \times 1^2 = \frac{1}{24}$,

$$\therefore \quad EI = 12000000.$$

Deflection due beam's own weight is, by (191),

$$D_w = \frac{27 \times 15^4 \times 12}{8 \times 12000000} = 0.17086 \text{ inch.}$$

From (194),

$$D_{w'} = \frac{100[3(5+4)^4 - 3 \times 5^4 - 4 \times 5^3 \times 4]}{24 \times 12000000} \times 12 = 0.06587 \text{ inch.}$$

Differentiating (193), we have

$$\frac{dy}{dx} = \tan \alpha_1$$

$$= \frac{w'}{24EI}[4x^3 - 12(a+b)x^3 + 12(a+b)^2 x - 4a^3]$$

$$= \frac{100 \times 2416}{24 \times 12000000}$$

when $x = a + b = 9$;

$$\therefore \quad (l - a - b)\tan \alpha_1 = \frac{6 \times 100 \times 2416 \times 12}{24 \times 12000000} = 0.0604 \text{ inch,}$$

$$\therefore \quad D_1 = 0.06587 + 0.06040 = 0.12627 \text{ inch at end due } bw'.$$

From (189),

$$D_W = \frac{100 \times 15^3 \times 12}{3 \times 12000000} = 0.1125 \text{ inch.}$$

Differentiating (188), we find

$$\frac{dy}{dx} = \tan\alpha_2 = \frac{W}{EI}\left(d'x - \frac{x^2}{2}\right) = \frac{100 \times \frac{1}{2} \times 121}{12000000}$$

when $x = d' = 11$;

$$\therefore \quad (l - d')\tan\alpha_2 = \frac{4 \times 100 \times \frac{1}{2} \times 121 \times 12}{12000000} = 0.0242 \text{ inch,}$$

$$\therefore \quad D_2 = 0.1125 + 0.0242 = 0.1367 \text{ inch at end due } W.$$

Therefore total deflection at free end is

$$D = D_w + D_1 + D_2 = 0.17086 + 0.12627 + 0.1367 = 0.43383 \text{ inch.}$$

68. If we have any number, r, equal weights, W, placed at equal intervals, $\frac{l}{n}$, along the semi-girder, the first weight being a full interval from the fixed end, and the n^{th} or last weight being at the free end when the beam is fully loaded, then the total deflection at the free end due to the first r equal weights may be found as follows:—

In equation (188) put $x = a' = \frac{lr}{n}$; then

$$y = \frac{W}{3EI}\left(\frac{l}{n}\right)^3 r^3, \tag{195}$$

which is the deflection at the r^{th} weight due to that weight alone.

But the deflection at the free end due the r^{th} weight is greater than the deflection at the r^{th} weight by the product of the tangent of the slope α of the beam at and beyond the r^{th} weight, multiplied by the horizontal distance between this weight and the free end; and this horizontal distance is practically equal to $\left(l - \frac{lr}{n}\right)$ = the part of the beam's length beyond the r^{th} weight.

By differentiating (188), we find

$$\frac{dy}{dx} = \tan\alpha = \frac{W}{EI} \times \frac{1}{2}\left(\frac{lr}{n}\right)^2$$

if $x = d' = \frac{lr}{n}$, and

$$\left(l - \frac{lr}{n}\right)\tan\alpha = \frac{W}{EI}\left(\frac{l}{n}\right)^3(\tfrac{1}{2}nr^2 - \tfrac{1}{2}r^3).$$

Add value of y in (195) $= \frac{W}{EI}\left(\frac{l}{n}\right)^3\frac{r^3}{3}$, and we have deflection at free end due the r^{th} weight,

$$D_r = \frac{Wl^3}{EI}\left(\frac{r^2}{2n^2} - \frac{r^3}{6n^3}\right). \qquad (196)$$

Hence the end deflection due the first r weights is

$$D_{\Sigma r} = \frac{Wl^3}{EI}\left\{\frac{1}{2n^2}(1^2+2^2+3^2+4^2\ldots+r^2) - \frac{1}{6n^3}(1^3+2^3+3^3+4^3\ldots+r^3)\right\}$$
$$= \frac{W}{24EI}\left(\frac{l}{n}\right)^3[2nr(r+1)(2r+1) - r^2(r+1)^2]. \quad (197)$$

When $r = n$, (197) becomes

$$D = \frac{Wl^3}{24EI}\frac{(n+1)(3n+1)}{n}, \qquad (198)$$

which is the deflection at the free end due n equal weights at equal intervals, $\frac{l}{n}$.

69. If the weight at the free end is but $\frac{1}{2}W$, while every other weight is W, as is generally the case with a uniform load of panel weights, we must diminish the deflection last found by the deflection due $\frac{1}{2}W$ at the free end, which, according to (189), is

$$D = \frac{\frac{1}{2}Wl^3}{3EI}.$$

This quantity taken from the value of D in (198) leaves

$$D = \frac{Wl^3}{24EI}\left\{\frac{(n+1)(3n+1)}{n} - 4\right\}, \qquad (199)$$

which is the deflection at the free end of a semi-girder of uniform cross-section when the load is distributed in equal panel weights; there being but the half panel weight at the free end.

70. To find the Deflection at the r^{th} Interval due to all the n Equally Distributed Equal Weights, W.

For the first r weights, (198) applies if we put r for n, and l_1 for l; l_1 being the distance of the r^{th} weight from the fixed end. That is,

$$D = \frac{Wl_1^3}{24EI}\frac{(r+1)(3r+1)}{r}. \qquad (200)$$

For the $(n - r)$ weights beyond the r^{th}, we have, from (188), since x now becomes l_1,

$$y = \frac{W}{EI}(\tfrac{1}{2}a'l_1^2 - \tfrac{1}{6}l_1^3), \qquad (201)$$

the deflection at the r^{th} weight due to any one weight at the distance a' from the fixed end,

$$a' > l_1.$$

Therefore, by summing (201), we find

$$D_{n-r} = \frac{W}{EI}\left\{\tfrac{1}{2}l_1^2\left(\frac{l}{n}[(r+1)+(r+2)+(r+3)\ldots(r+n-r)]\right) - \frac{(n-r)l_1^3}{6}\right\}$$

$$= \frac{W}{EI}\left\{\frac{1}{2}\frac{l^2r^2}{n^2}\left[\frac{l}{n}\left(r(n-r)+\frac{(n-r)(n-r+1)}{2}\right)\right] - \frac{(n-r)l^3r^3}{6n^3}\right\}$$

$$= \frac{W}{EI}\left\{\frac{1}{3}\frac{l^3r^3}{n^3}(n-r) + \frac{1}{4}\frac{l^3r^2}{n^3}(n-r)(n-r+1)\right\}, \qquad (202)$$

the deflection at the r^{th} weight due all the weights beyond.

Adding (200) and (202), there results

$$D = \frac{Wl^3r^2}{24EIn^3}(6n^2 + r^2 - 4rn - 2r + 6n + 1), \qquad (203)$$

which is the deflection at the r^{th} weight due to all the n given weights, W.

71. If the weight at the free end is the m^{th} part of W, instead of W, the deflection due $\frac{W}{m}$ at the r^{th} point of division is, by putting $d' = l$, and $l_1 = \frac{lr}{n}$, in (201),

$$y = \frac{W}{mEI}\left(\frac{l^3r^2}{2n^2} - \frac{l^3r^3}{6n^3}\right) = \frac{Wl^3r^2}{24EIn^3}\left(\frac{12n - 4r}{m}\right).$$

Subtracting this value of y from the deflection in (203), we have, finally,

$$D = \frac{Wl^3r^2}{24EIn^3}\left(6n^2 + r^2 - 4rn - 2r + 6n + 1 - \frac{12n - 4r}{m}\right)$$

$$= \frac{Wl^3r^2}{24EIn^3}(6n^2 - 4nr + r^2 + 1) \qquad (204)$$

if $m = 2$; so that (204) is the deflection at the r^{th} panel point due to a full load of equal panel weights.

EXAMPLE. — Take the oak semi-beam of the example in article 67, and suppose it loaded with $n = 5$ equal weights, of 100 pounds each, at intervals of 3 feet. $EI = 12{,}000{,}000$.

Then the deflection at the free end due the 5 weights is, by equation (198),

$$D = \frac{100 \times 15^3}{24 \times 12000000} \times \frac{(5+1)(3 \times 5 + 1)}{5}$$
$$= 0.0225 \text{ foot} = 0.27 \text{ inch};$$

to which add 0.17086, the deflection due the beam's own weight, for the total deflection at the free end = 0.44086 inch.

If the fifth or end weight is but $\frac{1}{2} \times 100 = 50$ pounds, then, by (199),

$$D = \frac{100 \times 15^3}{24 \times 12000000}\left\{\frac{(5+1)(3 \times 5 + 1)}{5} - 4\right\}$$
$$= 0.0178125 \text{ foot} = 0.21375 \text{ inch};$$

and the total deflection = 0.21375 + 0.17086 = 0.38461 inch.

To find the deflection at the third loaded point due the 5 equal weights in their positions, we use (203), taking $r = 3$; thus,

$$D = \frac{100 \times 15^3 \times 3^2}{24 \times 12000000 \times 5^3}(6 \times 5^2 + 3^2 - 4 \times 3 \times 5 - 2 \times 3 + 6 \times 5 + 1)$$
$$= 0.0104625 \text{ foot} = 0.12555 \text{ inch}.$$

And the deflection at this third interval, due the beam's own weight of 27 pounds per linear foot, is, by equation (190), putting $x = 9$, $w = 27$,

$$y = \frac{27}{24 \times 12000000}(6 \times 15^2 \times 9^2 - 4 \times 15 \times 9^3 + 9^4)$$
$$= 0.006766 \text{ foot} = 0.081192 \text{ inch}.$$

Therefore the total deflection at the third interval is

$$0.125550 + 0.081192 = 0.206742 \text{ inch}.$$

SECTION 2.

Deflection of a Beam of Uniform Cross-Section, supported at its Free Unfixed Ends.

72. **Deflection due the Beam's own Weight, supposed to be Uniform.**—For the cases in this section we employ the same notation as that given in article 17, Fig. 9, excepting that we take the origin of co-ordinates at $O_{,,}$ a point in the neutral surface, instead of using O as before, in order that y may be the deflection of the neutral line, as it is in the expression for the moment of the internal forces, $R = -EI\dfrac{d^2y}{dx^2} = M$, of article 63. We now have x positive to the right, and y positive downwards.

From equations (49) and (187),

$$-\tfrac{1}{2}w(l - x)x = EI\frac{d^2y}{dx^2}.$$

Integrating, with the condition that $\dfrac{dy}{dx} = 0$ when $x = \tfrac{1}{2}l$,

$$EI\frac{dy}{dx} = \tfrac{1}{2}w\left\{\frac{x^3 - (\tfrac{1}{2}l)^3}{3} - \tfrac{1}{2}l[x^2 - (\tfrac{1}{2}l)^2]\right\}.$$

Integrating again, with the condition that $y = 0$ when $x = 0$, we have, after reducing,

$$y = \frac{w}{24EI}(x^4 - 2lx^3 + l^3x), \qquad (205)$$

which is the deflection of the uniformly loaded beam at any point, w being the load per unit of the beam's length, l.

Putting this value of $\tan\alpha$ in (207), we get, after reducing,

$$y = \frac{W(l - d')}{6EIl}[(2l - d')d'x - x^3], \qquad (209)$$

which is the deflection at any point between the origin and the weight W.

If $x = d'$, we have the deflection at the loaded point,

$$D = \frac{Wd'^2}{3EIl}(l - d')^2. \qquad (210)$$

And if $x = d' = \frac{1}{2}l$,

$$\tan\alpha = 0;$$

and

$$\text{Deflection at centre} = D = \frac{Wl^3}{48EI}, \qquad (211)$$

which is a maximum, since W is now at the centre.

Comparing (211) and (206), where $wl = W =$ entire load on the beam, we see that the deflection at the centre due the load, lw, uniformly distributed continuously, is five-eighths of the deflection at the centre due the same amount of load concentrated at that point.

By putting $l - d'$ for d', and $l - x$ for x, in (209), or by substituting the value of $\tan\alpha = \frac{Wd'}{EIl}\left(\frac{l^2}{3} - d'l + \frac{2}{3}d'^2\right)$ in (208), we have

$$y = \frac{Wd'}{6EIl}[(l^2 - d'^2)(l - x) - (l - x)^3], \qquad (212)$$

which is the deflection when $x > d'$; that is, at any point between the weight W and the right-hand support.

EXAMPLE. — Take a beam of pine weighing 40 pounds per cubic foot, of rectangular cross-section. Depth $= h = 18\frac{1}{2}$ inches, breadth $= b = 15$ inches, length $= 12\frac{1}{2}$ feet $= 150$

inches. Call $E = 1,680,000$ pounds per square inch, $I = \frac{1}{12}bh^3 = \frac{15 \times 18.5^3}{12} = 7,914.53125$, $EI = 168 \times 79,145,312.5$; beam's own weight per inch of length $= w = \frac{18.5 \times 15 \times 40}{12^3}$ $= 6.42361/9$ pounds. Deflection due beam's own weight, lw, at a point 48 inches from one end is, by (205),

$$y = \frac{6.42361/9}{24 \times 168 \times 79145312.5}(48^4 - 2 \times 150 \times 48^3 + 150^3 \times 48)$$
$$= 0.0027 \text{ inch.}$$

Deflection at centre, from beam's weight, by (206), is

$$D = \frac{5 \times 6.42361/9 \times 150^4}{384 \times 168 \times 79145312.5} = 0.0032 \text{ inch,}$$

which is a maximum.

Deflection at the point of application, due weight $W = 17,935$ pounds placed $d' = 48$ inches from end of beam, is, by (210),

$$D = \frac{17935 \times 48^2}{3 \times 168 \times 79145312.5} \times (150 - 48)^2 = 0.07185 \text{ inch.}$$

Deflection at the centre when $W_1 = 17,935$ pounds is placed 48 inches from one end, is found by equation (212), making $x = \frac{1}{2}l$,

$$y = \frac{17935 \times 48}{12 \times 168 \times 79145312.5}\left(150^2 - 48^2 - \frac{150^2}{4}\right) = 0.078617 \text{ inch.}$$

And when W is at the centre, the central deflection is, from (211),

$$D = \frac{17935 \times 150^3}{48 \times 168 \times 79145312.5} = 0.0948 \text{ inch.}$$

Add deflection due beam's own weight for total maximum deflection $= 0.098$ inch.

74. Deflection due a Partial Load, $w'b$, Uniformly Distributed Continuously over the Length, b, beginning at the Horizontal Distance, a, from the Origin, O_1, or Left End of the Beam, Fig. 9.

To find this deflection, we use, when $x < a$, equations (53) and (187), giving

$$EI\frac{d^2y}{dx^2} = -w'b\frac{l - (a + \frac{1}{2}b)}{l}x = -\varepsilon x \text{ (say)}.$$

Let α be the angle of inclination, or slope, of the beam at the distance a from the left end; then integrating, with the condition that $\frac{dy}{dx} = \tan\alpha$ when $x = a$,

$$\therefore \quad EI\left(\frac{dy}{dx} - \tan\alpha\right) = -\frac{\varepsilon}{2}(x^2 - a^2).$$

Again, $y = 0$ when $x = 0$,

$$\therefore \quad EI(y - x\tan\alpha) = -\frac{\varepsilon}{2}\left(\frac{x^3}{3} - a^2x\right). \qquad (213)$$

Let $y = y_1$ when $x = a$,

$$\therefore \quad EI(y_1 - a\tan\alpha) = \frac{\varepsilon a^3}{3} = \frac{w'a^3b}{3l}(l - a - \tfrac{1}{2}b). \quad (214)$$

But when $x > a$ and $< (a + b)$, equations (55) and (187) are to be employed, yielding

$$EI\frac{d^2y}{dx^2} = -\varepsilon x + \frac{w'}{2}(x^2 - 2ax + a^2)$$
$$= \frac{w'}{2}x^2 - (\varepsilon + aw')x + \frac{a^2w'}{2}.$$

And, if β is the angle of inclination at the distance $(a + b)$ from the origin, we may integrate as follows:

$$\frac{dy}{dx} = \tan\alpha \text{ when } x = a,$$

$$\therefore \quad EI\left(\frac{dy}{dx} - \tan\alpha\right)$$

$$= \frac{w'}{2}\left(\frac{x^3 - a^3}{3}\right) - \frac{\epsilon + aw'}{2}(x^2 - a^2) + \frac{a^2w'}{2}(x - a).$$

$y = y_1$ when $x = a$,

$$\therefore \quad EI[y - y_1 - (x - a)\tan\alpha] = \frac{w'}{6}\left\{\frac{x^4 - a^4}{4} - a^3(x - a)\right\}$$

$$- \frac{\epsilon + aw'}{2}\left\{\frac{x^3 - a^3}{3} - a^2(x - a)\right\} + \frac{a^2w'}{2}\left\{\frac{x^2 - a^2}{2} - a(x - a)\right\}. \quad (215)$$

Let $y = y_2$ when $x = a + b$, Fig. 9; so that, after reducing, (215) becomes

$$EI(y_2 - y_1 - b\tan\alpha) = \frac{w'b^3}{l}\left(\frac{a^2}{2} + \frac{5ab}{12} + \frac{b^2}{12} - \frac{al}{2} - \frac{bl}{8}\right). \quad (216)$$

Or, we may integrate in a different manner; first, with the condition $\frac{dy}{dx} = \tan\beta$ when $x = a + b$,

$$\therefore \quad EI\left(\frac{dy}{dx} - \tan\beta\right)$$

$$= \frac{w'}{2}\left\{\frac{x^3 - (a + b)^3}{3}\right\} - \frac{\epsilon + aw'}{2}\left\{x^2 - (a + b)^2\right\} + \frac{a^2w'}{2}(x - a - b).$$

Also $y = y_2$ when $x = a + b$,

$$\therefore \quad EI[y - y_2 - (x - a - b)\tan\beta]$$

$$= \frac{w'}{6}\left\{\frac{x^4 - (a + b)^4}{4} - (a + b)^3(x - a - b)\right\}$$

$$- \frac{\epsilon + aw'}{2}\left\{\frac{x^3 - (a + b)^3}{3} - (a + b)^2(x - a - b)\right\}$$

$$+ \frac{a^2w'}{2}\left\{\frac{x^2 - (a + b)^2}{2} - (a + b)(x - a - b)\right\}. \quad (217)$$

But in (217) $y = y_1$ when $x = a$; therefore, after reducing, we have

$$EI(y_1 - y_2 + b\tan\beta) = \frac{w'b^3}{l}\left(\frac{a^2}{2} + \frac{7ab}{12} + \frac{b^2}{6} - \frac{al}{2} - \frac{5bl}{24}\right). \quad (218)$$

For the remaining part of the beam, Fig. 9, that is, when $x > (a + b)$, equations (57) and (187) give

$$EI\frac{d^2y}{dx^2} = -\varepsilon x + w'b(x - a - \tfrac{1}{2}b) = (w'b - \varepsilon)x - w'b(a + \tfrac{1}{2}b).$$

$$\frac{dy}{dx} = \tan\beta \text{ when } x = a + b,$$

$$\therefore \quad EI\left(\frac{dy}{dx} - \tan\beta\right)$$
$$= \frac{w'b - \varepsilon}{2}[x^2 - (a + b)^2] - w'b(a + \tfrac{1}{2}b)(x - a - b).$$

$$y = 0 \text{ when } x = l,$$

$$\therefore \quad EI[y - (x - l)\tan\beta]$$
$$= \frac{w'b - \varepsilon}{2}\left\{\frac{x^3 - l^3}{3} - (a + b)^2(x - l)\right\} - \frac{w'b(a + \frac{1}{2}b)}{2}(x^2 - l^2)$$
$$+ w'b(a + b)(a + \tfrac{1}{2}b)(x - l), \quad (219)$$

which becomes, if we put y_2 for y, and $a + b$ for x, and reduce,

$$EI[y_2 - (a + b - l)\tan\beta] = \frac{w'b}{3l}(l - a - b)^3(a + \tfrac{1}{2}b). \quad (220)$$

From equations (214), (216), (218), and (220), we may now determine the four quantities, $\tan\alpha$, $\tan\beta$, y_1, y_2, so that they can be eliminated from (213), (215), (217), and (219).

$$\tan\alpha = \frac{w'b}{EIl}\left(\frac{2a^3}{3} + \frac{a^2b}{2} + \frac{ab^2}{6} + \frac{b^3}{24} - a^2l - \frac{abl}{2} - \frac{b^2l}{6} + \frac{al^2}{3} + \frac{bl^2}{6}\right), \quad (221)$$

$$\tan\beta=\frac{w'b}{EIl}\left(\frac{2a^3}{3}+\frac{3a^2b}{2}+\frac{7ab^2}{6}+\frac{7b^3}{24}-a^2l-\frac{3abl}{2}-\frac{b^2l}{2}+\frac{al^2}{3}+\frac{bl^2}{6}\right), \quad (222)$$

$$y_1=\frac{w'ab}{EIl}\left(\frac{a^3}{3}+\frac{a^2b}{3}+\frac{ab^2}{6}+\frac{b^3}{24}-\frac{2a^2l}{3}-\frac{abl}{2}-\frac{b^2l}{6}+\frac{al^2}{3}+\frac{bl^2}{6}\right), \quad (223)$$

$$y_2=\frac{w'b}{EIl}\left(\frac{a^4}{3}+a^3b+\frac{7a^2b^2}{6}+\frac{5ab^3}{8}+\frac{b^4}{8}-\frac{2a^3l}{3}\right.$$
$$\left.-\frac{3a^2bl}{2}-\frac{7ab^2l}{6}-\frac{7b^3l}{24}+\frac{a^2l^2}{3}+\frac{abl^2}{2}+\frac{b^2l^2}{6}\right). \quad (224)$$

We have, then, from (213), where x is not greater than a,

$$y=\frac{w'b}{6EIl}(l-a-\tfrac{1}{2}b)(3a^2x-x^3)+x\tan\alpha, \quad (225)$$

which is the deflection due $w'b$ at any point between the origin and the beginning of the partial continuous uniform load $w'b$, Fig. 9.

For the uniformly loaded part, b, of the beam, we find, from (215),

$$y=\frac{w'}{2EI}\left\{\frac{x^4-a^4}{12}-\frac{a^3}{3}(x-a)\right.$$
$$\left.-\left[\frac{b(l-a-\frac{1}{2}b)}{l}+a\right]\left[\frac{x^3-a^3}{3}-a^2(x-a)\right]+a^2\left[\frac{x^2-a^2}{2}-a(x-a)\right]\right\}$$
$$+y_1+(x-a)\tan\alpha, \quad (226)$$

which is the deflection due $w'b$ at any point of the loaded portion b, since x is here not less than a, nor greater than $a+b$, Fig. 9.

Equation (219) gives the deflection for the remaining part of the beam, that is, where x is not less than $a+b$; and we find

$$y=\frac{w'b(a+\frac{1}{2}b)}{2EI}\left\{\frac{x^3-l^3}{3l}-\frac{(a+b)^2(x-l)}{l}-(x^2-l^2)+2(a+b)(x-l)\right\}$$
$$+(x-l)\tan\beta, \quad (227)$$

which is the deflection due $w'b$ at any point between the right-hand end of the beam and the load $w'b$, Fig. 9.

If $x = a = b = \frac{1}{2}l$ in (225) or (226), we have the central deflection when one-half of the beam is uniformly loaded continuously; viz.,

$$y = \frac{5w'l^4}{2 \times 384EI},$$

which, if $w' = w$, is one-half the deflection found by (206) for the fully loaded beam.

The same result may be obtained from (226) or (227) by putting $x = b = \frac{1}{2}l$, and $a = 0$; for then the one-half load is upon the other end of the beam. The greatest deflection due a partial uniform load evidently occurs when the centre of the load and centre of the beam are in the same vertical line; that is, when $a + \frac{1}{2}b = \frac{1}{2}l$, $a = \frac{1}{2}(l - b)$, and $b = l - 2a$. Then, putting $x = \frac{1}{2}l$ in (226), we may find the greatest deflection a partial uniform load can produce.

But if it is required to find the maximum deflection of the beam when a given partial uniform continuous load has any given position upon it, we may differentiate (225), (226), or (227), put $\frac{dy}{dx} = 0$, and so find a value of x that shallr ender y a maximum. If we then add the deflection at the point so found, due the beam's own weight, we have the total deflection.

75. An important application of (226) and (227) may be made if we take $a = 0$; for in that case the partial uniform continuous load begins at the left end of the beam, so that, by assigning successively increasing values to b, we may find the deflection at any point due an advancing continuous uniform load $w'b$.

If $a = 0$,

$$y_1 = 0, \qquad \tan\alpha = \frac{w'b^2}{24EIl}(b^2 - 4bl + 4l^2),$$

and (226) becomes

$$y = \frac{w'}{24EIl}\left\{lx^4 - 4b(l - \tfrac{1}{2}b)x^3 + (b^4 - 4b^3l + 4b^2l^2)x\right\}, \quad (228)$$

which is the deflection at any point of the loaded part of the beam, where x is not greater than b.

Also, if $a = 0$,

$$\tan\beta = \frac{w'b^2}{24EIl}(7b^2 - 12bl + 4l^2),$$

and (227) becomes

$$y = \frac{w'b^2}{24EIl}\left\{2(x^3 - l^3) - 6(x^2 - l^2)l + (b^2 + 4l^2)(x - l)\right\}, \quad (229)$$

which is the deflection at any point of the unloaded part of the beam, where x is not less than b.

EXAMPLES. — Partial uniform continuous load, $w'b$. Wrought-iron 15-inch I-beam. Length 30 feet = 360 inches = l. Moment of inertia $I = \frac{1}{12}(b_2h_2^3 - b_1h_1^3)$, by Table III. 5.

Let $h_2 = 15$ inches, $b_2 = 5\frac{3}{8}$ inches, $h_1 = 12\frac{3}{4}$ inches, $b_1 = 4\frac{3}{4}$ inches, Fig. 91; putting h_2 for h, and b_2 for b, to avoid confusion here. Beam supported at ends. Load $w' = 75$ pounds per inch of the length b,

$$\therefore \quad I = \tfrac{1}{12}(5.375 \times 15^3 - 4.75 \times 12.75^3) = 691.$$

Take $E = 24{,}000{,}000$, Table II.,

$$\therefore \quad EI = 16584000000,$$

all dimensions to be in inches. Let the load cover the first 10 feet of the beam.

1st, What is the deflection at the end of the load?

We have $x = b = \frac{1}{3}l = 120$ inches; and (228) applies, giving

$$y = \frac{75 \times 360^4}{24 \times 16584000000}\left(\frac{1}{81} - 4 \times \frac{1}{3} \times \frac{5}{6} \times \frac{1}{27} + \frac{1}{243} - 4 \times \frac{1}{81} + 4 \times \frac{1}{27}\right)$$

$= 0.23444$ inch.

2d, What is the deflection at the centre of the beam?

We have from (229), if $x = \frac{1}{2}l$, and $b = \frac{1}{8}l$,

$$y = \frac{75 \times \frac{1}{8} \times 360^4}{24 \times 16584000000}\left\{-2 \times \frac{7}{8} + 6 \times \frac{3}{4} - \frac{1}{2}\left(\frac{1}{9} + 4\right)\right\}$$
$$= 0.24421 \text{ inch.}$$

3d, What is the deflection 10 feet from the unloaded end of the beam?

Here we use (229) also, putting $x = \frac{2}{3}l$, and $b = \frac{1}{8}l$,

$$\therefore \quad y = \frac{75 \times \frac{1}{8} \times 360^4}{24 \times 16584000000}\left\{-2 \times \frac{19}{27} + 6 \times \frac{5}{9} - \frac{1}{3}\left(\frac{1}{9} + 4\right)\right\}$$
$$= 0.19176 \text{ inch.}$$

4th, Suppose it is now required to find the point of greatest deflection due this same load of 75 pounds to the inch on 10 feet of one end of the beam.

Differentiating (229), we find, since $b = \frac{1}{8}l$,

$$\frac{dy}{dx} = 6x^2 - 12lx + \frac{37}{9}l^2,$$

omitting constants. Putting this value of $\frac{dy}{dx}$ equal to zero, we at once have $x = 0.43892l$, which is the point of greatest deflection; and, by placing this value of x in (229), there results $y = 0.24847$ inch, which is the greatest deflection of the beam due this load along one end.

5th, But if this same load be moved along to the centre, so that we have $a = \frac{1}{3}l = b$, we find the greatest deflection the

load can produce, by putting $x = \frac{1}{2}l$ in equation (226), where y_1 becomes $= \dfrac{11wl^4}{1944EI}$, and $\tan\alpha = \dfrac{7wl^3}{648EI}$, from (223) and (221). Thus (226) becomes

$$y = \frac{75 \times 360^4}{2 \times 16584000000}\left\{\frac{1}{12}\left(\frac{1}{16} - \frac{1}{81}\right) - \frac{1}{81}\left(\frac{1}{2} - \frac{1}{3}\right)\right.$$
$$- \left[\frac{1}{3}\left(1 - \frac{1}{3} - \frac{1}{6}\right) + \frac{1}{3}\right]\left[\frac{1}{3}\left(\frac{1}{8} - \frac{1}{27}\right) - \frac{1}{9}\left(\frac{1}{2} - \frac{1}{3}\right)\right].$$
$$\left. + \frac{1}{9}\left[\frac{1}{2}\left(\frac{1}{4} - \frac{1}{9}\right) - \frac{1}{3}\left(\frac{1}{2} - \frac{1}{3}\right)\right]\right\} + y_1 + \left(\frac{1}{2} - \frac{1}{3}\right)\tan\alpha.$$

Deflection at centre $= y = 0.50063$ inch, $\frac{1}{8}wl$ central; deflection at centre $= y = 0.24421$ inch, $\frac{1}{8}wl$ at either end;

$$\therefore \quad y = 0.50063 + 2 \times 0.24421 = 0.98905 \text{ inch},$$

equals deflection at centre when the given 15-inch I-beam of 30 feet between supports is loaded with 13.5 tons, uniformly distributed continuously. And this result accords exactly with that given by (206); thus,

$$D = \frac{5 \times 75 \times 360^4}{384 \times 16584000000} = 0.98905 \text{ inch};$$

where, as in other values of the deflection, we have retained several unnecessary decimals, in order to test the accuracy of the solutions.

6th, If this beam is half loaded with 75 pounds to the inch, we have in (228), for the deflection at the centre, $x = b = \frac{1}{2}l = 180$ inches; and

$$y = \frac{75 \times 360^4}{24 \times 16584000000}\left\{\frac{1}{16} - 4 \times \frac{1}{2} \times \frac{3}{4} \times \frac{1}{8} + \left(\frac{1}{16} - 4 \times \frac{1}{8} + 4 \times \frac{1}{4}\right)\frac{1}{2}\right\}$$
$$= 0.494525 \text{ inch},$$

which is half the deflection due the fully loaded beam, as just found.

7th, The maximum deflection due this half-load on one end of the beam is found, both in position and magnitude, by differentiating (228), putting $\frac{dy}{dx} = 0$, and solving the resulting cubic equation, putting $b = \frac{1}{2}l$, $l = 360$. Thus, omitting constant factors,

$$\frac{dy}{dx} = 4x^3 - \frac{9}{2} \times 360x^2 + \frac{9}{16} \times 360^3 = 0,$$

$$\therefore \quad x^3 - 405x^2 + 6561000 = 0.$$

Solving this equation by Horner's Method, we find the three values,

$$x = 165.52 \text{ inches},$$

$$x = 352.08 \text{ inches},$$

$$x = -112.60 \text{ inches}.$$

But, since x must be positive and not greater than $\frac{1}{2}l = 180$, the value here sought is

$$x = 165.51995,$$

retaining decimals. Hence the point of greatest deflection is within the loaded part, and is $180 - 165.51995 = 14.48005$ inches from the centre of the beam.

Putting this value of x in (228), we find the maximum deflection $y = 0.49855$ inch.

8th, The beam's own weight per inch of length, calling wrought-iron five-eighteenths pound to the cubic inch, is $\frac{5}{18} \times$ area of cross-section $= \frac{5}{18}(b_2h_2 - b_1h_1) = \frac{5}{18}(5.375 \times 15 - 4.75 \times 12.75) = 5.573$ pounds, which, substituted for w in (206), gives the deflection at the centre due the beam's own weight $= 0.07349$ inch; so that the total central deflection for the fully loaded beam is $0.98905 + 0.07349 = 1.06254$ inches.

76. To find the Deflection at any Point, x, due any Number, $(r_1 - r_2)$, Equal Weights, W, placed at Equal Intervals, c, along the Beam, the First Weight being Distant by One or More Entire Intervals, c, from the Origin or Left End of the Beam. — For all the weights, $(r - r_2)$ in number, between the left end and the point x, we use equation (212), which reduces to

$$y = \frac{W}{6EIl}[(l^2a' - a'^3)(l - x) - a'(l - x)^3].$$

Now let a' take the successive values $c(r_2 + 1)$, $c(r_2 + 2)$, $c(r_2 + 3)$, . . . $c(r_2 + r - r_2)$, and we have, by summing,

$$\begin{aligned}\Sigma a' &= c(\overline{r_2 + 1} + \overline{r_2 + 2} + \overline{r_2 + 3} + \ldots + r)\\ &= \tfrac{1}{2}c(r - r_2)(r + r_2 + 1),\\ \Sigma(a'^3) &= c^3(\overline{r_2 + 1}^3 + \overline{r_2 + 2}^3 + \overline{r_2 + 3}^3 + \ldots + r^3)\\ &= \tfrac{1}{4}c^3[r^2(r + 1)^2 - r_2^2(r_2 + 1)^2].\end{aligned}$$

Hence (212) becomes

$$\begin{aligned}y = \frac{W}{24EIl}\Big\{&\{2c(r - r_2)(r + r_2 + 1)l^2\\ &- c^3[r^2(r + 1)^2 - r_2^2(r_2 + 1)^2]\}(l - x)\\ &- 2c(r - r_2)(r + r_2 + 1)(l - x)^3\Big\},\end{aligned} \tag{230}$$

which is the deflection due $r - r_2$ equal weights, W, at any point, x, between the r^{th} weight and the right end of the beam; r_2 being the number of full intervals vacant at the left end, and x being not less than cr.

For the $r_1 - r$ equal weights between the point x and the right end of the beam, we employ (209), which reduces to

$$y = \frac{W}{6EIl}[(2l^2a' - 3la'^2 + a'^3)x + (a' - l)x^3].$$

If in this equation a' takes the successive values $c(r+1)$, $c(r+2)$, $c(r+3)$, $c(r+4)$, . . . $c(r+r_1-r)$, then, by summing, we find

$$\begin{aligned}\Sigma a' &= c[(r+1)+(r+2)+(r+3)+\ldots+r_1]\\ &= \tfrac{1}{2}c(r_1-r)(r_1+r+1),\end{aligned}$$

$$\begin{aligned}\Sigma a'^2 &= c^2[(r+1)^2+(r+2)^2+(r+3)^2+\ldots r_1^2]\\ &= \tfrac{1}{6}c^2[r_1(r_1+1)(2r_1+1)-r(r+1)(2r+1)],\end{aligned}$$

$$\begin{aligned}\Sigma a'^3 &= c^3[(r+1)^3+(r+2)^3+(r+3)^3+\ldots r_1^3]\\ &= \tfrac{1}{4}c^3[r_1^2(r_1+1)^2-r^2(r+1)^2],\end{aligned}$$

$$\Sigma a'^0 = r_1 - r.$$

Hence, for this case, (209) becomes

$$\begin{aligned}y = \frac{W}{24EIl}\Big\{&\{4l^2c(r_1-r)(r_1+r+1)\\ &\qquad - 2lc^2[r_1(r_1+1)(2r_1+1)-r(r+1)(2r+1)]\\ &+ c^3[r_1^2(r_1+1)^2-r^2(r+1)^2]\}x\\ &\qquad + [2c(r_1-r)(r_1+r+1)-4(r_1-r)l]x^3\Big\},\end{aligned} \tag{231}$$

which is the deflection due the $r_1 - r$ equal weights on the beam at any point, x, between the left end and the $(r+1)^{\text{th}}$ weight; x not being greater than $c(r+1)$.

Adding the deflections given by (230) and (231), and calling their sum also y, we have

$$\begin{aligned}y = \frac{W}{24EIl}\Big\{&[2c(r_1-r_2)(r_1+r_2+1)-4(r_1-r)l]x^3\\ &\qquad -6c(r-r_2)(r+r_2+1)lx^2\\ &+ \{4l^2c(r_1-r_2)(r_1+r_2+1)+c^3[r_1^2(r_1+1)^2-r_2^2(r_2+1)^2]\\ &- 2lc^2[r_1(r_1+1)(2r_1+1)-r(r+1)(2r+1)]\}x\\ &\qquad - c^3l[r^2(r+1)^2-r_2^2(r_2+1)^2]\Big\},\end{aligned} \tag{232}$$

which is the deflection at any point, x, due the $r_1 - r_2$ equal weights, W; where r_1 denotes the number of intervals between the last weight and the left end of the beam, r a number of full intervals not less than r_2, the number of unloaded intervals at the left end of the beam, nor greater than r_1.

If in (232) we put $c = l \div n$, $x = \frac{1}{2}l$, $r_2 = 0$, $r_1 = n - 1$, and $r = \frac{1}{2}n$ when n is even, but $r = \frac{1}{2}(n - 1)$ when n is odd, we shall find

$$D \left\{ \begin{aligned} &= \frac{Wl^3}{384EIn}(5n^2 - 4), \ n \text{ even}, \\ \\ &= \frac{Wl^3}{384EIn^3}(5n^4 - 4n^2 - 1), \ n \text{ odd}, \end{aligned} \right\} \qquad (233)$$

which is the deflection at the centre due the $r_1 = n - 1$ equal weights, W, covering the beam of n equal intervals ($l \div n$).

EXAMPLES. — Let us take the same 15-inch I-beam we employed in the examples of article 75, for which $I = 691$, $E = 24{,}000{,}000$, $l = 360$ inches. Take 3 weights of 4,500 pounds each, placed at intervals of 60 inches, beginning at one end of the beam; then the deflection at the centre is given by (230) if we put $W = 4{,}500$, $l = 360$, $c = \frac{1}{6}l = 60$, $r_2 = 0$, $r = 3$, $x = \frac{1}{2}l$, $EI = 16{,}584{,}000{,}000$. Thus,

$$y = \frac{4500 \times 360^3}{24 \times 16584000000}\left(2 \times \frac{1}{6} \times 3 \times 4 \times \frac{1}{2} - \frac{1}{6^3} \times 9 \right.$$
$$\left. \times 16 \times \frac{1}{2} - 2 \times \frac{1}{6} \times 3 \times 4 \times \frac{1}{8}\right) = 0.6154 \text{ inch.}$$

If 2 more equal weights are added at the same interval, so as to cover the beam, the central deflection due these last 2 is, by (231), where $r_1 = 5$, $r = 3$,

$$y = \frac{4500 \times 360^3}{24 \times 16584000000}\Big\{4 \times \frac{1}{6}\Big(25 - 9 + 5 - 3\Big)\frac{1}{2}$$

$$- 2 \times \frac{1}{36}\Big(5 \times 6 \times 11 - 3 \times 4 \times 7\Big)\frac{1}{2} + \frac{1}{6^3}\Big(25 \times 36 - 9 \times 16\Big)\frac{1}{2}$$

$$+ 2 \times \frac{1}{6}\Big(25 - 9 + 5 - 3\Big)\frac{1}{8} - 4 \times 2 \times \frac{1}{8}\Big\} = 0.3517 \text{ inch.}$$

If we compute the central deflection due these 5 equal weights by (233), we have $n = 6$, and

$$D = \frac{4500 \times 360^3}{384 \times 16584000000}\Big(\frac{5 \times 36 - 4}{6}\Big) = 0.9671 \text{ inch,}$$

which is the sum of the deflections found by (230) and (231).

Again, if there are 8 weights upon the beam, each equal to $W = 3{,}000$ pounds, at intervals of 40 inches, we have $n = 9$, $l = 360$ inches; and (233) gives the central deflection,

$$D = \frac{3000 \times 360^3}{384 \times 16584000000}\Big(\frac{5 \times 9^4 - 4 \times 9^2 - 1}{9^3}\Big) = 0.97926 \text{ inch.}$$

In these examples the weight has been purposely chosen equal to 75 pounds to the inch for the entire length of the beam, except a half-interval, $(l \div 2n)$, at each end; so that we may compare the results with the central deflection of the same beam, computed by (206) for the continuous uniform load of 75 pounds to the inch, which deflection we have found to be 0.98905 inch.

Now it will be found that the central deflection due the discontinuous load, $(n - 1)W$, at equal intervals, $(l \div n)$, will be less than that due the continuous uniform load, lw, until n becomes infinite, and $W = \frac{lw}{n}$ infinitesimal, when (233) becomes identical with (206).

The greatest difference between the central deflections of these two loads, $(n - 1)W$ and lw, manifestly occurs when $n = 2$; that is, when there is but one weight, and that at the centre, and equal to $W = \frac{lw}{2}$. Equation (233) then becomes $D = \frac{4wl^4}{384EI}$, which is four-fifths of the deflection due lw continuously distributed uniformly, as shown by (206).

From these considerations it appears, that, in practice, the formulæ found in article 75, for a uniform continuous load, are applicable to a uniform load distributed, as above, discontinuously, or by panel weights, each equal to $(lw \div n)$, provided n is large.

But in any case, whether there be many or few intervals, we may find, by means of equation (232), the greatest deflection due any partial or complete load of equal panel weights, W, and the point where it occurs.

For this purpose, differentiate (232) with respect to x, and put $\frac{dy}{dx} = 0$. This gives

$$\begin{aligned}[6c(r_1 - r_2)(r_1 + r_2 + 1) - 12(r_1 - r)l]x^2 \\ - 12c(r - r_2)(r + r_2 + 1)lx \\ + 4l^2c(r_1 - r_2)(r_1 + r_2 + 1) + c^3[r_1^2(r_1 + 1)^2 - r_2^2(r_2 + 1)^2] \\ - 2lc^2[r_1(r_1 + 1)(2r_1 + 1) - r(r + 1)(2r + 1)] = 0, \quad (234)\end{aligned}$$

from which we find

$$x = A \pm \sqrt{A^2 + B}, \qquad (235)$$

where

$$A = \frac{c(r - r_2)(r + r_2 + 1)l}{c(r_1 - r_2)(r_1 + r_2 + 1) - 2(r_1 - r)l},$$

and

$$B = \frac{2lc^2[r_1(r_1 + 1)(2r_1 + 1) - r(r + 1)(2r + 1)] - 4l^2c(r_1 - r_2)(r_1 + r_2 + 1) - c^3[r_1^2(r_1 + 1)^2 - r_2^2(r_2 + 1)^2]}{6c(r_1 - r_2)(r_1 + r_2 + 1) - 12(r_1 - r)l}.$$

Now put cr for x in (234); and find r by trial, easily, since it is an integer, and the point of greatest deflection is approximately known, by inspection, for any given load. Then, having r, compute x in (235), after which the greatest deflection, y, may be found by equation (232).

EXAMPLE 1. — Given the wrought-iron I-beam of article 75, where $I = 691$, $E = 24{,}000{,}000$, $l = 360$ inches; and let there be upon it 5 weights of 4,500 pounds each, at equal intervals of $c = \frac{1}{6}l = 60$ inches. We then have $W = 4{,}500$, $r_1 = 5$, $r_2 = 0$; and, by putting cr for x in equation (234), we find

$$2r^3 - 18r^2 + r + 105 = 0.$$

By trial, we see that $r = 3$, as we should also infer from the symmetrical load. Making $r = 3$ in (235), there results $x = \frac{1}{2}l$. This value of x placed in (232) gives the deflection $y = 0.9671$ inch, as by (233).

EXAMPLE 2. — If on this same beam we have 4,500 pounds at the end of the second and third intervals, we have $W = 4{,}500$, $r_1 = 3$, $r_2 = 1$, $c = \frac{1}{6}l = 60$ inches; and, by putting cr for x in equation (234), we find

$$6r^3 - 48r^2 + 39r + 143 = 0,$$

where, by trial, we see that r lies between 2 and 3. Making $r = 2$ in (235), we find $x = 0.48141l$. With this value of x, (232) gives the maximum deflection due the 2 given weights, $y = 0.48934$ inch.

EXAMPLE 3. — If these 2 equal weights are at the end of the third and fourth intervals, then $r_1 = 4$, $r_2 = 2$, $c = 60$; and we shall find $r = 3$, $x = 0.51859l$, and the maximum deflection, as before, $y = 0.48934$ inch.

SECTION 3.

The Influence of Fixed Ends upon the Deflection of a Beam of Uniform Cross-Section, Supported at its Two Extremities, which are Assumed to be Level, and One or Both of Them Fixed Horizontally or Otherwise. Determination of the End Moments and Points of Contrary Flexure.

77. The influence of the end couples upon the moments due the other forces has already been found, by equation (93), to be

$$M_x = \frac{M_2 - M_1}{l}x + M_1,$$

where M_1 is the left end moment, and M_2 the right end moment, of the fixed beam, Fig. 12.

Wherefore, to find the deflection due these end couples, (187) becomes

$$EI\frac{d^2y}{dx^2} = \frac{M_1 - M_2}{l}x - M_1;$$

giving the first member the positive sign, since M_1 and M_2 are here assumed to have a tendency to deflect the beam upward, and are negative relatively to the moments tending to deflect it downwards.

If $\alpha =$ the slope of the beam at the centre, then $\frac{dy}{dx} = \tan\alpha$ when $x = \frac{1}{2}l$, and the first integration yields

$$EI\left(\frac{dy}{dx} - \tan\alpha\right) = \frac{M_1 - M_2}{2l}\left(x^2 - \frac{l^2}{4}\right) - M_1(x - \tfrac{1}{2}l).$$

Again, since $y = 0$ when $x = 0$,

$$\therefore\ EI(y - x\tan\alpha) = \frac{M_1 - M_2}{2l}\left(\frac{x^3}{3} - \frac{l^2}{4}x\right) - M_1\left(\frac{x^2}{2} - \frac{l}{2}x\right).$$

Also $y = 0$ when $x = l$,

$$\therefore \quad \tan\alpha = -\frac{(M_1 - M_2)l}{24EI}.$$

Therefore

$$y = \frac{1}{6EI}\left\{\frac{M_1 - M_2}{l}(x^3 - l^2x) - 3M_1(x^2 - lx)\right\}, \quad (236)$$

which is the deflection due the end moments in terms of these unknown end moments. Now, since (236) has been found without assuming the ends of the beam tangent to the line drawn through the two points of support, we may suppose M_1 or M_2 to vanish, or to be equal to each other.

If $M_1 = 0$, $$y = \frac{M_2}{6EI}\left(lx - \frac{x^3}{l}\right). \quad (237)$$

If $M_2 = 0$, $$y = \frac{M_1}{6EI}\left(\frac{x^3}{l} - 3x^2 + 2lx\right). \quad (238)$$

If $M_1 = M_2 = M$, $$y = \frac{M}{2EI}(lx - x^2). \quad (239)$$

In order to determine the end moments in particular cases, we must consider the particular mode of loading.

78. Load Continuous and Uniform throughout, $= w$ per Unit of Length, l.—If we add equations (236) and (205), calling the result y, we have

$$y = \frac{1}{24EI}\left\{w(x^4 - 2lx^3 + l^3x) + \frac{M_1 - M_2}{l}(4x^3 - 4l^2x) - 12M_1(x^2 - lx)\right\}, \quad (240)$$

which is the deflection at any point of the uniformly loaded beam of fixed ends.

If the ends are both fixed horizontally (that is, if the tangent to the curve is horizontal at each end of the level beam), we must have $M_1 = M_2$, since the load is uniform. And, differentiating (240),

$$\frac{dy}{dx} = \frac{1}{24EI}\left\{w(4x^3 - 6lx^2 + l^3) + \frac{M_1 - M_2}{l}(12x^2 - 4l^2) - 12M_1(2x - l)\right\}.$$

But, now, $\frac{dy}{dx} = 0$ when $x = 0$,

$$\therefore \quad wl^3 + 12M_1l = 0,$$

$$M_1 = M_2 = -\tfrac{1}{12}wl^2. \qquad (241)$$

Putting this value of the end moments into (240), there results

$$y = \frac{w}{24EI}(l - x)^2x^2, \qquad (242)$$

which is the deflection at any point, x, of a beam with ends fixed horizontally, under a continuous uniform load, w, per unit of length, l.

If $x = \frac{1}{2}l$, we have the central deflection

$$D = \frac{wl^4}{384EI}, \qquad (243)$$

which is one-fifth that due the same load on the same beam with its ends not fixed, as given by (206).

Since $M_1 = M_2$, the total moment due lw at any point, is, by (49) and (93),

$$M_x = \tfrac{1}{2}w(l - x)x + M_1 = w\left(\frac{lx - x^2}{2} - \tfrac{1}{12}l^2\right).$$

And, if we put this moment $M_x = 0$, we shall have x representing the distance from the left end of the beam to the *points of contrary flexure*, as those points are called where the curvature changes from convex to concave upward.

Therefore

$$x^2 - lx + \tfrac{1}{6}l^2 = 0,$$

$$x = 0.21133l \quad \text{or} \quad 0.78867l. \tag{244}$$

79. But if the right end of the beam is fixed horizontally, while the left end is not fixed at all, we have $M_1 = 0$, and (240) becomes

$$y = \frac{1}{24EI}\left\{w(x^4 - 2lx^3 + l^3x) - 4M_2\frac{x^3 - l^2x}{l}\right\}; \tag{245}$$

and

$$\frac{dy}{dx} = \frac{1}{24EI}\left\{w(4x^3 - 6lx^2 + l^3) - \frac{M_2}{l}(12x^2 - 4l^2)\right\}$$

equal to 0 when $x = l$.

$$\therefore \quad M_2 = -\tfrac{1}{8}wl^2. \tag{246}$$

Hence, from (245),

$$y = \frac{w}{48EI}(2x^4 - 3lx^3 + l^3x), \tag{247}$$

which is the deflection at any point of a beam horizontally fixed at one end, and simply supported at the other, under a uniform load w per unit of length, l; x to be measured from the unfixed end.

Since, now, $M_1 = 0$, the total moment due lw at any point is, from (49) and (93),

$$M_x = \tfrac{1}{2}w(l - x)x + \frac{M_2x}{l} = w(\tfrac{1}{2}lx - \tfrac{1}{2}x^2 - \tfrac{1}{8}lx).$$

If $M_x = 0$,

$$x = \tfrac{3}{4}l, \tag{248}$$

which is the distance of the point of contrary flexure from the free end of the beam, under the load lw uniformly distributed continuously.

EXAMPLES. — Suppose the wrought-iron 15-inch I-beam 30 feet in length, of the examples in article 75, to be fixed horizontally at both ends, and loaded uniformly with 75 pounds to each inch of its length; what is the deflection 10 feet from either end? We now have $I = 691$, $E = 24{,}000{,}000$, $l = 360$, $x = \tfrac{1}{3}l$ or $\tfrac{2}{3}l$. Hence (242) gives the deflection

$$y = \frac{75 \times 360^4 \times 4}{24 \times 16584000000 \times 81} = 0.1563 \text{ inch.}$$

At the centre, where $x = \tfrac{1}{2}l$, (242) gives

$$y = \frac{75 \times 360^4 \times 1}{24 \times 16584000000 \times 16} = 0.19781 \text{ inch,}$$

which is one-fifth of that given by (206) for beam with free ends.

If only one end of the beam is fixed, (247) gives,

When $x = \tfrac{1}{3}l$, $\quad y = \dfrac{75 \times 360^4 \times 10}{24 \times 16584000000 \times 81} = 0.39074$ inch.

$x = \tfrac{1}{2}l$, $\quad y = \dfrac{75 \times 360^4 \times 1}{24 \times 16584000000 \times 8} = 0.39563$ inch.

$x = \tfrac{2}{3}l$, $\quad y = \dfrac{75 \times 360^4 \times 7}{24 \times 16584000000 \times 81} = 0.27352$ inch.

$x = 151.7846$ inches, $y = 0.41141$ inch, a maximum.

80. Deflection of a Beam fixed Horizontally at Both Ends, due to a Concentrated Load, W, placed at the Horizontal Distance a' from the Left End of the Beam.—From equations (40), (187), and (93), we have the total moment due W when x is not greater than a',

$$M_x = -EI\frac{d^2y}{dx^2} = W\frac{l-a'}{l}x - \frac{M_1 - M_2}{l}x + M_1, \quad (249)$$

$$EI\frac{d^2y}{dx^2} = \frac{1}{l}[W(a'-l) + M_1 - M_2]x - M_1.$$

Integrating, as in article 73, $\frac{dy}{dx} = \tan\alpha$ when $x = a'$,

$$\therefore \quad EI\left(\frac{dy}{dx} - \tan\alpha\right)$$
$$= \frac{W(a'-l) + M_1 - M_2}{2l}(x^2 - a'^2) - M_1(x - a'). \quad (250)$$

Again, $y = 0$ when $x = 0$,

$$\therefore \quad EI(y - x\tan\alpha)$$
$$= \frac{W(a'-l) + M_1 - M_2}{2l}\left(\frac{x^3}{3} - a'^2x\right) - M_1\left(\frac{x^2}{2} - a'x\right). \quad (251)$$

But when x is not less than a', use (43) with (93) and (187), giving

$$M_x = -EI\frac{d^2y}{dx^2} = \frac{Wa'}{l}(l - x) - \frac{M_1 - M_2}{l}x + M_1, \quad (252)$$

$$EI\frac{d^2y}{dx^2} = \frac{1}{l}(Wa' + M_1 - M_2)x - (Wa' + M_1).$$

$\frac{dy}{dx} = \tan\alpha$ when $x = a'$,

$$\therefore \quad EI\left(\frac{dy}{dx} - \tan\alpha\right)$$
$$= \frac{Wa' + M_1 - M_2}{2l}(x^2 - a'^2) - (Wa' + M_1)(x - a'). \quad (253)$$

$y = 0$ when $x = l$,

$$\therefore\quad EI[y - (x - l)\tan\alpha]$$

$$= \frac{Wa' + M_1 - M_2}{2l}\left\{\frac{x^3 - l^3}{3} - a'^2(x - l)\right\}$$

$$- (Wa' + M_1)\left\{\frac{x^2 - l^2}{2} - a'(x - l)\right\}. \quad (254)$$

Now y in (251) is equal to y in (254) when $x = a'$; therefore, from (251) and (254), we find

$$\tan\alpha = \frac{1}{EIl}[Wa'(\tfrac{2}{3}a'^2 + \tfrac{1}{3}l^2 - a'l)$$

$$+ M_1(\tfrac{1}{3}l^2 + \tfrac{1}{2}a'^2 - a'l) - M_2(\tfrac{1}{2}a'^2 - \tfrac{1}{6}l^2)]. \quad (255)$$

But in (250) we now have $\dfrac{dy}{dx} = 0$ when $x = 0$,

$$\therefore\quad \tan\alpha = \frac{1}{EIl}\left\{Wa'^2\frac{a' - l}{2} + M_1\left(\frac{a'^2}{2} - a'l\right) - M_2\frac{a'^2}{2}\right\}. \quad (256)$$

Also, in (253) $\dfrac{dy}{dx} = 0$ when $x = l$,

$$\therefore\quad \tan\alpha = \frac{1}{EIl}[W(\tfrac{1}{2}a'^3 + \tfrac{1}{2}a'l^2 - a'^2l)$$

$$+ \tfrac{1}{2}(M_1 - M_2)(a'^2 - l^2) + M_1l(l - a')]. \quad (257)$$

From (255), (256), and (257), we find

$$M_1 = -\frac{W}{l^2}(l - a')^2a', \quad (258)$$

$$M_2 = -\frac{W}{l^2}(l - a')a'^2, \quad (259)$$

which are the end moments developed by the weight W in any position, a'.

If the weight W is at the centre, $a' = \frac{1}{2}l$, and

$$M_1 = M_2 = -\tfrac{1}{8}Wl. \qquad (260)$$

Eliminating M_1, M_2, and $\tan\alpha$ from equation (251), we find, x not being greater than a',

$$y = \frac{W}{6EIl^3}[(3a'^2l - l^3 - 2a'^3)x^3 + (3a'^3l - 6a'^2l^2 + 3a'l^3)x^2], \qquad (261)$$

which is the deflection at any point between the weight W and the left end of the fixed beam with ends horizontal.

Again, eliminating M_1, M_2, and $\tan\alpha$ from (254), we find, x not being less than a',

$$y = \frac{W}{6EIl^3}[(3a'^2l - 2a'^3)x^3 + (3a'^3l - 6a'^2l^2)x^2 + 3a'^2l^3x - a'^3l^3], \qquad (262)$$

which is the deflection due W at any point between W and the right-hand end of the fixed beam.

If $x = a' = \frac{1}{2}l$, both (261) and (262) reduce to

$$D = \frac{Wl^3}{192EI}, \qquad (263)$$

which is the central deflection when the weight W is at the centre of the fixed beam, and is one-fourth of that due the same load on the same beam with its ends not fixed, as seen by equation (211).

To find one point of contrary flexure, we put $M_x = 0$ in equation (249), and, after eliminating M_1 and M_2, have

$$x = \frac{a'l(l - a')^2}{2a'^3 - 3a'^2l + l^3}. \qquad (264)$$

If $a' = \frac{1}{2}l$,

$$x = \tfrac{1}{4}l. \tag{265}$$

For the other point of contrary flexure, put $M_x = 0$ in (252), and the result is

$$x = \frac{l(2l - a')}{3l - 2a'}. \tag{266}$$

If $a' = \frac{1}{2}l$,

$$x = \tfrac{3}{4}l. \tag{267}$$

From (265) and (267) it appears that when the concentrated load, W, is at the centre of the fixed beam, the points of contrary flexure are each midway between the centre and end of the beam.

81. If the beam is fixed horizontally at the right-hand end, but only supported at the left end, we have $M_1 = 0$; while M_2 may be found from (255) and (257), since the condition that $\frac{dy}{dx} = 0$ when $x = 0$, on which (256) depends, does not now exist.

$$\therefore\ M_2 = \frac{W}{2l^2}(a'^2 - l^2)a'. \tag{268}$$

This value of M_2 placed in either (255) or (257), while $M_1 = 0$, gives

$$\tan\alpha = \frac{W}{EIl^3}(a'^3l^2 - a'^2l^3 + \tfrac{1}{4}a'l^4 - \tfrac{1}{4}a'^5), \tag{269}$$

which is the tangent of the angle of inclination of the beam at any point where the load W may be, while only the right end is fixed; a' to be measured from the free end. With these values of M_1, M_2, and $\tan\alpha$ substituted in equation (251), we find

$$y = \frac{W}{12EIl^3}[(3a'l^2 - a'^3 - 2l^3)x^3$$
$$+ (3a'^3l^2 - 6a'^2l^3 + 3a'l^4)x], \tag{270}$$

which is the deflection at any point between the weight W and the unfixed end of the beam, from which end a' and x are to be measured.

In the same manner, from equation (254) we find, x being not less than a',

$$y = \frac{W}{12EIl^3}[(3a'l^2 - a'^3)x^3 - 6a'l^3x^2 + (3a'^3l^2 + 3a'l^4)x - 2a'^3l^3], \quad (271)$$

which is the deflection at any point between the weight W and the horizontally fixed end of the beam; a' and x being measured from the free end.

If in either (270) or (271) we put $x = a' = \frac{1}{2}l$, we have the central deflection

$$D = \frac{7Wl^3}{768EI} \quad (272)$$

due the concentrated load W applied at the centre of the beam horizontally fixed at one end.

If we differentiate (270), and put $\frac{dy}{dx} = 0$, we shall find

$$x = \pm l\left(\frac{2a'^2l - a'^3 - a'l^2}{3a'l^2 - a'^3 - 2l^3}\right)^{\frac{1}{2}}, \quad (273)$$

which is the point of maximum deflection between the weight W and the free end.

If the weight is at the centre, $a' = \frac{1}{2}l$, and

$$x = \pm l\sqrt{\tfrac{1}{5}} = 0.44721l. \quad (274)$$

In a similar manner, differentiating equation (271), and putting $\frac{dy}{dx} = 0$, we find

$$x = \frac{l^3 + a'^2l}{3l^2 - a'^2}, \quad (275)$$

the point of maximum deflection between W and fixed end, where x cannot be less than a'; that is, a' in this formula cannot be greater than x.

Putting a' for x in (275), we may find easily, by trial, that $a' = 0.414213l$ is the greatest value a' can have in this case of a maximum value of y between the weight W and the horizontally fixed end of the beam.

The point of contrary flexure may be found from (252) by putting $M_x = 0$, $M_1 = 0$, and M_2 as in (268). This substitution gives

$$x = \frac{2l^3}{3l^2 - a'^2}. \tag{276}$$

If the weight W is at the centre, $a' = \frac{1}{2}l$, and

$$x = \tfrac{8}{11}l, \tag{277}$$

which is the distance of the point of contrary flexure from the free end of the beam.

If $a' = 0$, $x = \frac{2}{3}l$; and if $a' = l$, $x = l$: which are the limits to the range of the point of contrary flexure, for a concentrated load W, on a beam fixed horizontally at one end, and free at the other; x being measured from the free end.

Examples.—Take the 15-inch I-beam of article 75, and suppose it bears a concentrated load $W = 27{,}000$ pounds, and that both ends are fixed horizontally. We have, as before, $I = 691$, $E = 24{,}000{,}000$, $l = 360$ inches.

When W is at the centre, what is the deflection halfway between the centre and either end of the beam?

Put $a' = \frac{1}{2}l$ and $x = \frac{1}{4}l$ in (261), or $a' = \frac{1}{2}l$ and $x = \frac{3}{4}l$ in (262), and find

$$y = \frac{27000 \times 360^3}{6 \times 16584000000}\left[\left(\tfrac{3}{4} - 1 - \tfrac{2}{8}\right)\tfrac{1}{64} + \left(\tfrac{3}{8} - \tfrac{6}{4} + \tfrac{3}{2}\right)\tfrac{1}{16}\right] = 0.19781 \text{ inch.}$$

At the centre the deflection is, by (263),

$$D = \frac{27000 \times 360^3}{192 \times 16584000000} = 0.39562 \text{ inch},$$

which is one-fourth of $1.58248 =$ the deflection due the same load on the same beam with free ends. And this 1.58248 is, again, eight-fifths of 0.98905, the deflection found by (206) for the same load continuously distributed uniformly over the same beam with free ends.

The points of contrary flexure are given, by (265) and (267), at 90 inches and 270 inches from either end. Now, since the deflection at the quarter-points is just one-half that at the centre, it follows that, in this case, the end of the neutral line, the point of contrary flexure, and the centre are in the same straight line.

When W is at the distance $a' = \frac{1}{4}l$ from the left end of the beam, what is the maximum deflection?

Differentiating (262), and putting $\frac{dy}{dx} = 0$, we find

$$x = 0.4l,$$

$$\therefore \quad y = 0.2136 \text{ inch.}$$

Or, if $a' = \frac{3}{4}l$, we find in the same way, from (261),

$$x = 0.6l,$$

$$\therefore \quad y = 0.2136 \text{ inch.}$$

If $a' = \frac{3}{4}l$, the points of contrary flexure are, by (264) and (266), $x = \frac{3}{10}l$, $x = \frac{5}{6}l$.

But if $a' = \frac{1}{4}l$,

$$x = \frac{1}{6}l, \qquad x = \frac{7}{10}l.$$

Let us now suppose that this beam is fixed horizontally at the right-hand end, but is simply supported at the left end.

When $W = 27{,}000$ pounds is at the centre, what is the deflection at the quarter-points?

Putting $a' = \frac{1}{2}l$, and $x = \frac{1}{4}l$, we find, from (270),

$$y = 0.5316 \text{ inch.}$$

But if $a' = \frac{1}{2}l$, and $x = \frac{3}{4}l$, (271) gives

$$y = 0.3091 \text{ inch.}$$

W remaining at the centre, the central deflection is, from (272), $D = 0.69234$ inch.

Also, from (274) and (270), the maximum deflection due W at the centre is $y = 0.70769$ inch.

If we place the weight $W = 27{,}000$ pounds at the distance $a' = 0.414213l$ from the free end for the maximum value of the deflection y, we shall find, by (275), $x = a' = 0.414213l$; and from (270) or (271), $y = 0.74534$ inch, which is the greatest deflection W can produce on this beam, since it is at the point of maximum deflection.

Putting $a' = 0.414213l$ in (276), we find

$$x = 0.707106l,$$

the point of contrary flexure when W is at the lowest point of the beam fixed horizontally at one end; x to be measured from the free end.

82. Any Number, $r_1 - r_2$, Equal Weights, W, placed at Equal Intervals, c, along the Beam; the First Weight being $(r_2 + 1)$ Intervals from the Left End, and the Beam being fixed Horizontally at Both Ends.—Let $r - r_2$ denote the number of equal weights, and r equal the number of full intervals, between the point x and the origin or left end of the beam, Fig. 12; then $r_1 - r$ = the number of weights between the point x and the right end, if any.

The deflection at the point x due any one of the $r - r_2$ equal weights, W, is given by equation (262). Let d' in that equation take the successive values $c(r_2 + 1)$, $c(r_2 + 2)$, $c(r_2 + 3)$, . . . $c(r_2 + r - r_2)$; then, by summing, we have

$$\Sigma d'^2 = c^2[(r_2 + 1)^2 + (r_2 + 2)^2 + (r_2 + 3)^2 + \ldots + r^2]$$
$$= \frac{c^2}{6}[r(r + 1)(2r + 1) - r_2(r_2 + 1)(2r_2 + 1)],$$

$$\Sigma d'^3 = c^3[(r_2 + 1)^3 + (r_2 + 2)^3 + (r_2 + 3)^3 + \ldots + r^3]$$
$$= \frac{c^3}{4}[r^2(r + 1)^2 - r_2^2(r_2 + 1)^2],$$

which values, put in the place of d'^2 and d'^3 in (262), give

$$y = \frac{W}{6EIl^3}\Big\{\{\tfrac{1}{2}c^2[r(r+1)(2r+1)-(r_2+1)(2r_2+1)r_2]l-\tfrac{1}{2}c^3[r^2(r+1)^2-(r_2+1)^2r_2^2]\}x^3$$
$$+\{\tfrac{3}{4}c^3[r^2(r+1)^2-(r_2+1)^2r_2^2]l-c^2[r(r+1)(2r+1)-(r_2+1)(2r_2+1)r_2]l^2\}x^2$$
$$+\tfrac{1}{2}c^2[r(r+1)(2r+1)-(r_2+1)(2r_2+1)r_2]l^3x-\tfrac{1}{4}c^3[r^2(r+1)^2-(r_2+1)^2r_2^2]l^3\Big\}, \quad (278)$$

which is the deflection due $r - r_2$ equal weights at any point, x, between the r^{th} interval and the right end of the beam having both ends horizontally fixed; x being not less than cr.

If in (278) we make $x = cr$, and $r_2 = 0$, then

$$y = \frac{Wc^3r^2(r + 1)}{24EIl^3}[(3r + 1)l^3 - 4cr(2r + 1)l^2$$
$$+ c^2r^2(7r + 5)l - 2c^3r^3(r + 1)], \quad (279)$$

which is the deflection, at the r^{th} weight, due r equal weights, W, along the left end of the beam at equal intervals, c.

Again, the deflection at the point x due any one of the $r_1 - r$ equal weights beyond the point x, is given by equation (261).

Let a' in that equation take the successive values $c(r+1)$, $c(r+2)$, $c(r+3)$, . . . cr_1; then summing as in article 76, and putting the values of $\Sigma a'^0$, $\Sigma a'$, $\Sigma a'^2$, $\Sigma a'^3$, into equation (261), we find

$$y = \frac{W}{6EIl^3}\Big\{\{\tfrac{1}{2}c^2[r_1(r_1+1)(2r_1+1)-(r+1)(2r+1)r]l \\ -(r_1-r)l^3-\tfrac{1}{4}c^3[r_1^2(r_1+1)^2-(r+1)^2r^2]\}x^3 \\ +\{\tfrac{3}{4}c^3[r_1^2(r_1+1)^2-(r+1)^2r^2]l-c^2[r_1(r_1+1)(2r_1+1) \\ -(r+1)(2r+1)r]l^2+\tfrac{3}{2}c(r_1-r)(r_1+r+1)l^3\}x^2\Big\}, \qquad (280)$$

which is the deflection due $r_1 - r$ equal weights, W, at any point, x, between the $(r+1)^{\text{th}}$ interval and the left end of the beam; x being not greater than $c(r+1)$.

Adding equations (278) and (280), and calling the result y still, we have

$$y = \frac{W}{6EIl^3}\Big\{\{\tfrac{1}{2}c^2[r_1(r_1+1)(2r_1+1)-(r_2+1)(2r_2+1)r_2]l \\ -(r_1-r)l^3-\tfrac{1}{4}c^3[r_1^2(r_1+1)^2-(r_2+1)^2r_2^2]\}x^3 \\ +\{\tfrac{3}{4}c^3[r_1^2(r_1+1)^2-(r_2+1)^2r_2^2]l-c^2[r_1(r_1+1)(2r_1+1) \\ -(r_2+1)(2r_2+1)r_2]l^2+\tfrac{3}{2}c(r_1-r)(r_1+r+1)l^3\}x^2 \\ +\tfrac{1}{2}c^2[r(r+1)(2r+1)-(r_2+1)(2r_2+1)r_2]l^3x \\ -\tfrac{1}{4}c^3[r^2(r+1)^2]-(r_2+1)^2r_2^2)l^3\Big\}, \qquad (281)$$

which is the deflection due all the $r_1 - r_2$ equal weights at any point, x, between the r^{th} and the $(r+1)^{\text{th}}$ intervals; x being not less than cr, nor greater than $c(r+1)$, while r here is not greater than r_1, nor less than r_2.

Beam fixed horizontally at both ends. If we now suppose the beam divided into n full intervals, each $= c = \dfrac{l}{n}$, and a

weight, W, at each point of division; and further, if we require the central deflection due such a load, we have $x = \frac{1}{2}l$, $c = \frac{l}{n}$, $r_1 = n - 1$, $r_2 = 0$, $r = \frac{1}{2}n$ when n is even, but $r = \frac{1}{2}(n - 1)$ when n is odd.

Placing these values in (281), we obtain

$$\left.\begin{aligned} D &= \frac{Wl^3 n}{384EI}, \qquad n \text{ even}, \\ D &= \frac{Wl^3(n^4 - 1)}{384EIn^3}, \; n \text{ odd}, \end{aligned}\right\} \tag{282}$$

which is the deflection at the centre due the $r_1 = n - 1$ equal weights, W, covering the beam of n equal intervals, $\frac{l}{n}$; beam fixed horizontally at both ends.

The end moments, M_1, M_2, due a single weight, W, are given by (258) and (259), which reduce to

$$M_1 = -\frac{W}{l^2}(d'l^2 - 2d'^2l + d'^3),$$

$$M_2 = -\frac{W}{l^2}(d'^2l - d'^3).$$

Now let d' take the successive values $c(r_2 + 1)$, $c(r_2 + 2)$, $c(r_2 + 3)$, . . . $c(r_2 + r_1 - r_2)$, so that we have

$$\Sigma d'^0 = r_1 - r_2,$$

$$\begin{aligned} \Sigma d' &= c(\overline{r_2 + 1} + \overline{r_2 + 2} + \overline{r_2 + 3} + \ldots + r_1) \\ &= \tfrac{1}{2}c(r_1 - r_2)(r_1 + r_2 + 1), \end{aligned}$$

$$\begin{aligned} \Sigma d'^2 &= c^2[(r_2 + 1)^2 + (r_2 + 2)^2 + (r_2 + 3)^2 + \ldots + r_1^2] \\ &= \tfrac{1}{6}c^2[r_1(r_1 + 1)(2r_1 + 1) - r_2(r_2 + 1)(2r_2 + 1)], \end{aligned}$$

$$\Sigma a'^3 = c^3[(r_2 + 1)^3 + (r_2 + 2)^3 + (r_2 + 3)^3 + \ldots + r_1^3]$$
$$= \tfrac{1}{4}c^3[r_1^2(r_1 + 1)^2 - r_2^2(r_2 + 1)^2],$$

$$\therefore \quad M_1 = \frac{-W}{l^2}\{\tfrac{1}{2}c(r_1 - r_2)(r_1 + r_2 + 1)l^2$$
$$- \tfrac{1}{3}c^2[r_1(r_1 + 1)(2r_1 + 1) - r_2(r_2 + 1)(2r_2 + 1)]l$$
$$+ \tfrac{1}{4}c^3[r_1^2(r_1 + 1)^2 - r_2^2(r_2 + 1)^2]\}, \quad (283)$$

$$M_2 = \frac{-W}{l^2}\{\tfrac{1}{6}c^2[r_1(r_1 + 1)(2r_1 + 1) - r_2(r_2 + 1)(2r_2 + 1)]l$$
$$- \tfrac{1}{4}c^3[r_1^2(r_1 + 1)^2 - r_2^2(r_2 + 1)^2]\}, \quad (284)$$

which are the end moments due $r_1 - r_2$ equal weights, W; both ends of beams fixed horizontally.

The greatest deflection due $r_1 - r_2$ equal weights, W, placed at equal consecutive intervals anywhere along the beam, may be found by the following method:—

If in equation (281) we provisionally make $x = cr$, we shall have y_r. Then, putting $r + 1$ for r in this value of y_r, we find y_{r+1}; and therefore

$$\Delta y = y_{r+1} - y_r.$$

Now, by making $\Delta y = 0$, we obtain a value of r the integral part of which, not less than r_2 nor greater than r_1, will be the value of r in (281) when y is a maximum. Then, differentiating (281), and putting $\frac{dy}{dx} = 0$, we find a value of x which renders y a maximum.

Although this solution is rigorous, it need not often be employed, since (281) gives the deflection at as many points as we please, and a close approximation to the greatest value of y may be found by a few trials. An example will be given.

any point, x, between the r^{th} interval and the horizontally fixed end of the beam is given by (271), provided we put therein

For a',

$$\tfrac{1}{2}c(r - r_2)(r + r_2 + 1).$$

For a'^3,

$$\tfrac{1}{4}c^3[r^2(r + 1)^2 - r_2^2(r_2 + 1)^2].$$

But if the first weight is at the distance $c(r + 1)$ from the free end, and if there are $r_1 - r$ equal weights at equal intervals, c, beyond, then the deflection at any point, x, between the $(r + 1)^{\text{th}}$ weight and the free end of the beam is given by equation (270) if there we substitute

For a'^0,

$$r_1 - r.$$

For a',

$$\tfrac{1}{2}c(r_1 - r)(r_1 + r + 1).$$

For a'^2,

$$\tfrac{1}{6}c^2[r_1(r_1 + 1)(2r_1 + 1) - r(r + 1)(2r + 1)].$$

For a'^3,

$$\tfrac{1}{4}c^3[r_1^2(r_1 + 1)^2 - r^2(r + 1)^2].$$

If, then, we add the two deflections thus derived from (271) and (270), we shall have

$$
\begin{aligned}
y = \frac{W}{12EIl^3}\Big\{ & \{\tfrac{3}{2}c(r_1 - r_2)(r_1 + r_2 + 1)l^2 \\
& - \tfrac{1}{4}c^3[r_1^2(r_1 + 1)^2 - r_2^2(r_2 + 1)^2] - 2(r_1 - r)l^3\}x^3 \\
& \qquad - 3c(r - r_2)(r + r_2 + 1)l^3x^2 \\
+ & \{\tfrac{3}{4}c^3[r_1^2(r_1 + 1)^2 - (r_2 + 1)^2r_2^2]l^2 + \tfrac{3}{2}c(r_1 - r_2)(r_1 + r_2 + 1)l^4 \\
& - c^2[r_1(r_1 + 1)(2r_1 + 1) - (r + 1)(2r + 1)r]l^3\}x \\
& \qquad - \tfrac{1}{2}c^3[r^2(r + 1)^2 - (r_2 + 1)^2r_2^2]l^3\Big\}, \qquad (292)
\end{aligned}
$$

which is the deflection due all the $r_1 - r_2$ equal weights at any point, x, between the r^{th} and the $(r+1)^{\text{th}}$ points of division; x being not less than cr nor greater than $c(r+1)$, but r from r_2 to r_1.

In this case, where $M_1 = 0$, M_2 is derived from (268), which reduces to $M_2 = \dfrac{W}{2l^2}(a'^3 - a'l^2)$.

For a' put

$$\tfrac{1}{2}c(r_1 - r_2)(r_1 + r_2 + 1).$$

For a'^3 put

$$\tfrac{1}{4}c^3[r_1^2(r_1+1)^2 - r_2^2(r_2+1)^2].$$

We then have

$$M_2 = \frac{W}{2l^2}\{\tfrac{1}{4}c^3[r_1^2(r_1+1)^2 - r_2^2(r_2+1)^2] - \tfrac{1}{2}c(r_1 - r_2)(r_1 + r_2 + 1)l^2\}, \quad (293)$$

which is the end moment due $r_1 - r_2$ equal weights, W, uniformly distributed at equal intervals, c, on any part of the beam fixed horizontally at one end and simply supported at the other; r_1 and r_2 to be counted from the free end.

84. Deflection, End Moments, and Points of Contrary Flexure, due a Partial Uniform Load continuously distributed, when Both Ends of the Beam are fixed Horizontally. — We might proceed in this case as in article 74, using equations (53), (187), and (93); but, as the process is tedious, we employ the following method instead, utilizing results already obtained.

Let n denote, as heretofore, the whole number of intervals, each equal to $(l \div n)$. Let r_2 denote a certain part of n, which we will call $\dfrac{a}{l}n$; let $r_1 = \dfrac{a+b}{l}n$, where neither a nor $a+b$ can exceed l.

$$c = \frac{l}{n}.$$

Now, for a uniform continuous load we must have in the values of M_1 and M_2, equations (290) and (291), n, r_2, and r_1 infinite, and W infinitesimal; so that we must put $nW = w'l$ if w' = the weight per unit of the length.

Making these substitutions in (290) and (291), they become

$$M_1 = \frac{w'}{12l^2}\{8[(a+b)^3 - a^3]l$$
$$- 6[(a+b)^2 - a^2]l^2 - 3[(a+b)^4 - a^4]\}, \quad (294)$$

$$M_2 = \frac{w'}{12l^2}\{3[(a+b)^4 - a^4] - 4[(a+b)^3 - a^3]l\}, \quad (295)$$

which are the end moments due the uniform continuous load w' per unit on the length b, measured to the right from a point at the distance a from the left end of the beam fixed horizontally at both extremities.

Now, if $a = 0$ (that is, if the continuous uniform load begins at the left end, and extends over the length b), equations (294) and (295) reduce to

$$M_1 = \frac{w'b^2}{12l^2}(8bl - 3b^2 - 6l^2), \quad (296)$$

$$M_2 = \frac{w'b^2}{12l^2}(3b^2 - 4bl), \quad (297)$$

which are the end moments due the continuous uniform load w' per unit, on the length b, measured from the left end.

It may be noted here, that if in (296) and (297), while $a = 0$, we suppose $b = l$, these values of M_1 and M_2 become each equal to $-\frac{1}{12}w'l^2$, which accords with equation (241) for the fully loaded beam.

Let us now put the values of M_1 and M_2, as given by (294) and (295), into equation (236); we shall then have

$$y = \frac{w'}{24EIl^3}\Big\{\{4[(a+b)^3 - a^3]l$$
$$- 2[(a+b)^2 - a^2]l^2 - 2[(a+b)^4 - a^4]\}(x^3 - l^2x)$$
$$- \{8[(a+b)^3 - a^3]l^2 - 6[(a+b)^2 - a^2]l^3$$
$$- 3[(a+b)^4 - a^4]l\}(x^2 - lx)\Big\}, \qquad (298)$$

which is that part of the deflection due to the influence of the end moments, the beam horizontally fixed at both ends being loaded with w' per unit for any part, b, of the beam's length, l; x varying from 0 to l.

If x be now restricted so as not to exceed a, and we add y in (298) to y in (225), the sum will be the deflection due $w'b$ at any point between the origin and the beginning of the partial continuous uniform load $w'b$.

If again we limit x between the values a and $a + b$, and add the values of y in equations (298) and (226), the sum will be the deflection due in $w'b$ at any point of the loaded portion b.

Finally, by making x not less than $a + b$ in (298), and adding that equation to (227), the sum of the second members will be the deflection due $w'b$ at any point between the right-hand end of the beam and the load $w'b$.

It is evident, that, by assigning the proper values to a and b, we may place the load anywhere upon the beam, and give it any magnitude not exceeding $w'l$. Also, we may put many partial uniform continuous loads, w_1b_1, w_2b_2, w_3b_3, etc., upon the beam, by so choosing the values of a_1, a_2, a_3, etc., b_1, b_2, b_3, etc., that the partial loads shall take desired positions, whether they are required to be equal to each other, or to overlap, or to have intervals between them.

But it is not necessary to formulate the deflection for such totals here.

It remains to find the points of contrary flexure for partial continuous uniform loads, $w'b$, when the beam is fixed horizontally at both ends.

If there is a point of contrary flexure between the left end of the beam and the beginning of the partial load (that is, within the length a), we use equations (53) and (93), giving

$$M_x = w'b\frac{l - a - \frac{1}{2}b}{l}x - \frac{M_1 - M_2}{l}x + M_1.$$

If $M_x = 0$,

$$x = \frac{M_1 l}{M_1 - M_2 - w'b(l - a - \frac{1}{2}b)}, \qquad (299)$$

where the values of M_1 and M_2 are to be taken from (294) and (295), and x cannot be greater than a. Should (299) yield a value of x either negative or greater than a, there is no point of contrary flexure in the part a.

For the loaded part of the beam b, we have equations (55) and (93), giving

$$M_x = w'b\frac{l - a - \frac{1}{2}b}{l}x - \frac{1}{2}w'(x - a)^2 - \frac{M_1 - M_2}{l}x + M_1 = 0,$$

$$\therefore \quad x = \varepsilon \pm \sqrt{\frac{2M_1}{w'} - a^2 + \varepsilon^2}, \qquad (300)$$

where $\varepsilon = \frac{b(l - a - \frac{1}{2}b)}{l} - \frac{M_1 - M_2}{w'l} + a.$

M_1 and M_2 are given by (294) and (295).

When, in (300), either value of x is less than a or greater than $a + b$, it must be rejected; and when both values of x are in this condition, there is no point of contrary flexure in the loaded part b.

In finding the point of contrary flexure between the right end of the beam and the load $w'b$, we employ equations (57) and (93); taking, as before, the values of M_1 and M_2 from (294) and (295). Thus,

$$M_x = w'b(a + \tfrac{1}{2}b)\frac{l - x}{l} - \frac{M_1 - M_2}{l}x + M_1 = 0,$$

$$\therefore \quad x = \frac{M_1 l + w'b(a + \frac{1}{2}b)l}{M_1 - M_2 + w'b(a + \frac{1}{2}b)}. \qquad (301)$$

Equations (299) and (301) show that there can be but one point of contrary flexure between either end of the beam and the adjacent end of the load, while (300) indicates that there may be two such points within the length b covered by the uniform load $w'b$.

85. Partial or Full Continuous Uniform Load, $w'b$, on any Portion of a Beam fixed Horizontally at the Right End, but simply Supported at the Left.—Proceeding as in article 84, we make $c = \frac{l}{n}$, $r = \frac{a}{l}n$, $r_1 = \frac{a + b}{l}n$, and substitute these values in (293), which, when n is infinite, and W infinitesimal and $= \frac{w'l}{n}$, becomes

$$M_2 = \frac{w'}{8l^2}\{(a + b)^4 - a^4 - 2l^2[(a + b)^2 - a^2]\}, \qquad (302)$$

which is the moment at the fixed end due the uniform continuous load, $w'b$, anywhere on the beam. Here, if $a = 0$, and $b = l$, the beam is fully covered by the load, and $M_2 = -\frac{1}{8}w'l^2$, in agreement with equation (246).

If in (302) $a = 0$, we have as the moment at the fixed end, when the partial load $w'b$, begins at the free end,

$$M_2 = \frac{w'}{8l^2}(b^4 - 2b^2l^2). \qquad (303)$$

Substituting the value of M_2 as given by (302), in equation (237), we obtain

$$y = \frac{w'}{48EIl^3}\left\{\{(a+b)^4-a^4-2l^2[(a+b)^2-a^2]\}(l^2x-x^3)\right\}, \quad (304)$$

which is the deflection due the end moment M_2 when $M_1 = 0$, and the load is $w'b$ in any position; x varying from 0 to l.

If, as in article 84, x be now limited so as not to exceed a, and we add y in (304) to y in (225), the algebraic sum will be the deflection due $w'b$ at any point, x, between the free end of the beam and the beginning of the load $w'b$.

If, again, x be limited between the values a and $a + b$, and we add algebraically the values of y in equations (304) and (226), the result will be the deflection due $w'b$ at any point, x, of the loaded portion b.

Also, by making x not less than $a + b$ in (304), and adding that equation to (227), the sum of the second members will be the deflection due $w'b$ at any point, x, between the right or fixed end of the beam and the load $w'b$; x measured, as usual, from the free end of the beam.

The point of contrary flexure for the beam fixed horizontally at one end and simply supported at the other, which is taken as the origin, is found for a partial continuous uniform load, $w'b$, by means of equations (299), (300), and (301) for their respective cases, by putting $M_1 = 0$, and taking M_2 from (302).

86. Examples illustrating Articles 82, 83. — For the sake of comparing the deflection of the same beam when one or both its ends are fixed, with its deflection when both ends are simply supported, we further consider the 15-inch rolled wrought-iron I-beam of 30 feet clear span, whose moment of inertia $I = 691$, and whose modulus of elasticity $E = 24{,}000{,}000$, as given in article 75.

1st, Take 3 weights, of 4,500 pounds each, placed at intervals of 60 inches, beginning at the left end of the beam fixed horizontally at both ends; then the deflection at the centre is given by (278) if we put $W = 4{,}500$ pounds, $l = 360$ inches, $c = \frac{1}{6}l = 60$, $r = 3$, and $x = \frac{1}{2}l$; EI being 16,584,000,000. Thus,

$$y = \frac{4500 \times 360^3}{6 \times 16584000000}\left\{\left(\frac{1}{2} \times \frac{1}{36} \times 84 - \frac{1}{2} \times \frac{1}{216} \times 144\right)\frac{1}{8}\right.$$

$$\left.+ \left(\frac{3}{4} \times \frac{144}{216} - \frac{84}{36}\right)\frac{1}{4} + \frac{1}{2} \times \frac{84}{36} \times \frac{1}{2} - \frac{1}{4} \times \frac{144}{216}\right\} = 0.13187 \text{ inch.}$$

2d, If 2 other equal weights, 4,500 pounds each, be added at the same interval of 60 inches, so as to cover the beam with concentrated loads, the central deflection due these last 2 is, by (280), where $r = 3$, and $r_1 = 5$, or by (278), making $r = 2$,

$$y = \frac{4500 \times 360^3}{6 \times 16584000000}\left\{\left[\frac{1}{2} \times \frac{1}{36}(330 - 84) - 2 - \frac{1}{2} \times \frac{1}{216} \times 756\right]\frac{1}{8}\right.$$

$$\left.+ \left(\frac{3}{4} \times \frac{756}{216} - \frac{246}{36} + \frac{3 \times 18}{12}\right)\frac{1}{4}\right\} = 0.06594 \text{ inch.}$$

3d, For the 5 equal weights now on this beam, (281) gives the deflection

$$y = \frac{4500 \times 360^3}{6 \times 16584000000}\left\{\left(\frac{1}{2} \times \frac{1}{36} \times 330 - 2 - \frac{900}{432}\right)\frac{1}{8}\right.$$

$$\left.+ \left(\frac{3}{4} \times \frac{900}{216} - \frac{330}{36} + \frac{54}{12}\right)\frac{1}{4} + \frac{1}{2} \times \frac{1}{36} \times \frac{84}{2} - \frac{1}{4} \times \frac{144}{216}\right\}$$

$$= 0.19781 \text{ inch};$$

or, (282) gives the same much more simply.

This value is, as it should be, the sum of the two deflections last found.

4th, Suppose the fifth weight removed from the beam, what is the deflection at the fourth weight? Use (279), making $r = 4$, $c = \frac{1}{6}l = 60$ inches, $W = 4,500$ pounds;

$$\therefore \quad y = \frac{4500 \times 360^3 \times 16 \times 5}{24 \times 16584000000 \times 216}\Big(13 - 4 \times \frac{1}{6} \times 36$$

$$+ \frac{16 \times 33}{36} - \frac{2 \times 64 \times 5}{216}\Big) = 0.13748 \text{ inch.}$$

5th, These 4 equal weights remaining on the beam, what is the deflection at the third weight, or centre?

In equation (281), put $x = \frac{1}{2}l = cr$, $r = 3$, $r_1 = 4$, $c = \frac{1}{6}l$;

$$\therefore \quad y = \frac{4500 \times 360^3}{6 \times 16584000000}\Big\{\Big(\frac{180}{72} - 1 - \frac{400}{432}\Big)\frac{1}{8}$$

$$+ \Big(\frac{3 \times 400}{4 \times 216} - \frac{180}{36} + \frac{24}{12}\Big)\frac{1}{4} + \frac{84}{72} - \frac{144}{4 \times 216}\Big\} = 0.18072 \text{ inch.}$$

6th, The same 4 weights remaining, what is the deflection at the second weight?

Use (281), calling $r_1 = 4$, $r = 2$, $x = rc = \frac{1}{3}l$, $c = \frac{1}{6}l$;

$$\therefore \quad y = \frac{4500 \times 360^3}{6 \times 16584000000}\Big\{\Big(\frac{180}{72} - 2 - \frac{400}{432}\Big)\frac{1}{27}$$

$$+ \Big(\frac{3}{4} \times \frac{400}{216} - \frac{180}{36} + \frac{42}{12}\Big)\frac{1}{9} + \frac{30}{216} - \frac{9}{216}\Big\} = 0.1458 \text{ inch.}$$

7th, What are the end moments due these 4 weights in the same position as above?

Use (283) and (284), making $r = 4$, $c = \frac{1}{6}l = 60$, $W = 4,500$;

$$\therefore \quad M_1 = \frac{4500 \times 360}{12} \times \frac{20}{6}\Big(4 \times \frac{1}{6} \times 9 - 6 - \frac{3}{36} \times 20\Big)$$

$$= -750000 \text{ inch-pounds.}$$

$$M_2 = \frac{4500 \times 360}{12} \times \frac{20}{36}\Big(\frac{3}{6} \times 20 - 2 \times 9\Big) = -600000 \text{ inch-pounds.}$$

8th, When all the 5 weights are on the beam uniformly distributed as above, $r = 5$, $c = \frac{1}{6}l = 60$, $W = 4{,}500$. Then, by (283) and (284),

$$M_1 = \frac{4500 \times 360}{12} \times \frac{30}{6}\left(\frac{44}{6} - 6 - \frac{90}{36}\right) \quad = -787500 \text{ inch-pounds.}$$

$$M_2 = \frac{4500 \times 360}{12} \times \frac{30}{36}\left(\frac{3}{6} \times 30 - 2 \times 11\right) = -787500 \text{ inch-pounds.}$$

9th, The 4 equal weights of 4,500 pounds still occupying the first 4 intervals on this beam, where are the points of contrary flexure? Here we have $M_1 = -750{,}000$, $M_2 = -600{,}000$, $W = 4{,}500$, $c = \frac{1}{6}l = 60$, $r_1 = 4$, $r_2 = 0$.

These values put in (289) give

$$r = 4.6595 \text{ or } 1.1923.$$

We have then, rejecting the decimals, $r = 1$ or 4. Hence (288) becomes

$$x = \frac{-750000 + \frac{1}{2} \times 4500 \times \frac{360}{6} \times 2}{-\frac{150000}{360} - 4500(3 - \frac{1}{2} \times \frac{1}{6} \times 4 \times 5)} = \quad 74.806 \text{ for } r = 1,$$

$$x = \frac{-750000 + \frac{1}{2} \times 4500 \times \frac{360}{6} \times 20}{-\frac{150000}{360} - 4500(0 - \frac{1}{2} \times \frac{1}{6} \times 4 \times 5)} = 275.294 \text{ for } r = 4.$$

10th, If these 4 weights occupy the last 4 intervals, leaving the first vacant, we shall have $M_1 = -600{,}000$, $M_2 = -750{,}000$, $r_1 = 5$, $r_2 = 1$, $c = \frac{1}{6}l = 60$, $W = 4{,}500$; so that, from (289), we find $r = 4.80372$ or 1.34442, that is, 4 or 1.

These values placed in (288) give

$$x = 84.706 \text{ for } r = 1,$$

$$x = 285.194 \text{ for } r = 4,$$

which accords with example 9th, since $360 - 84.706 = 275.294$, and $360 - 285.194 = 74.806$.

11th, When all 5 weights are on the beam at equal intervals, $M_1 = M_2 = -787,500$ by example 8th. Also, $c = \frac{1}{6}l = 60$, $r_1 = 5$, $r_2 = 0$, $W = 4,500$. From (289), we find that r must be 1 or 4; and therefore (288) gives, as the points of contrary flexure,

$$x = 76\tfrac{2}{3} \text{ for } r = 1,$$

$$x = 283\tfrac{1}{3} \text{ for } r = 4.$$

The sum of these values of x is 360, as it should be, since the load is symmetrical.

12th, Let there be on this beam weights at the end of the second and third intervals, and find the end moments and points of contrary flexure. We now have $W = 4,500$, $c = \frac{1}{6}l = 60$ inches, $r_1 = 3$, $r_2 = 1$; so that (290) and (291) become

$$M_1 = \frac{-4500 \times 360}{12}\left\{\frac{6}{6} \times 2 \times 5 - 4 \times \frac{1}{36}(3 \times 4 \times 7 - 1 \times 2 \times 3) + \frac{3}{216}(144 - 4)\right\} = -442500,$$

$$M_2 = \frac{-4500 \times 360}{12}\left\{\frac{2}{36}(84 - 6) - \frac{3}{216}(144 - 4)\right\} = -322500.$$

Using these moments in (289), we find $r = 1.2460$ or 4.2354; we use $r = 1$ or 4. Therefore, from (288), $x = 79.254$ or 251.053, which are the points of contrary flexure sought.

13th, When these 2 equal weights are at the second and third points of division, as in the twelfth example, what is the maximum deflection of the beam, and at what point does it occur?

Using (281), where now $W = 4,500$, $c = 60 = \frac{1}{6}l$, $r_1 = 3$, $r_2 = 1$, and provisionally putting $x = cr$, we find y_r; then, putting $r + 1$ for r in the value of y_r, we find y_{r+1}; and then, making $\Delta y = y_{r+1} - y_r = 0$, we obtain

$$r^3 - 6.722r^2 + 9.111r + 2.676 = 0,$$

from which we easily see, as was suspected, that a positive value of r lies between 2 and 3 for a maximum y.

Making, therefore, $r = 2$ in equation (281), and differentiating with respect to x, then putting $\frac{dy}{dx} = 0$, we find

$$x = 0.47391l,$$

which, substituted in (281), r being 2, gives

$$y = 0.11553 \text{ inch, a maximum.}$$
$$\text{At centre, } y = 0.11478 \text{ inch, at second weight.}$$
$$\text{At } \tfrac{1}{3}l, \quad y = 0.09514 \text{ inch, at first weight.}$$

87. When the uniform discontinuous load is applied at equal consecutive intervals, the first weight being placed at no integral number of times the common interval from the left end of the beam, we may proceed in finding the deflection, end moments, and points of contrary flexure as in article 20, where r, r_{11} and r_2 need not be integral, but where the differences, $r_1 - r_2$, $r - r_2$, $r_1 - r$, each denoting a number of weights, must be integral. In this way the deflection formulæ already established in this chapter for full intervals, r, r_{11}, r_2, being whole numbers, also apply to the case now under consideration, where r, r_{11} and r_2 have the same fractional part, except that, when r_2 is negative, its value is less, by unity, than the common decimal part of r and r_{11} as before shown.

Example 1. — Beam fixed horizontally at right-hand end, simply supported at left end. Length = 360 inches = l, $c = \frac{1}{6}l = 60$ inches; 6 weights, each = $W = 4{,}500$ pounds, applied at the intervals $\frac{1}{2}c$, c, c, c, c, c, $\frac{1}{2}c$; depth and cross-section of I-beam as in the example of article 75, where the moment of inertia of section = $I = 691$, and $E = 24{,}000{,}000$. What is the deflection at the centre under this load?

Using equation (292), where now $r_2 = -\frac{1}{2}$, $r_1 = 5\frac{1}{2}$, $r = 2\frac{1}{2}$, and $x = \frac{1}{2}l = 180$, we find central deflection

$$y = \frac{4500 \times 360^3}{12 \times 16584000000}\left\{\frac{1}{8}\left(\frac{3}{2} \cdot \frac{1}{6} \cdot 6^2 - \frac{1}{4} \cdot \frac{1}{216}\left[\left(\frac{11}{2}\right)^2\left(\frac{13}{2}\right)^2 - \frac{1}{16}\right] - 2 \times 3\right) - \frac{1}{4}\left(\frac{3^3}{6}\right) + \frac{1}{2}(4.4375 + 9 - 10.4583) - 0.1771\right\} = 0.3993 \text{ inch.}$$

And the greatest deflection due this full load on the beam fixed horizontally at the right-hand end is found by putting $x = cr$ provisionally in (292), and making $y_{r+1} - y_r = 0 = \Delta y$. This equation indicates a value of r between $\frac{3}{2}$ and $\frac{5}{2}$.

Calling $r = \frac{3}{2}$ in (292), and putting $\frac{dy}{dx} = 0$, we find $x = 0.42077l = 151.477$ inches, which is greater than $c(r + 1) = 60(\frac{3}{2} + 1) = 150$ inches, an inadmissible result. Hence we see that the approximate equation $\Delta y = 0$ gave r too small. Now, calling $r = \frac{5}{2}$ in (292), and making $\frac{dy}{dx} = 0$, we find $x = 0.417404l = 150.265$ inches, which is between cr and $c(r + 1)$, as it should be.

With $r = \frac{5}{2}$, and $x = 0.417404l$, (292) gives greatest deflection $y = 0.41463$ inch; while at the centre it was 0.3993 inch. The end moment in this case removes the point of greatest deflection $180 - 150.265 = 29.735$ inches from the centre.

The end moment due this load is given by (293), where $r_1 = \frac{11}{2}$, $r_2 = -\frac{1}{2}$, $c = \frac{1}{6}l = 60$, and $W = 4{,}500$ pounds; and it is, in inch-pounds,

$$M_2 = \frac{4500 \times 360}{2}\left\{\frac{1}{4} \times \frac{1}{216}\left[\left(\frac{11}{2}\right)^2\left(\frac{13}{2}\right)^2 - \frac{1}{16}\right] - \frac{1}{2} \cdot \frac{1}{6} \cdot 6^2\right\}$$
$$= -1231875.$$

The point of contrary flexure is found by adding equations (62) and (93), and equating the sum of the second members to zero.

Thus, since $M_1 = 0$, we have

$$M_x = \left\{ \frac{W}{2l}[2(r_1 - r)l - c(r_1 - r_2)(r_1 + r_2 + 1)] + \frac{M_2}{l} \right\} x$$
$$+ \tfrac{1}{2}Wc(r - r_2)(r + r_2 + 1) = 0. \quad (305)$$

$$\therefore \quad x = \frac{-c(r - r_2)(r + r_2 + 1)l}{2l(r_1 - r) - c(r_1 - r_2)(r_1 + r_2 + 1) + \frac{2M_2}{W}}. \quad (306)$$

Making $x = rc$ provisionally in (306), we find $r = 4.5353$. Calling $r = \frac{9}{2}$, and $M_2 = -1,231,875$, (306) yields the point of contrary flexure

$$x = 0.754717l,$$

which is between rc and $(r + 1)c$ (that is, between $0.75l$ and $\frac{11}{12}l$), though very close to the former.

If both ends of this beam are free under this load of 6 equal weights, we find by (232), at the point $x = 0.417404l$, $y = 0.9676$ inch.

And, by (237), the deflection due $M_2 = 1,231,875$ is $y = -0.5530$ inch, which added to 0.9676 gives $y = 0.4146$ inch, as found by (292) above.

88. Continuous Uniform Load, w/b, on Beam fixed Horizontally at Both Ends.—Take the examples of article 75, and apply to the deflections there found the effects of the end moments as given by equation (298).

1st, In the first example of article 75, for beam with free ends, the deflection, when $x = b = \frac{1}{8}l = 120$ inches, and $a = 0$, was found to be $y = 0.23444$ inch.

Now, by (298), the effect of end moments on the deflection in this case is

$$y = \frac{75 \times 360^4}{24 \times 16584000000}\left\{\left(\frac{4}{27} - \frac{2}{9} - \frac{2}{81}\right)\left(\frac{1}{27} - \frac{1}{3}\right) - \left(\frac{8}{27} - \frac{6}{9} - \frac{3}{81}\right)\left(\frac{1}{9} - \frac{1}{3}\right)\right\} = -0.19392 \text{ inch.}$$

Therefore the deflection sought is

$$y = 0.23444 - 0.19392 = 0.04052 \text{ inch};$$

the left third of the 15-inch I-beam bearing 75 pounds to the inch, both ends being fixed horizontally.

2d, Again, in the second example of article 75 the central deflection $= 0.24421$ inch, under the same conditions. If, now, in (298) we make $a = 0$, $b = \frac{1}{3}l = 120$ inches, $x = \frac{1}{2}l$, we get the effect of end moments on deflection

$$y = \frac{75 \times 360^4}{24 \times 16584000000}\left\{-\frac{8}{81}\left(\frac{1}{8} - \frac{1}{2}\right) + \frac{33}{81}\left(\frac{1}{4} - \frac{1}{2}\right)\right\} = -0.20514 \text{ inch.}$$

Therefore the required deflection is

$$y = 0.24421 - 0.20514 = 0.03907 \text{ inch}$$

at the centre of the beam fixed horizontally at both ends.

3d, Applying the value of y in (298) to the deflection found e third example of article 75, where $a = 0$, $b = \frac{1}{3}l$, $x = \frac{2}{3}l$, nd, for the beam with fixed ends,

$$y = 0.19176 - 0.17077 = 0.02099 \text{ inch.}$$

4th, The greatest deflection due 75 pounds per inch on the left third of this I-beam fixed at both ends, is found by adding equations (229) and (298), and in the resulting equation making $\frac{dy}{dx} = 0$ when $a = 0$, and $b = \frac{1}{3}l$.

This gives $x = \frac{2}{5}l$, whence the greatest deflection $y = 0.042199$ inch at $\frac{2}{5}l$ from the left end of the beam, which is $(\frac{2}{5} - \frac{1}{3})l = \frac{1}{15}l$ beyond the end of the load.

5th, The end moments for this load of 75 pounds per inch on the left third of this 15-inch I-beam 30 feet long, where $I = 691$, $E = 24{,}000{,}000$, $a = 0$, $b = \frac{1}{3}l = 120$ inches, are given by equations (296) and (297), as follows:

$$M_1 = \frac{75 \times \frac{1}{9}l^4}{12l^2}\left(\frac{8}{3} - \frac{3}{9} - 6\right) = -330000 \text{ inch-pounds},$$

$$M_2 = \frac{75 \times 360^2}{108}\left(\frac{3}{9} - \frac{4}{3}\right) = -80000 \text{ inch-pounds}.$$

6th, With these values of M_1 and M_2, equation (300) gives the first point of contrary flexure,

$$x = 100.926 \pm 37.229 = 63.697 \text{ inches},$$

since in (300) x cannot be greater than $(a + b) = (0 + \frac{1}{3}l) = 120$ inches.

The second point of contrary flexure is derived from (301), where we find $x = 260.69$ inches.

The mode of procedure when only one end of the beam is fixed horizontally is so similar to that just exemplified for two fixed ends, that further examples seem to be unnecessary.

SECTION 4.

Deflection of a Girder of Variable Cross-Section in Terms of the Constant Unit Strain upon the Extreme Fibres of the Section; that is, Deflection of a Beam of Uniform Strength. End Moments for Fixed Beams.

89. Economy in the construction of built beams or framed girders requires that the cross-sections of the various members, as well as that of the whole structure, should be proportioned to the greatest strains allowed upon the sections; and, when the dimensions of parts are so adjusted, it is clear that the unit strain of tension, compression, or bending will be constant throughout the girder.

The complete realization of this condition is, for obvious considerations, probably seldom attained; but it is a condition so nearly approximated in practice as to require examination here.

For this case we employ equation (186); viz.,

$$-E\frac{d^2y}{dx^2} = \frac{2B_1}{h},$$

which is independent of I, the moment of inertia of the cross-section, and in which B_1 is constant for a given load, and equal to the mean of the unit strains upon the fibres at the upper and lower surfaces of the beam, and h = height of cross-section.

90. **Deflection of Semi-Girder of Uniform Height, h, and Uniform Strength.** — Using the notation of article 64, as illustrated by Fig. 8, and integrating (186), with the sign of $E\frac{d^2y}{dx^2}$ positive for the semi-girder, first, with the condition that $\frac{dy}{dx} = 0$ when $x = 0$,

$$\therefore \quad E\frac{dy}{dx} = \frac{2B_1}{h}x;$$

secondly, $y = 0$ when $x = 0$,

$$\therefore \quad Ey = \frac{B_1 x^2}{h},$$

$$\therefore \quad y = \frac{B_1 x^2}{Eh}, \qquad (307)$$

which is the deflection at any point, x, of the semi-girder of uniform height and strength.

If $x = l$,

$$D = \frac{B_1 l^2}{Eh}, \qquad (308)$$

which is the deflection at the free end of the semi-girder of uniform height and strength.

It may be observed that (307) is the equation of a parabola with its vertex at the origin of co-ordinates.

EXAMPLE. — Take an open-webbed semi-girder of wrought-iron whose effective height, h, is 20 feet = 240 inches, length, l, = 50 feet = 600 inches; and suppose the allowed unit strain in the top chord is C_1 = 8,000 pounds per square inch, and in the bottom chord T_1 = 10,000 pounds per square inch. Then calling, as we may do without sensible error, the top and bottom chords extreme fibres of the cross-section, we have

$$B_1 = \tfrac{1}{2}(C_1 + T_1) = 9000. \qquad (309)$$

Take E = 25,000,000, then

$$\text{Deflection at free end} = D = \frac{9000 \times 600^2}{240 \times 25000000} = 0.54 \text{ inch},$$

$$\text{Deflection at centre} = \frac{9000 \times 300^2}{240 \times 25000000} = 0.135 \text{ inch}.$$

It should be remembered that the deflection of a framed girder due to its first full load is likely to be greater than that computed by these formulæ, by reason of the yielding of the joints and probable straightening of some of the parts in tension. It is customary, therefore, in computing the deflection of a girder under its first loads until the frame becomes "set," to take E ranging from 15,000,000 to 20,000,000 for wrought-iron, according to the accuracy of the joint fittings and general workmanship; afterwards the ordinary value of E may be used.

91. Deflection of the Semi-Girder of Uniform Strength but of Variable Height.—(*a*) Let the semi-girder be like either half of Fig. 64, 65, 67, 33, 34, 39, or 83; that is, let it slope uniformly from the fixed end, whose height we will call h_1, to the free end, whose height is h_0.

Then the height at any point, x, is

$$h = h_1 - \frac{h_1 - h_0}{l}x; \tag{310}$$

and

$$dh = \frac{h_0 - h_1}{l}dx,$$

$$dx^2 = \left(\frac{l}{h_0 - h_1}dh\right)^2.$$

Hence (186) becomes, for this semi-girder,

$$\frac{E(h_0 - h_1)^2}{2B_1 l^2} \cdot \frac{d^2y}{dh^2} = \frac{1}{h}.$$

Integrating, with the condition that $\dfrac{dy}{dh} = 0$ when $h = h_1$,

$$\therefore \quad \frac{E(h_0 - h_1)^2}{2B_1 l^2} \cdot \frac{dy}{dh} = \log_e \frac{h}{h_1},$$

where $\log_e$ denotes the Napierian logarithm.

Again, $y = 0$ when $h = h_1$,

$$\therefore \quad \frac{E(h_0 - h_1)^2}{2B_1l^2}y = \int_{h_1}^{h} \log_e \frac{h}{h_1} \,.\, dh = h\left(\log_e \frac{h}{h_1} - 1\right) + h_1,$$

$$\therefore \quad y = \frac{2B_1l^2}{E(h_0 - h_1)^2}\left\{h_1 - h\left(2.302585 \log \frac{h_1}{h} + 1\right)\right\}, \qquad (311)$$

which is the deflection of the uniformly sloping semi-girder at any point where the height is h; log denoting the common logarithm, and the girder being of uniform strength.

Putting for h in (311) its value as taken from (310), we have y in terms of x; thus,

$$y = \frac{2B_1l^2}{E(h_0 - h_1)^2}\left\{h_1 - \left(h_1 - \frac{h_1 - h_0}{l}x\right)\left(2.302585 \log \frac{h_1 l}{h_1 l + (h_0 - h_1)x} + 1\right)\right\}, \quad (312)$$

which is the same as (311).

If the semi-girder of uniform slope and strength comes to a point at the free end, we have at that end $h_0 = 0 = h$; and therefore (311) becomes

$$D = \frac{2B_1l^2}{Eh_1}, \qquad (313)$$

which is twice the deflection given by (308) for semi-beam of the same length but of the uniform height h_1.

When $h_0 = h_1 = h$, the value of y in (311) and (312) is indeterminate, but is given by (307).

EXAMPLE.. — Length of semi-girder $l = 50$ feet; height at fixed end $= 20$ feet, at free end 10 feet; $B_1 = 9{,}000$; $E = 25{,}000{,}000$ pounds per square inch. What is the deflection at the free end? $h_1 = 240$ inches, $h_0 = h = 120$ inches, $l = 600$ inches.

By (311),

$$y = \frac{2 \times 9000 \times 600^2}{25000000 \times (-120)^2}[240 - 120(2.302585 \log 2 + 1)]$$

$$= 0.6628 \text{ inch.}$$

(*b*) *Semi-Girder with Either or Both Chords Parabolic. Open Frame.* — First, take a case like the half of Fig. 63, supposing the top chord parabolic, and, as in all these cases, the members formed as for a semi-beam. Let l = length of semi-girder, h_1 = its height at the fixed end, h_0 = height at free end and h = variable height. Then, by equation (136), putting for the h in that equation $h_1 - h_0$, and adding h_0 to the second member for our present case, we have, l also being put for $\frac{1}{2}l$,

$$h = h_1 - \frac{h_1 - h_0}{l^2}x^2. \qquad (314)$$

This value of h placed in (186) gives, after reducing, and making $m^2 = \frac{h_1}{h_1 - h_0}$,

$$\frac{(h_1 - h_0)E}{2B_1l^2} \cdot \frac{d^2y}{dx^2} = \frac{1}{m^2l^2 - x^2}. \qquad (315)$$

Integrating (315), first with the condition $\frac{dy}{dx} = 0$ when $x = 0$,

$$\therefore \quad \frac{m(h_1 - h_0)E}{B_1l} \cdot \frac{dy}{dx} = \log_e \frac{ml + x}{ml - x}, \qquad (316)$$

where $\log_e$ means Napierian logarithm.

Integrating again, with the condition $y = 0$ when $x = 0$,

$$\therefore \quad \frac{m(h_1 - h_0)E}{B_1l}y = 2.302585[(ml + x)\log(ml + x) + (ml - x)\log(ml - x) - 2ml\log ml].$$

$$y = \frac{2.302585 B_1 l}{m(h_1 - h_0)E}[(ml + x)\log(ml + x)$$
$$+ (ml - x)\log(ml - x) - 2ml\log ml], \quad (317)$$

where log denotes common logarithm, and y is the deflection at any point, x, of the semi-girder of uniform strength, and of the form of one-half of Fig. 63, when the top chord is parabolic.

EXAMPLE. — Let $B_1 = 9{,}000$, $E = 25{,}000{,}000$, $l = 600$ inches, $h_1 = 240$ inches, $h_0 = 120$ inches.

$$m = \sqrt{\frac{h_1}{h_1 - h_0}} = \sqrt{2}.$$

If, now, $x = l$, we have the deflection at the free end of the girder, from (317),

$$y = \frac{2.302585 \times 9000 \times 600}{25000000 \times 120\sqrt{2}}(203.9) = 0.59757 \text{ inch},$$

which is, as it manifestly should be, less than the deflection just found by (311) for the semi-beam of equal length and depth of ends, but of uniform slope, and greater than the deflection of semi-beam of same length and uniform depth $= h_1$, found by equation (308).

If this girder comes to a point at the free end (that is, if it is the half of the parabolic bowstring), we have, in (317), $h_0 = 0$, $m = 1$;

$$\therefore\ y = \frac{2.302585 B_1 l}{Eh_1}[(l + x)\log(l + x)$$
$$+ (l - x)\log(l - x) - 2l\log l], \quad (318)$$

which is the deflection at any point, x.

When, in (318), $x = l$, we have the deflection at the free end of the parabolic semi-bowstring; thus,

$$D = \frac{1.386295 B_1 l^2}{E h_1}, \tag{319}$$

which, according to (313), is $\frac{1.386295}{2}$ of the deflection at the free end of the semi-girder of same length and height at fixed end, but sloping uniformly to a point.

From the identity in the form of equations (136), (137), and (138), and from the manner in which (317), (318), and (319) have been derived from (136), it follows that the deflection of any parabolic semi-girder of uniform strength, whether the half-crescent, or the half double bowstring, may be found from (317), (318), and (319), provided we make $h_1 =$ the height of girder at fixed end, and $h_0 =$ its height at the free end.

(*c*) *Semi-Girder with Circular Arc for Top Chord. Uniform Strength.* — Let, as before, $h_1 =$ height at fixed end, $h_0 =$ height at free end, for a girder like the right half of Fig. 63, fixed at the vertical plane through the centre; the top chord being now supposed circular.

If R is the radius of the circle, the height of the semi-girder at any point, x, is given by equation (125),

$$h = h_1 - h_0 + h_0 + \sqrt{R^2 - x^2} - R,$$

$$h = h_1 - R + \sqrt{R^2 - x^2}, \tag{320}$$

$$h = h_1 - R + R\cos\theta; \tag{321}$$

θ being the arc between the point (x, y) of equation (125) and the fixed end of the girder.

Therefore (186) becomes

$$\frac{E}{2B_1}\frac{d^2y}{dx^2} = \frac{1}{h_1 - R + R\cos\theta}. \qquad (322)$$

But $x = R\sin\theta$,

$$\therefore \quad dx = R\cos\theta\, d\theta,$$

$$\therefore \quad \frac{E}{2B_1}\frac{d^2y}{dx} = \frac{\cos\theta\, d\theta}{a + \cos\theta}, \qquad (323)$$

if $a = \dfrac{h_1 - R}{R}$.

Integrating first with the condition that $\dfrac{dy}{dx} = 0$ when $\theta = 0$, we have (for this first integration, see Price, "Infinitesimal Calculus," vol. ii. p. 85), after reducing, and putting $a = \cos\alpha$,

$$\frac{E}{2B_1}\frac{dy}{dx} = \theta - \frac{a}{\sin\alpha}\log_e\frac{\cos\dfrac{\alpha-\theta}{2}}{\cos\dfrac{\alpha+\theta}{2}} \qquad (324)$$

where, as usual, $\log_e$ means Napierian logarithm.

For the second integration, between the limits 0 and y, 0 and θ, (324) takes the form

$$\frac{E}{2B_1R}dy = \theta\cos\theta\, d\theta - \frac{a}{\sin\alpha}\log_e\frac{\cos\dfrac{\alpha-\theta}{2}}{\cos\dfrac{\alpha+\theta}{2}}\cos\theta\, d\theta. \quad (325)$$

The first term is easily integrated thus:

$$\begin{aligned}\int_0^\theta \theta\cos\theta\, d\theta &= \theta\sin\theta - \int_0^\theta \sin\theta\, d\theta \\ &= \Big[\theta\sin\theta + \cos\theta\Big]_0^\theta \\ &= \theta\sin\theta + \cos\theta - 1.\end{aligned}$$

Integrate the second term also by parts, according to the form

$$\int u\,dv = uv - \int v\,du. \tag{326}$$

Take $u = \log_e \dfrac{\cos\dfrac{\alpha-\theta}{2}}{\cos\dfrac{\alpha+\theta}{2}} = \log_e \cos\dfrac{\alpha-\theta}{2} - \log_e \cos\dfrac{\alpha+\theta}{2}.$

$$dv = \cos\theta\,d\theta, \qquad \therefore \quad v = \sin\theta.$$

$$du = \frac{\frac{1}{2}\sin\dfrac{\alpha-\theta}{2}}{\cos\dfrac{\alpha-\theta}{2}}d\theta + \frac{\frac{1}{2}\sin\dfrac{\alpha+\theta}{2}}{\cos\dfrac{\alpha+\theta}{2}}d\theta = \frac{1}{2}\left(\tan\frac{\alpha-\theta}{2} + \tan\frac{\alpha+\theta}{2}\right)d\theta.$$

Therefore the second term of the second member of (325) becomes

$$-\frac{a}{\sin\alpha}\left\{\sin\theta\log_e\frac{\cos\dfrac{\alpha-\theta}{2}}{\cos\dfrac{\alpha+\theta}{2}} - \frac{1}{2}\int\left(\tan\frac{\alpha-\theta}{2} + \tan\frac{\alpha+\theta}{2}\right)\sin\theta\,d\theta\right\}.$$

But

$$\tan\frac{\alpha-\theta}{2} + \tan\frac{\alpha+\theta}{2} = \frac{2\sin\alpha}{\cos\alpha + \cos\theta},$$

and the second term reduces to

$$-\frac{a\sin\theta}{\sin\alpha}\log_e\frac{\cos\dfrac{\alpha-\theta}{2}}{\cos\dfrac{\alpha+\theta}{2}} + a\int\frac{\sin\theta\,d\theta}{\cos\alpha + \cos\theta}.$$

Now

$$a\int\frac{\sin\theta\,d\theta}{\cos\alpha+\cos\theta} = -a\int\frac{d(\cos\alpha+\cos\theta)}{\cos\alpha+\cos\theta} = -a\log_e(\cos\alpha+\cos\theta).$$

Whence, finally, the integral of (325) is

$$\frac{E}{2B_1R}y = \left\{\theta\sin\theta + \cos\theta - \frac{a\sin\theta}{\sin\alpha}\log_e\frac{\cos\frac{\alpha-\theta}{2}}{\cos\frac{\alpha+\theta}{2}} - a\log_e(\cos\alpha + \cos\theta)\right\}_0^\theta,$$

$$\therefore\ y = \frac{2B_1R}{E}\left\{\theta\sin\theta + \cos\theta - 1 - \frac{a}{m_1}\left(\frac{\sin\theta}{\sin\alpha}\log\frac{\cos\frac{\alpha-\theta}{2}}{\cos\frac{\alpha+\theta}{2}} + \log\frac{a+\cos\theta}{a+1}\right)\right\}, \quad (327)$$

where $m_1 = 0.4342945$, the modulus of common logarithms, log; and y is the deflection at any point, $x = R\sin\theta$, of the semi-girder having its top chord circular and bottom chord straight, like the truncated bowstring.

When $h_0 = 0$ (that is, when the semi-girder is half of the common bowstring girder), the last term of (327) becomes infinite for $a = -\cos\theta$, which is the case if $x = l =$ length of semi-girder, and $\sin\theta = l \div R$.

But in this case $\sin\theta = \sin\alpha$; and (327) is easily reduced to

$$y = \frac{2B_1R}{E}\left\{\theta\sin\theta + \cos\theta - 1 + 2.302585a\log\frac{a+1}{2\left(\cos\frac{\alpha-\theta}{2}\right)^2}\right\}, \quad (328)$$

which is the deflection at the free end of the circular semi-bow-string of uniform strength.

EXAMPLE I. — Semi-bowstring. $l = 600$ inches, $h_1 = 240$ inches, $B_1 = 9{,}000$, $E = 25{,}000{,}000$, wrought-iron.

$$\therefore\ R = \frac{l^2 + h_1^2}{2h_1} = 870 \text{ inches} = 72.5 \text{ feet},$$

$$\sin\theta = l \div R = \frac{60}{87} = 0.689655, \quad \theta = 43^\circ\,36'\,10''.15,$$

$$\cos\theta = \frac{R - h_1}{R} = \frac{63}{87} = 0.724138.$$

$$\text{In arc, } \theta = \frac{43.60282}{180}\pi = 0.761013,$$

$$a = \frac{h_1 - R}{R} = -\frac{63}{87} = -0.724138 = \cos\alpha = -\cos\theta,$$

$$\therefore \quad \alpha = 180^\circ - 43^\circ\,36'\,10''.15 = 136^\circ\,23'\,49''.85,$$

$$\tfrac{1}{2}(\alpha - \theta) = 46^\circ\,23'\,49''.85.$$

Therefore the deflection at the free end of this semi-bowstring of uniform strength is, by (328),

$$y = \frac{2 \times 9000 \times 870}{25000000}(0.524836 + 0.724138 - 1 + 1.145371)$$
$$= 0.71745 \text{ inch},$$

which is a little less than $\frac{1.386295}{2} \times 1.08 = 0.7486$ inch = deflection at free end of parabolic semi-bowstring, by (319). And this should be so, since the top chord of the parabolic girder lies just below that of the circular bowstring of the same central height and same span.

EXAMPLE 2. — Semi-girder, truncated bowstring, circular. $l = 600$ inches, $h_1 = 240$, $h_0 = 120$, $B_1 = 9{,}000$, wrought-iron; $E = 25{,}000{,}000$;

$$\therefore \quad R = \frac{l^2 + (h_1 - h_0)^2}{2(h_1 - h_0)} = 130 \text{ feet} = 1560 \text{ inches},$$

Use equation (327).

$$\sin\theta = l \div R = \tfrac{5}{13},$$

$$\theta = 22^\circ\,37'\,11''.5 = \frac{22.619861}{180}\pi = 0.39479 \text{ in arc.}$$

$$\cos\theta = 0.923077, \qquad a = \cos\alpha = \frac{20 - 130}{130} = -\frac{11}{13};$$

$$\sin\alpha = 0.532939, \qquad \alpha = 180^\circ - 32^\circ\,12'\,15''.3 = 147^\circ\,47'\,44''.7;$$

$$\frac{\alpha + \theta}{2} = 85^\circ\,12'\,28''.1, \qquad \frac{\alpha - \theta}{2} = 62^\circ\,35'\,16''.6;$$

$$\therefore\ y = \frac{2 \times 9000 \times 1560}{25000000}(0.151842 + 0.923077 - 1 + 0.455710)$$
$$= 0.596003 \text{ inch,}$$

which is the deflection at the free end, and is, as was to be expected, a little less than that found by (317) for the parabolic semi-girder of the same length and end heights.

92. Equations (327) and (328) apply also to the double circular bowstring, truncated or otherwise, provided the radii of the two curves are the same. But when these radii are different, we may, without sensible error, employ the equations (317), (318), and (319), deduced for the deflection of the parabolic semi-girder of uniform strength, and applicable to all the cases, including the crescent and the double bow; the computed deflection being always a little greater than that due the circular semi-girder of the same end heights and span.

93. **Deflection of the Girder of Uniform Strength supported at Both Ends, either Fixed or Free, and the Height of the Girder being either Uniform or Variable.** — Since the deflection of a girder may be defined as the difference of level between the position of any one of its points before bending and the position of the same point after bending, under

the given load, it follows that the formulæ already established for the deflection of the semi-girder of uniform strength also apply to the present case, provided we take the origin of co-ordinates in the neutral axis at the centre of the span, and call y positive upward, and write $\frac{1}{2}l$ for l; l being the length of the girder in all cases, and the neutral axis taken horizontal.

EXAMPLES. — Take an open webbed girder of wrought-iron, height at the centre = 25 feet = 300 inches, span = 200 feet = 2,400 inches; therefore $h_1 = 300$, $\frac{1}{2}l = 1,200$. Let $B_1 = \frac{1}{2}(C_1 + T_1) = 9,000$, $E = 25,000,000$. What is the deflection at the centre?

EXAMPLE 1. — Height uniform $= h = h_1 = 300$; therefore central deflection is, by (308),

$$D = \frac{9000 \times 1200^2}{25000000 \times 300} = 1.728 \text{ inches.}$$

EXAMPLE 2. — Truncated circular bowstring. $h_1 = 300$, $h_0 = 180 =$ end height, $h_1 - h_0 = 120$.

$$R = \frac{(\frac{1}{2}l)^2 + (h_1 - h_0)^2}{2(h_1 - h_0)} = \frac{1200^2 + 120^2}{2 \times 120} = 6060 \text{ inches} = 505 \text{ feet.}$$

Use equation (327).

$$\sin\theta = \frac{\frac{1}{2}l}{R} = \frac{100}{505} = 0.198020, \qquad \theta = 11^\circ\, 25'\, 16''.3;$$

$$\cos\theta = \frac{R - (h_1 - h_0)}{R} = \frac{495}{505} = 0.980198.$$

$$\text{In arc, } \theta = \frac{11.4212}{180}\pi = 0.199338.$$

$$\cos\alpha = a = \frac{h_1 - R}{R} = -0.950495.$$

$\frac{1}{2}(\alpha + \theta) = 86° 39' 31''.15, \qquad \alpha = 180° - 18° 6' 14'' = 161° 53' 46''.$

$\frac{1}{2}(\alpha - \theta) = 75° 14' 14''.85.$

$$\theta \sin\theta = 0.039473, \qquad \log\frac{a + \cos\theta}{a + 1} = -0.2218488.$$

$$\log\frac{\cos\frac{1}{2}(\alpha - \theta)}{\cos\frac{1}{2}(\alpha + \theta)} = 0.6406695, \frac{\sin\theta}{\sin\alpha}\log\frac{\cos\frac{1}{2}(\alpha - \theta)}{\cos\frac{1}{2}(\alpha + \theta)} = -0.408267.$$

$$\frac{a}{m_1}(0.408267 - 0.2218488) = 0.407994.$$

$$y = \frac{2 \times 9000 \times 6060}{25000000}(0.039473 + 0.980198 - 1 + 0.407994)$$
$$= 1.86169 \text{ inches.}$$

EXAMPLE 3. — Truncated parabolic bowstring, equation (317).

$h_1 = 300$, $h_0 = 180$, $x = \frac{1}{2}l = 1{,}200$, $m^2 = \dfrac{h_1}{h_1 - h_0} = 2.5.$

$m(\frac{1}{2}l) = 1897.366, \ m(\frac{1}{2}l) + x = 3097.366, \ m(\frac{1}{2}l) - x = 697.366.$

$$\log\tfrac{1}{2}ml = 3.2781512, \qquad \log(\tfrac{1}{2}ml + x) = 3.4909925,$$
$$\log(\tfrac{1}{2}ml - x) = 2.8434608.$$

Therefore (317) becomes

$$y = \frac{2.302585 \times 9000 \times 1200}{1.58114 \times 120 \times 25000000}(10812.88 + 1982.93 - 12439.70)$$
$$= 1.86695 \text{ inches.}$$

EXAMPLE 4. — Chords of uniform slope. $h_1 = 300$, $h_0 = 180 = h$, $\frac{1}{2}l = 1{,}200$. Use equation (311).

$$y = \frac{2 \times 9000 \times 1200^2}{25000000 \times 120^2}\left\{300 - 180\left(2.302585 \log\frac{300}{180} + 1\right)\right\}$$
$$= 2.0197 \text{ inches.}$$

EXAMPLE 5. — Circular bowstring. $\frac{1}{2}l = 1{,}200$, $h_1 = 300$, $h_0 = 0$. Use equation (328).

$$R = \frac{(\frac{1}{2}l)^2 + h_1^2}{2h_1} = 212.5 \text{ feet} = 2550 \text{ inches}.$$

$$\sin\theta = \frac{\frac{1}{2}l}{R} = 0.470588, \qquad \theta = 28^\circ\, 4'\, 21''.$$

$$\cos\theta = \frac{R - h_1}{R} = 0.882353. \qquad \text{In arc, } \theta = 0.501346.$$

$$\cos\alpha = -\cos\theta = a = -0.882353.$$

$$\alpha = 180^\circ - 28^\circ\, 4'\, 21'' = 151^\circ\, 55'\, 39''.$$

$$\tfrac{1}{2}(\alpha - \theta) = 61^\circ\, 55'\, 39'', \quad \log\frac{a + 1}{2[\cos\frac{1}{2}(\alpha - \theta)]^2} = -0.5757318.$$

$$\theta\sin\theta = 0.23593.$$

$$y = \frac{2 \times 9000 \times 2550}{25000000}(0.23593 + 0.882353 - 1$$
$$+ 2.302585 \times 0.882353 \times 0.5757318) = 2.36477 \text{ inches.}$$

EXAMPLE 6. — Parabolic bowstring. $\frac{1}{2}l = 1{,}200$, $h_1 = 300$, $h_0 = 0$.

By equation (319),

$$D = \frac{1.386295 \times 9000 \times 1200^2}{25000000 \times 300} \doteq 2.39552 \text{ inches.}$$

EXAMPLE 7. — Girder sloping uniformly from centre to ends. $\frac{1}{2}l = 1{,}200$, $h_1 = 300$, $h_0 = 0$.

By equation (313),

$$D = \frac{2 \times 9000 \times 1200^2}{25000000 \times 300} = 3.456 \text{ inches.}$$

EXAMPLE 8. — Parabolic crescent. $h_1 = 300$, $h_0 = 0$, $\frac{1}{2}l = 1{,}200$.

The deflection in this case must be the same as that in the sixth example, for the parabolic bowstring.

$$\therefore \quad D = 2.39552 \text{ inches.}$$

EXAMPLE 9. — Girder like Fig. 53, sloping uniformly from centre to ends. $h_1 = 300$, $h_0 = 0$, $\frac{1}{2}l = 1{,}200$.

Deflection the same as in example 7, viz.,

$$D = 3.456 \text{ inches.}$$

EXAMPLE 10. — Girder like Fig. 66, polygonal.

Find the deflection for each part having a uniform slope, separately, and add the results for the total central deflection, after correcting.

Take $h_1 = 300$ at Z_4, and $h_0 = 240$ at Z_6, the quarter-section. Then $\frac{1}{2}l = 600$, and equation (311) gives the deflection at Z_6, thus,

$$y = \frac{2 \times 9000 \times 600^2}{25000000 \times 60^2}[300 - 240(2.302585 \log \tfrac{300}{240} + 1)]$$
$$= 0.46408 \text{ inch.}$$

Similarly, for the end quarter, equation (313) gives

$$D = \frac{2 \times 9000 \times 600^2}{25000000 \times 240} = 1.08 \text{ inches.}$$

But, before adding these results, we must find, as in article 67, how much the free end of the semi-beam is deflected by reason of the bending of the part between Z_4 and Z_6; that is, we must add to 0.46408 the quantity $\frac{1}{2}l \times \tan \alpha = 600 \tan \alpha = 600 \frac{dy}{dx}$.

From (311),

$$\frac{dy}{dx} = \tan\alpha = \frac{2B_1 l}{E(h_0 - h_1)}\log_e\frac{h}{h_1} = 0.0016066,$$

$$600 \times 0.0016066 = 0.96398 \text{ inch},$$

$$\therefore \quad \text{Total deflection} = 0.46408 + 0.96398 + 1.08 = 2.50806 \text{ inches},$$

which is greater than the deflection found in example 6, for the parabolic bowstring; and it will be found that although the girder, Fig. 66, is deeper at the quarter-points than the parabolic bow of example 6, yet at the $\frac{1}{8}$ and $\frac{3}{8}$ points the latter is the deeper.

In like manner may we proceed in all cases of irregular forms, whether there be two or more changes of slope; but, in general, we may use the formulæ already found for regular forms, with sufficient accuracy, always choosing the one most fitting for the case in hand.

94. We may arrange the results found in these examples according to the amount of the deflection, and thus the more clearly perceive the effect of form upon the bending of girders of uniform strength. All the girders here represented are 200 feet in length if supported at both ends, or 100 feet long if semi-girders; the deflection being the same in either case.

Since in all the formulæ the deflection varies directly as $\frac{B_1}{E} = \frac{\frac{1}{2}(C_1 + T_1)}{E}$, we may find the deflection of girders of the same dimensions, but of other material than wrought-iron, by substituting for E the proper value taken from Table II., and for C_1 and T_1 the allowed unit strain.

If, for pine, $T_1 = 1{,}200$, $C_1 = 552$, $E = 1{,}460{,}000$, then $B_1 = 876$, and $\frac{B_1}{E} = 0.0006$, but for wrought-iron $\frac{B_1}{E} = 0.00036$; hence, for a girder of uniform strength and of given span and

height, the deflection, if the material is pine, will be five-thirds of the deflection were the material wrought-iron; that is, allowing C_1 and T_1 the above values.

If the compressed chord be of pine, $C_1 = 552$, and the other of wrought-iron, $T_1 = 10,000$, and if $E = 13,230,000 = \frac{1}{2}(25,000,000 + 1,460,000)$, we have $B_1 = 5,276$, $\frac{B_1}{E} = 0.0004$. Hence a combination of pine and wrought-iron gives a deflection $\frac{4}{3.6} = \frac{10}{9}$ times that due wrought-iron alone, with these unit strains.

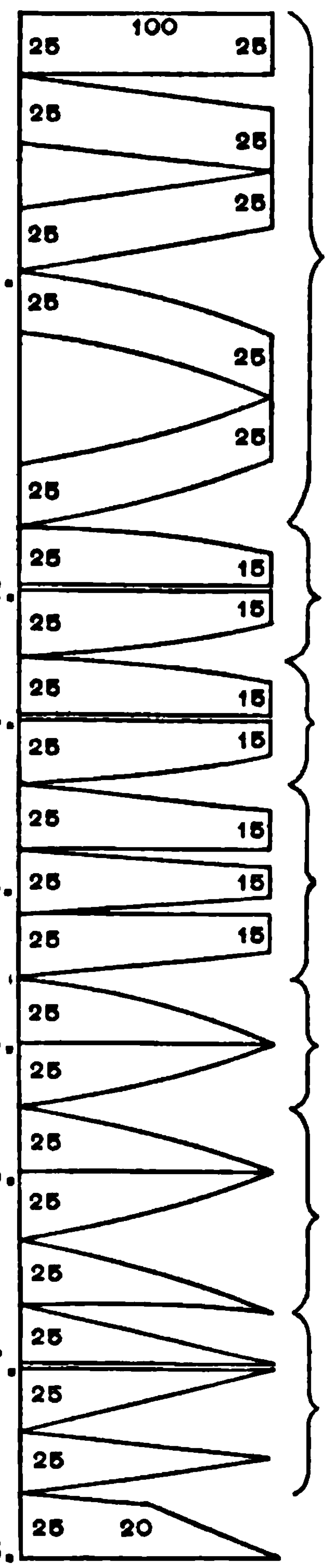
Were the compressed chord of cast-iron, for which $C_1 = 15,000$, while the other chord is of wrought-iron, $T_1 = 10,000$, and $E = \frac{1}{2}(25,000,000 + 12,000,000) = 18,500,000$, we should have $B_1 = 12,500$, $\frac{B_1}{E} = 0.00067567$, and the deflection would be 1.877 times that of the girder of same size in wrought-iron.

95. By inspecting the following table, we see that for open girders of the same central height, same length, and of uniform strength, *the total deflection is* NEARLY *in the inverse ratio of the areas of the figures of the girders.*

This is exactly the ratio of the deflections in case of the girder of uniform height and of that sloping uniformly to a point: viz., ratio of areas, $\frac{2}{1}$; ratio of deflections, $\frac{1}{2}$. We may, therefore, without appreciable error, employ this principle in finding the total deflection of open girders of uniform strength and variable height.

EXAMPLES.—DEFLECTION OF OPEN WEBBED GIRDERS OF UNIFORM STRENGTH.

Length $= l = 200$ feet, central height $= h_1 = 25$ feet.



Material.		Wrought-Iron.	Pine.	Wrt.-Iron and Pine.	Wrt. and Cast Iron.	Equation.
B_1		9,000	876	5,276	12,500	
E		25,000,000	1,460,000	13,230,000	18,500,000	
Form.	Description.	Def., ins.	Def., ins.	Def., ins.	Def., ins.	
1.	Uniform height	1.728	2.88	1.92	3.24	(308)
2.	One chord circular	1.86169	3.1028	2.0685	3.494	(327)
3.	One chord parabolic	1.86695	3.1116	2.0744	3.504	(317)
4.	Uniform slope	2.0197	3.366	2.2441	3.791	(311)
5.	Circular Bowstring	2.36477	3.9413	2.6275	4.438	(328)
6.	Parabolic curves	2.39552	3.9925	2.6617	4.496	(319)
7.	Uniform slope	3.456	5.76	3.84	6.48	(313)
8.	Polygonal chord	2.50806	4.1801	2.7867	4.703	(311)

We give below, the deflections of girders of wrought-iron for the eight cases just tabulated, but now computed by this *Method of Areas*:—

No.	Area of One-Half Girder.	Deflection.	Deflection by Formulæ.
1	2500 square feet.	1.7280 inches.	1.72800 inches.
2	2167 " "	1.9931 "	1.86169 "
3	2167 " "	1.9938 "	1.86695 "
4	2000 " "	2.1600 "	2.01970 "
5	1737 " "	2.4871 "	2.36477 "
6	1667 " "	2.5920 "	2.39552 "
7	1250 " "	3.4560 "	3.45600 "
8	1625 " "	2.6585 "	2.50806 "

The deflections of such girders as those shown in Figs. 19, 20, 29, 30, 31, 32, 33, 34, 39, 40, 53, 54, 55, etc., are therefore easily found by the method of areas.

It should be noticed that in the preceding table of deflections of the same girder in different materials, a factor of safety equal to 10 has been allowed for pine, while 5 is the factor allowed for wrought and for cast iron.

The modulus of elasticity for cast-iron, $E = 12{,}000{,}000$, is so small, that, in spite of its large resistance to compression, $C_1 = 15{,}000$, the open beam made of wrought and cast iron, and of uniform strength, has greater deflection than that of wrought-iron alone, and, indeed, greater than that of pine alone, with the low unit strain here allowed.

96. Finally, if the beam be of uniform strength, but have a continuous web, the formulæ already deduced for girders of uniform strength and of open web may be employed by assigning to B_1 its proper value derived from Table II.

EXAMPLE 1. — Plate girder of uniform strength and uniform height, wrought-iron. Take the length l = 50 feet = 600 inches, height h = 5 feet = 60 inches; the girder being supported at both ends.

By Table II., B = 42,000 = breaking unit strain for plate beams. Allowing a safety factor of 5, we have B_1 = 8,400; and calling E = 25,000,000, and putting $\frac{1}{2}l$ for l in equation (308), there results the central deflection,

$$D = \frac{8400 \times 300^2}{25000000 \times 60} = 0.504 \text{ inch.}$$

EXAMPLE 2. — Take a plate girder of the same length, 50 feet, and same central height, 5 feet, but sloping uniformly from centre to ends, where the height is 2 feet.

Then, if the girder is of uniform strength, we have, from equation (311),

$$y = \frac{2 \times 8400 \times 300^2}{25000000 \times (60 - 24)^2}[60 - 24(2.302585 \log \tfrac{60}{24} + 1)]$$
$$= 0.65376 \text{ inch.}$$

EXAMPLE 3. — Cast-iron beam of uniform strength, and height h = 3 feet = 36 inches, $\frac{1}{2}l$ = 6 feet = 72 inches. Take $B_1 = \frac{45900}{6}$ = 7,650, E = 17,000,000 (Table II.).

Then, by (308),

$$D = \frac{7650 \times 72^2}{17000000 \times 36} = 0.0648 \text{ inch.}$$

EXAMPLE 4. — If this cast-iron beam of uniform strength slope uniformly from centre to ends, where h = 12 inches, then, by (311),

$$y = \frac{2 \times 7650 \times 72^2}{17000000 (12 - 36)^2}[36 - 12(2.302585 \log \tfrac{36}{12} + 1)]$$
$$= 0.0876 \text{ inch.}$$

EXAMPLE 5. — Oak beam of uniform strength and height. Take $\frac{1}{2}l = 120$ inches, $h = 18$ inches, $B_1 = \frac{10600}{10}$, $E = 2,150,000$; then (308) gives

$$D = \frac{1060 \times 120^2}{2150000 \times 18} = 0.3944 \text{ inch.}$$

EXAMPLE 6. — If this oak beam of uniform strength slope uniformly from centre to ends, where $h = 12$ inches, then, by (311),

$$y = \frac{2 \times 1060 \times 120^2}{2150000 \times (12 - 18)^2}[18 - 12(2.302585 \log \tfrac{18}{12} + 1)]$$
$$= 0.4474 \text{ inch.}$$

EXAMPLE 7. — Beam of Bessemer hammered steel, uniform strength. $\frac{1}{2}l = 72$ inches; height at centre, $h_1 = 20$ inches, at ends, $h_0 = 10$ inches. Take $B_1 = \frac{128080}{5} = 25,616$, $E = 31,000,000$ (Table II.).

Then deflection at centre is, from (311),

$$y = \frac{2 \times 25616 \times 72^2}{31000000(10 - 20)^2}[20 - 10(2.302585 \log \tfrac{20}{10} + 1)]$$
$$= 1.1196 \text{ inches.}$$

For same beam of wrought-iron,

$$y = \frac{2 \times 9000 \times 72^2}{25000000(10 - 20)^2}[20 - 10(2.302585 \log \tfrac{20}{10} + 1)]$$
$$= 0.4878 \text{ inch,}$$

which is less than half the deflection of the same beam in steel.

But if we suppose this beam to be of rectangular cross-section, and to bear a concentrated weight, W, at its centre, where the height is $h_1 = 20$ inches, and the thickness $b = 2$

inches, the length being $l = 144$ inches, then, from equations (46) and (160), we have moment at centre,

$$M = \frac{1}{4}Wl = \frac{1}{6}Bbh^2 = \frac{1}{6}B_{\prime}bh^2 \text{ for safety,}$$

$$\therefore \quad W = \frac{2}{3} \times \frac{2 \times 20^2}{144}B_{\prime\prime}$$

$$W = 3.7037 \times 25616 = 94874 \text{ pounds for steel,}$$

$$W = 3.7037 \times 9000 = 33333 \text{ pounds for wrought-iron.}$$

Hence, under the assumed unit strains, the steel beam bears $\frac{25616}{9000} = 2.8462$ times the weight at the centre of the wrought-iron beam of the same dimensions, while the deflection of the steel beam is $\frac{1.1196}{0.4878} = 2.2953$ times that of the wrought-iron beam; that is, what is shown in all the formulæ, the weight W varies directly with the unit strain $B_{\prime\prime}$, while for the same unit strain the deflection varies inversely as the modulus of elasticity, E.

Therefore in the present case, so far as deflection is concerned, the advantage of steel over wrought-iron, under same load, is $\frac{2.8462}{2.2953} = \frac{31}{25}$, which is the simple ratio of the moduli of elasticity.

97. The thickness, b, of a continuous webbed girder of uniform strength at any rectangular section of given height, h, may be found, in general, by equating the moment, M, due the external forces, to the moment of resistance, R, of the internal forces of the beam at the given section, and solving with respect to b.

For a beam of rectangular cross-section, bearing a concentrated load at its centre, equations (45) and (160) give $M = \frac{1}{2} Wx = \frac{1}{6} B_1 bh^2$. B_1 = allowed unit strain.

$$\therefore \quad b = \frac{3Wx}{B_1 h^2}, \qquad (329)$$

where, if the height, h, be uniform, b varies as x; making the horizontal projection or ground plan of each half of the beam a triangle with a vertex at the end of the beam, where $b = x = 0$, and a base at the beam's centre, where $b = \frac{3}{2} \frac{Wl}{b_1 h^2}$.

EXAMPLE. — Oak beam of uniform strength, and height $h = 15$ inches, length = 15 feet, weight applied at centre $= W = 4{,}000$ pounds allowed unit strain $= \frac{10600}{10} = 1{,}060$ pounds per square inch. What must be the thickness of this beam at the centre?

Here $x = \frac{1}{2} l = 90$ inches,

$$\therefore \quad b = \frac{3 \times 4000 \times 90}{1060 \times 15^2} = 4.53 \text{ inches.}$$

It must be remembered that wherever the moment becomes zero, causing b, the thickness of the beam, to vanish by the formulæ, we must, nevertheless, have at all such points sufficient material to resist, with the proper margin of safety, the shearing-strain which may there be developed, and the re-action of the supports.

In this example the shearing-strain at each end of this beam is $\frac{4000}{2} = 2{,}000$ pounds. Now, by Table I., the ultimate resistance to shearing is, for oak, across the grain, 4,000 pounds;

one-tenth of which is 400 pounds, to be allowed to each square inch of the vertical section at each end.

Therefore $\frac{2000}{400} = 5$ square inches of section at least the beam must have at each end; that is, the depth being 15 inches, the thickness is $\frac{1}{3}$ inch. But there is another consideration to be attended to; viz., the bearing-surface at the ends must be sufficient to resist with safety and permanence the pressure coming upon it.

This beam as now estimated is $\frac{1}{3}$ inch thick at each end, and 4.53 inches at its centre. Hence it must have 8.903 inches of its length at each end upon the support, in order to secure a bearing of $3\frac{1}{3}$ square inches, required for 2,000 pounds with an allowed unit strain of 600 pounds to the square inch, in compression.

Again, a beam so thin at the ends would lack lateral stiffness unless it were walled in.

In practice, therefore, even when it is desired to use the least material possible, it is customary to make those parts of a beam which theoretically, or rather, by formula, are almost nothing, of such size as a just regard to all these requirements, as well as to the good appearance of the structure, may demand.

Let it not be inferred that theory and practice are at variance here, for such is not the case. The equations which determine the thickness of the beam do not pretend to take into the account all the conditions affecting the sufficiency of the beam for its purpose. And hence the theory is not complete till the modifying conditions are introduced.

98. If the beam of uniform strength be loaded uniformly with w units of weight to the unit of length, we have, from equations (49) and (160),

$$\tfrac{1}{2}w(l - x)x = \tfrac{1}{6}B_1bh^2,$$

putting B_1 for B, and the cross-section being rectangular;

$$\therefore \quad b = \frac{3w(l - x)x}{B_1 h^2}, \tag{330}$$

which is the thickness of the beam at any point, x, measured from the end. When h is constant, (330) is the equation of a parabola; the vertex being at the end of the beam.

$$\text{Thickness at end} \quad = b = 0. \qquad x = 0.$$

$$\text{Thickness at centre} = b = \frac{3wl^2}{4B_1 h^2}. \qquad x = \tfrac{1}{2}l.$$

Horizontal projection, two parabolas.

Example. — Oak beam, uniform strength. Height uniform $= h = 15$ inches, length $l = 180$ inches, $B_1 = \frac{10600}{10} = 1{,}060$, $w = \frac{8000}{180} = 44\frac{4}{9}$ pounds per inch.

Then thickness at centre is

$$b = \frac{3 \times 400 \times 180^2}{4 \times 9 \times 1060 \times 15^2} = 4.53 \text{ inches.}$$

99. If the cross-section of the beam of uniform strength be of either form, Fig. 91, then, by assigning values to three of the dimensions, h, h_{11}, b, b_{11}, we may, from equation (161) and the equation expressing the moment due the given load, find the fourth dimension of the cross-section, which, therefore, becomes known at every point.

In like manner may we determine any one dimension of any cross-section whose moment of resistance, R, is known.

EXAMPLE. — Take a tubular plate girder of the dimensions given in example 1, article 96; viz., $l = 50$ feet, $h = 5$ feet, $B_1 = 8{,}400$, uniform strength and height. Cross-section as in Fig. 91, where let $b = 12$ inches, $b_1 = 11\frac{1}{4}$ inches; the side plates being $\frac{3}{8}$ inch thick each, $h = 60$ inches.

From (49) and (161), we have

$$\tfrac{1}{2}w(l - x)x = \tfrac{1}{6}B_1\frac{bh^3 - b_1h_1^3}{h},$$

$$\therefore \quad h_1 = \left(\frac{bh^3}{b_1} - \frac{3hw(l - x)x}{B_1b_1}\right)^{\frac{1}{3}}, \qquad (331)$$

equal to 58 inches if $wl = 123{,}508$ pounds, the total uniform load on beam, and $x = \frac{1}{2}l = 300$ inches.

At the centre, therefore, the top and bottom plates must have the thickness of 1 inch each; while at the ends, where $x = 0$, (331) gives

$$h_1 = h\left(\frac{12}{11.25}\right)^{\frac{1}{3}} = 1.02174h = 61.3044 \text{ inches},$$

which renders $h - h_1 = -1.3044$ inches negative, showing that the cross-section of the side plates is more than sufficient at the ends to resist the moment.

We may find at what distance from either end of this beam the top and bottom plates begin to be needed, by putting $h_1 = h = 12$ in (331), and finding x. This gives $x = 69.19$ inches, for which the side plates alone are sufficient if properly braced laterally. Now, the shearing-strain at each end of the beam supporting this load is $\frac{1}{2} \times 123{,}508 = 61{,}754$ pounds; and, calling the allowed shearing-strain 8,000 pounds to the square inch, we require $\frac{61754}{8000} = 7.72$ inches in cross-section of the two plates, whereas we have $2 \times \frac{3}{8} \times 60 = 45$ square inches. But

in order to have sufficient bearing-surface on the abutments, allowing the iron to bear 8,000 pounds to the square inch in compression also, the beam must be supported for at least $\frac{7.72}{2 \times \frac{3}{8}} = 10.3$ inches of its length at each end.

The semi-girder of uniform strength and continuous web is to be treated in the same manner as the girder just considered when we seek its variable cross-section.

100. Beam of Uniform Strength fixed Horizontally at Both Ends.—By definition the beam of uniform strength is equally efficient at all sections to resist the strains generated by the external forces. Hence, when this beam is horizontally fixed at both ends, and loaded with a concentrated or with a continuous load, the points of contrary flexure are, for any style of beam or girder, practically midway between the centre of gravity of the load and the ends of the girder; since there is as much reason for their being on one side of this midway point as there is for their being upon the other side of it, and no more. And the beam of uniform strength is such only with reference to a particular mode of loading. That is, if the unit strain is uniform throughout the girder for a given position of the load, a change in the position of the load causes a change in the relative values of the total strains in the members or parts of the girder, and therefore a change in the unit strain on each member, if, as is assumed, the cross-sections of the members be not changed.

In general, we have for any girder, from equations (184) and (187),

$$\text{Moment due internal forces, } M_x = \frac{2B_1I}{h} = \frac{2B_1Sr^2}{h}, \quad (332)$$

where B_1 = allowed unit strain in bending, S = area of any cross-section, r = radius of gyration of the section about its neutral axis, h = height of section.

By equating the last member of (332) to the known moment due the external forces applied to the girder, any *one* of the four quantities B_1, S, r, h, may be found. But, when the girder is fixed at one or both ends, we need to know the point or points of contrary flexure, in order to determine the end moments.

101. For the girder of uniform height and strength, fixed at both ends, it follows from the uniformity of the unit strain and height, which causes a uniformity of curvature, that, as already stated, each point of contrary flexure is sensibly midway between the centre of gravity of the load and the corresponding end of the girder.

Assuming that the height and strength are uniform, and that, for any required form of cross-section, the necessary variation in its area is attained by varying the thickness of the beam only, we shall have, in (332), r, h, and B_1 constant, so that the variable area, S, may be found at once for any section of the beam; and from S the thickness is to be determined.

102. **Beam of Uniform Strength and Height fixed at Both Ends, and bearing a Concentrated Weight, W, at the Distance a' from the Left End.**—The moment at any point between the weight and left end of the beam, that is, when x is not greater than a', is given by equations (40), (93), and (332), thus,

$$M_x = W\frac{l - a'}{l}x - \frac{M_1 - M_2}{l}x + M_1 = \frac{2B_1Sr^2}{h}. \qquad (333)$$

Now, $M_x = 0$ when $x = \frac{1}{2}a'$,

$$\therefore \quad 0 = W\frac{l - a'}{2l}a' - \frac{a'}{2l}(M_1 - M_2) + M_1. \qquad (334)$$

Also, when x is not less than a', we have, from (43), (93), and (332),

$$M_x = W\frac{l-x}{l}a' - \frac{M_1 - M_2}{l}x + M_1 = \frac{2B_1Sr^2}{h}. \quad (335)$$

$M_x = 0$ when $x = \frac{1}{2}(l + a')$,

$$\therefore \quad 0 = W\frac{l-a'}{2l}a' - \frac{1}{2}(M_1 - M_2) - \frac{a'}{2l}(M_1 - M_2) + M_1. \quad (336)$$

From (334) and (336), we find

$$M_1 = M_2 = -\tfrac{1}{2}W\frac{a'}{l}(l - a'). \quad (337)$$

That is, the end moments are equal and negative for any given position of the load.

Eliminating M_1 and M_2 from (333) and (335), we obtain

$$x \leqq a', \qquad M_x = W\frac{l-a'}{l}(x - \tfrac{1}{2}a') = \frac{2B_1Sr^2}{h}, \quad (338)$$

$$S = \frac{Wh(l-a')}{2B_1lr^2}(x - \tfrac{1}{2}a'). \quad (339)$$

$$x \ngtr a', \qquad M_x = W\frac{a'}{l}\left(\frac{l+a'}{2} - x\right) = \frac{2B_1Sr^2}{h}, \quad (340)$$

$$S = \frac{Wha'}{2B_1lr^2}\left(\frac{l+a'}{2} - x\right) \quad (341)$$

EXAMPLE I. — If the varying cross-section is a rectangle of the breadth b, and constant height h, we have $r^2 = \frac{1}{12}h^2$, and (339) and (341) become

$$x \leqq a', \qquad S = bh = \frac{6W(l - a')}{B_1 lh}(x - \tfrac{1}{2}a'), \qquad (342)$$

$$b = \frac{6W(l - a')}{B_1 lh^2}(x - \tfrac{1}{2}a'). \qquad (343)$$

$$x \mathrel{\overline{\gt}} a', \qquad S = bh = \frac{6Wa'}{B_1 lh}\left(\frac{l + a'}{2} - x\right), \qquad (344)$$

$$b = \frac{6Wa'}{B_1 lh^2}\left(\frac{l + a'}{2} - x\right). \qquad (345)$$

If, further, the weight, W, is at the centre of the girder, $a' = \frac{1}{2}l$, and when

$$x \leqq a', \qquad b = \frac{3W}{B_1 h^2}(x - \tfrac{1}{4}l). \qquad (346)$$

$$x \mathrel{\overline{\gt}} a', \qquad b = \frac{3W}{B_1 h^2}(\tfrac{3}{4}l - x). \qquad (347)$$

In (346), for $x = 0$, $b = b_1 = -\dfrac{3Wl}{4B_1 h^2}$, at left end.

$x = \frac{1}{4}l$, $\quad b = 0$, at quarter point.

$x = \frac{1}{2}l$, $\quad b_c = \dfrac{3Wl}{4B_1 h^2}$, at centre.

In (347), for $x = \frac{1}{2}l$, $\quad b_c = \dfrac{3Wl}{4B_1 h^2}$, at centre.

$x = \frac{3}{4}l$, $\quad b = 0$, at quarter point.

$x = l$, $\; b = b_2 = -\dfrac{3Wl}{4B_1 h^2}$, at right end.

If the beam is of oak, and $B_1 = \frac{1}{10}B = 1{,}060$ pounds, $E = 2{,}150{,}000$, $l = 180$ inches, $h = 15$ inches, $W = 4{,}000$ pounds, then $b_1 = b_2 = -b_c = -\frac{3 \times 4000 \times 180}{4 \times 1060 \times 15^2} = -2.264$ inches; the algebraic sign only indicating the direction of the inclination of the vertical planes forming the sides, to the vertical longitudinal plane of the beam.

Fig. 99 shows this beam thus loaded, in plan and elevation.

It is evident that the deflection of the part $BD = \frac{1}{2}l$, or of the part $AB = \frac{1}{4}l$, as a semi-beam, is equal to the deflection of a beam of uniform strength and height supported but not fixed at the points B and D, and bearing the concentrated weight W. But, by equation (307), the deflection of the part AB or BD is, since for x we must put $\frac{1}{4}l$,

$$D = \frac{B_1 l^2}{16Eh}.$$

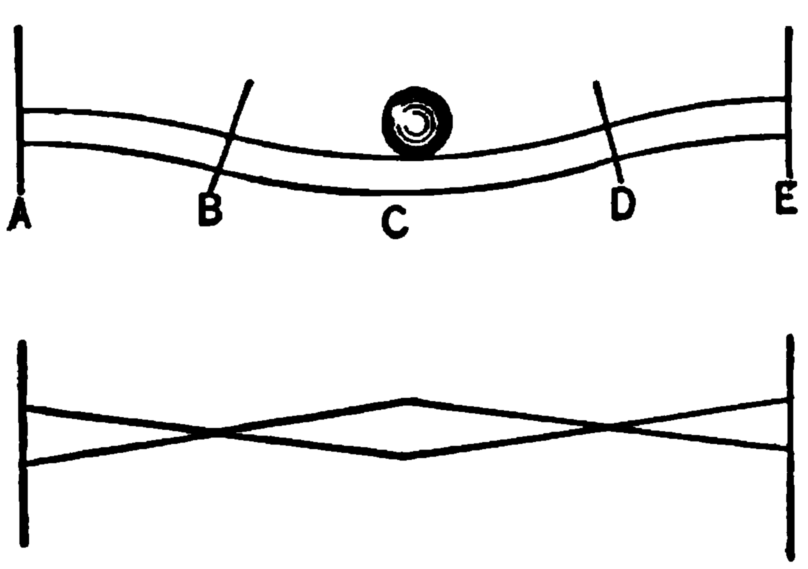


FIG. 99.

Therefore the total deflection at C, the centre of AE, is

$$2D = \frac{Bl^2}{8Eh}, \tag{348}$$

equal to $\frac{1060 \times 180^2}{8 \times 2150000 \times 15} = 0.1331$ inch in the present case.

At the points of contrary flexure, where $b = 0$, the beam, of course, must be enlarged, to resist with safety the shearing-strains.

The shearing-strain at each of these points is now $\frac{1}{2}W = 2,000$ pounds. By Table I., article 42, the ultimate shearing-strength of oak across the grain is 4,000 pounds to the inch; or the working-strength is 400 pounds to the square inch of cross-section.

We require, therefore, at least $\frac{2000}{400} = 5$ square inches of area at each point of contrary flexure; that is, the beam, being 15 inches deep, must be at least $\frac{1}{3}$ inch thick at these points, even when restrained from moving laterally.

103. Beam of Uniform Strength, Height, and Load, fixed Horizontally at Both Ends. Rectangular Cross-Section.

Equations (49) and (93) give

$$M_x = \tfrac{1}{2}w(l - x)x - \frac{M_1 - M_2}{l}x + M_1 = \tfrac{1}{6}B_1bh^2. \quad (349)$$

Make $x = 0$, then

$$M = M_2 = \tfrac{1}{6}B_1b_1h^2.$$

But, as in article 102, $b = 0$ when $x = \frac{1}{4}l$ or $\frac{3}{4}l$,

$$\therefore\ \tfrac{1}{2}w(l - \tfrac{1}{4}l)\tfrac{1}{4}l + M_1 = 0,$$

$$M_1 = M_2 = -\tfrac{3}{32}wl^2,$$

$$M_c = \tfrac{1}{2}w(l - \tfrac{1}{2}l)\tfrac{1}{2}l - \tfrac{3}{32}wl^2 = \tfrac{1}{32}wl^2,$$

$$M_x = \tfrac{1}{2}w(l - x)x - \tfrac{3}{32}wl^2 = \tfrac{1}{6}B_1bh^2. \quad (350)$$

$$b = \frac{3w}{B_1h^2}[(l - x)x - \tfrac{3}{16}l^2]. \quad (351)$$

When $x = 0$, or $x = l$, (351) gives

$$b = b_1 = b_2 = -\frac{9wl^2}{16B_1h^2},$$

which is the width of beam at either fixed end;

$$b_c = \frac{3w}{B_1h^2}\left\{\left(l - \frac{1}{2}l\right)\frac{1}{2}l - \frac{3}{16}l^2\right\} = \frac{3wl^2}{16B_1h^2},$$

which is the width of the beam at centre, being one-third of the width at either end.

Equation (351) being that of a parabola with respect to the variables b and x, the plan of this uniformly loaded beam, of uniform strength and height, fixed at its ends, is shown in Fig. 100.

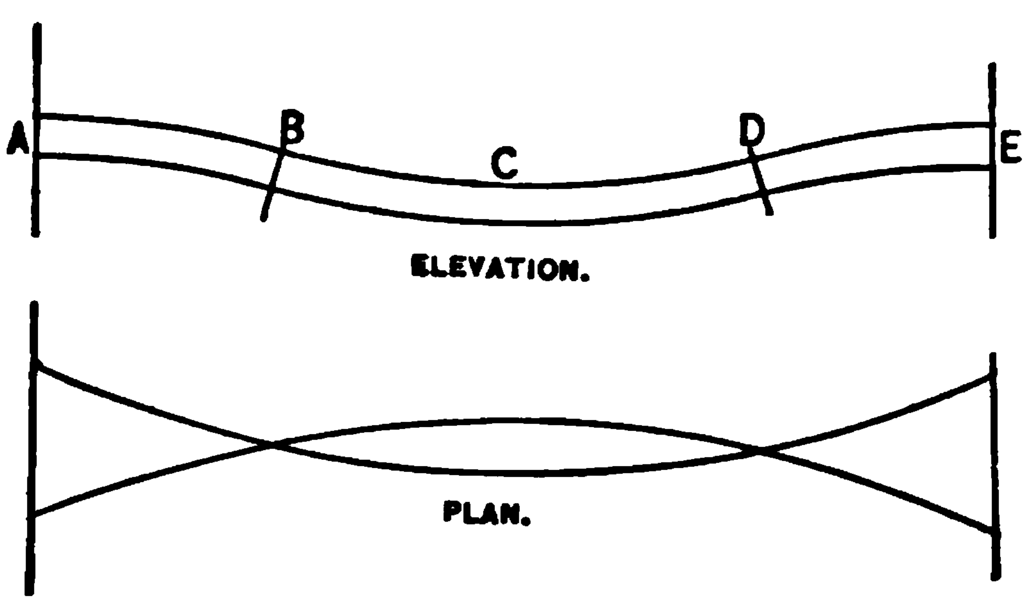


FIG. 100.

EXAMPLE. — Oak beam. $l = 180$ inches, $h = 15$ inches, ends fixed.

Take $B_1 = 1{,}060$, $E = 2{,}150{,}000$, $w = \frac{8000}{180} = 44\frac{4}{9}$ pounds to the linear inch.

Then

$$M_1 = M_2 = -\frac{3}{32} \times \frac{400}{9} \times 180^2 = -135000 \text{ inch-pounds.}$$

$$M_c = 45000 \text{ inch-pounds.}$$

$$b_1 = b_2 = -\frac{9 \times \frac{400}{8} \times 180^2}{16 \times 1060 \times 15^2} = -3.3962 \text{ in.} = \text{thickness at ends.}$$

$$b_c = 1.1321 \text{ inches, at centre.}$$

The maximum deflection is evidently given, as in article 102, by equation (348), and is

$$D = 0.1331 \text{ inch.}$$

104. Concentrated Weight, W, at the distance a' from the Unfixed End of a Beam of Uniform Strength, fixed at the Right End, but simply supported at the Left.—The point of contrary flexure must be at the distance $x_0 = \frac{1}{2}(l + a')$ from the unfixed end, in order that the greatest positive moment may be equal to the greatest negative moment.

Equations (43) and (93) apply, giving, since $M_1 = 0$,

$$M_x = W\frac{l - x}{l}a' + M_2\frac{x}{l} = \frac{1}{6}B_1bh^2 \qquad (352)$$

if the cross-section be rectangular, and $x \mathrel{\overline{>}} a'$.

For $x = l$, $\qquad M_2 = \frac{1}{6}B_1b_2h^2.$

For $x = \frac{1}{2}(l + a')$, $\quad M_2 = -W\frac{a'(l - a')}{l + a'}$ when $b = 0$.

For $x = a'$, $\qquad M_{a'} = W\frac{a'(l - a')}{l + a'}.$

$$b = \frac{6W}{B_1h^2}\left\{\frac{a'}{l + a'}(l + a' - 2x)\right\}. \qquad (353)$$

When $x = l$, $\qquad b = b_2 = -\frac{6W}{B_1h^2}\left(\frac{a'(l - a')}{l + a'}\right).$

When $x = a'$, $$b = \frac{6W}{B_1h^2}\left(\frac{a'(l-a')}{l+a'}\right).$$

Which shows that the width at D, Fig. 101, is the same as the width at C for h constant.

When $x \overline{<} a'$, use (40) and (93), giving

$$M_x = W\frac{l-a'}{l}x + M_2\frac{x}{l} = \tfrac{1}{6}B_1bh^2. \qquad (354)$$

To find the lowest point, E, in the curve, Fig. 101, we equate the deflection, D_1, between the lowest point and left end of the beam, to the total deflection, $D_2 + D_3$, between the same point and the right end of the beam, and solve the equation $D_1 = D_2 + D_3$.

For the length $AE = z$, (307) gives

$$D_1 = \frac{B_1z^2}{Eh}.$$

$$EB = \tfrac{1}{2}(l + a') - z, \qquad D_2 = \frac{B_1\left(\frac{l+a'}{2} - z\right)^2}{Eh}.$$

$$BD = \tfrac{1}{2}(l - a'), \qquad D_3 = \frac{B_1[\frac{1}{2}(l-a')]^2}{Eh}.$$

$$\therefore \quad z^2 = \left(\frac{l+a'}{2} - z\right)^2 + \tfrac{1}{4}(l-a')^2,$$

$$z = \frac{l^2 + a'^2}{2(l+a')}.$$

$$D_1 = D_2 + D_3 = \frac{B_1(l^2 + a'^2)^2}{4Eh(l+a')^2}, \qquad (355)$$

which is the deflection at the lowest point, E.

EXAMPLE. — Given $W = 4{,}000$ pounds at the distance $a' = \frac{1}{8}l$ from the unfixed end, A, Fig. 101; $l = 180$ inches; $h = 15$ inches = uniform height of beam; $B_1 = \frac{1}{10}B = 1{,}060$ pounds = working inch strain for oak; $E = 2{,}150{,}000$. Cross-section rectangular.

Then width of beam is,

At left end, (354),

$$x = 0, \qquad b = 0.$$

At the weight, (353),

$$x = \tfrac{1}{8}l, \qquad b = \frac{4000 \times 180}{1060 \times 15^2} = 3.019 \text{ inches.}$$

(353),

$$x = \tfrac{3}{8}l, \qquad b = 0.$$

At fixed end, (353),

$$x = l, \qquad b = -\frac{4000 \times 180}{1060 \times 15^2} = -3.019 \text{ inches};$$

the negative sign simply showing that the lines cd, c_1d_{11} have crossed somewhere between $x = \frac{1}{8}l$ and $x = l$.

Moment at fixed end,

$$M_2 = -4000\frac{\frac{1}{8} \times \frac{7}{8}l}{1 + \frac{1}{8}} = -120000.$$

Moment at the weight,

$$M_{a'} = 120000 \text{ inch-pounds.}$$

The deflection at the lowest point, E, is given by (355),

$$D_1 = \frac{1060 \times (\frac{10}{9})^2 \times 180^2}{4 \times 2150000 \times 15 \times (\frac{4}{3})^2} = 0.1849 \text{ inch.}$$

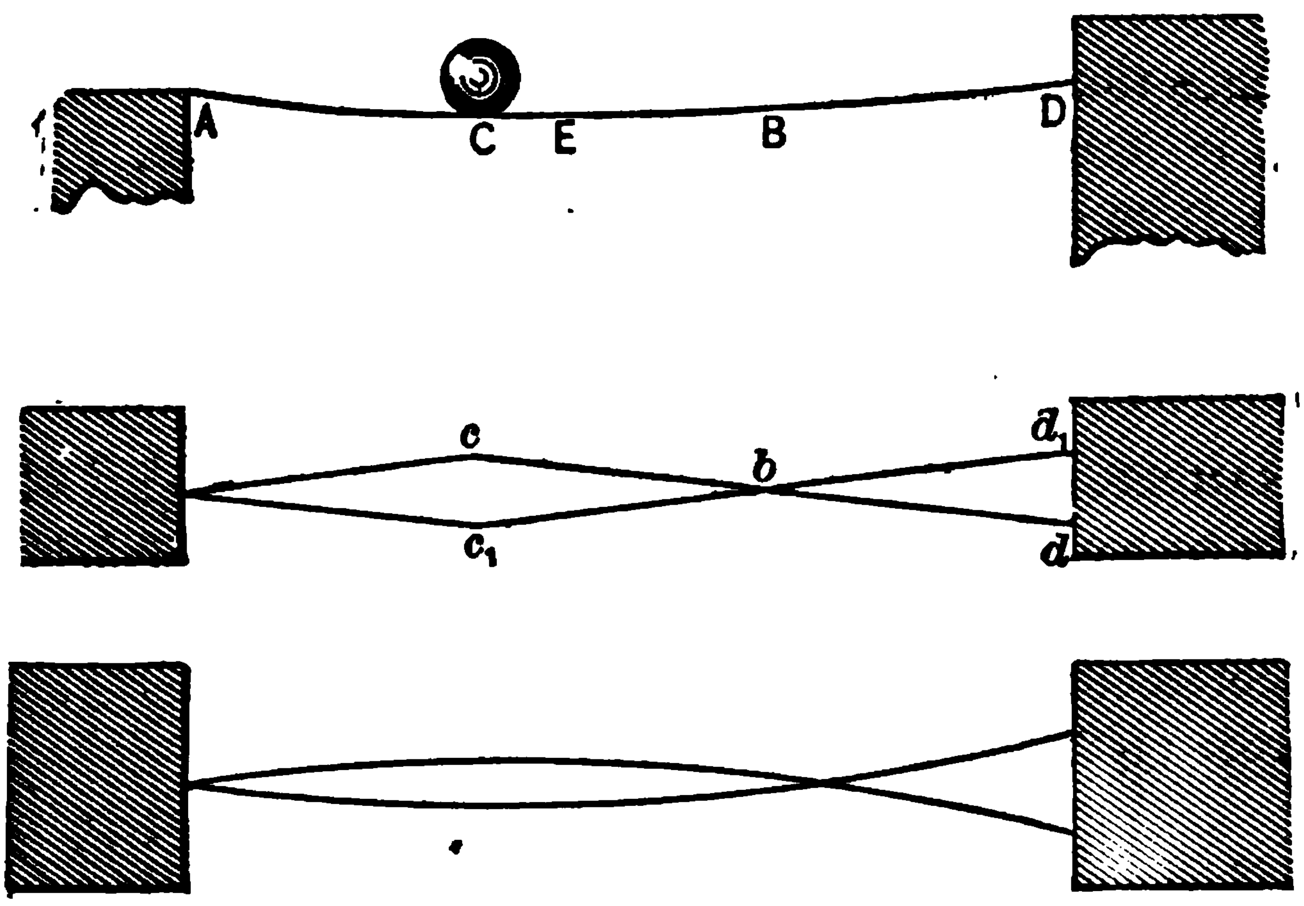


FIG. 101.

105. Continuous Uniform Load, wl, on a Beam of Uniform Strength, fixed at the Right End, and simply supported at the Left, Fig. 101.—The figure shows the curvature and the plan, when the section is rectangular; and equations (49) and (93) give, since $M_1 = 0$,

$$M_x = \tfrac{1}{2}w(l - x)x + M_2\frac{x}{l} = \tfrac{1}{6}B_1bh^2 \qquad (356)$$

for these conditions.

For $x = l$, $\qquad M_2 = \tfrac{1}{6}B_1b_2h^2$, $\qquad b_2 = -\dfrac{wl^2}{B_1h^2}$.

For $x = \tfrac{3}{4}l$, $\qquad M_2 = -\tfrac{1}{8}wl^2$, $\qquad b = 0$.

For $x = \frac{1}{3}l$, $\qquad M = \frac{1}{18}wl^2$, $\qquad b = \frac{wl^2}{3B_1h^2}$.

$$b = \frac{w}{B_1h^2}(2l - 3x)x. \tag{357}$$

Hence the breadth at C is one-third of that at D when the height is uniform, as seen in the parabolic plan, Fig. 101, derived from equation (357).

The deflection at E is given, by (355), for this case also, provided we put $\frac{1}{3}l$ for a'.

EXAMPLE. — Let $l = 180$ inches, $h = 15$, $B_1 = \frac{1}{10}B = 1{,}060$ pounds = working unit strain for bending oak, $E = 2{,}150{,}000$. Cross-section rectangular; load $wl = 8{,}000$ pounds uniformly distributed continuously, $44\frac{4}{9}$ pounds to the inch.

From (356) and (357),

$x = 0, \quad M_1 = 0.$

$x = \frac{1}{3}l, \quad M = \frac{1}{18} \times \frac{400}{9} \times 180^2 = 80000$ inch-pounds.

$x = \frac{2}{3}l, \quad M = 0.$

$x = l, \quad M_2 = -\frac{1}{6} \times \frac{400}{9} \times 180^2 = -240000$ inch-pounds.

Width at left end,

$$b = 0,$$

by (357).

Width at $\frac{1}{3}l$,

$$b = \frac{\frac{400}{9} \times 180^2}{3 \times 1060 \times 15^2} = 2.0126 \text{ inches.}$$

Width at $\frac{2}{3}l$,

$$b = 0.$$

Width at right end,

$$b_2 = 6.0378 \text{ inches.}$$

Put $\frac{1}{8}l$ for d in (355), and find the deflection $D = 0.1849$ at lowest point.

106. Beam of Uniform Strength and Uniformly Varying Height, fixed at Both Ends. — The end moments $M_1 = M_2$ are determined for this case as in articles 102 and 103, for the same kind of load.

To find the deflection of this beam, we may regard it as composed of four semi-beams, Fig. 102.

1st, AB, fixed at A;	deflection D_1.
2d, BC, fixed at C, the lowest point;	deflection D_2.
3d, CE, fixed at C, the lowest point;	deflection D_3.
4th, EF, fixed at F;	deflection D_4.

Now we must have $D_1 + D_2 = D_3 + D_4$, from which the lowest point and its deflection may be found.

EXAMPLE I. — Take one-half of the girder shown in Fig. 65, and suppose the ends of this half to be immovably fixed. Call the length $l = 100$ feet $= 1{,}200$ inches; and height at left end, $h_0 = 180$ inches; height at right end, $h_1 = 300$ inches. Let the girder be of wrought-iron, and, as in article 93, take $B_1 = \frac{1}{2}(C_1 + T_1) = 9{,}000$ pounds, $E = 25{,}000{,}000$; and suppose the load to be a concentrated weight, $W = 200{,}000$ pounds at the centre, no account being here taken of the girder's own weight.

By article 100, the points of contrary flexure are $\frac{100}{4} = 25$ feet from the centre of the beam; and from equation (337), since $d = \frac{1}{2}l$,

$$M_1 = M_2 = -M_c = -\tfrac{1}{2} \times 200000 \times \tfrac{1}{2} \times \tfrac{1}{2} \times 1200,$$
$$= -30000000 \text{ inch-pounds.}$$

The area S of any cross-section on the left of the weight is given by (339), and on the right by (341). But these equations suppose the section of the top chord to be equal to that of the bottom chord in the same vertical plane of section, and at the centre give

$$S_c = \frac{200000 \times 240 \times 600 \times \frac{1}{2} \times 1200}{2 \times 9000 \times 1200 \times \frac{1}{4} \times 240^2} = 27.777 \text{ inches};$$

at left end, (339),

$$S_1 = \frac{200000 \times 180 \times 600 \times -\frac{1}{2} \times 1200}{2 \times 9000 \times 1200 \times \frac{1}{4} \times 180^2} = -37.037 \text{ inches};$$

at right end, (341),

$$S_2 = \frac{200000 \times 300 \times 600 \times \frac{1}{2} \times 1200}{2 \times 9000 \times 1200 \times \frac{1}{4} \times 300^2} = 22.222 \text{ inches};$$

the negative sign indicating only a difference in the direction of the lateral faces of the chords, that is, change of slope laterally.

But if $S' =$ area of section of chord in compression, $S'' =$ area of section of chord in tension, we have

$$C_1S' = P = \frac{H}{\cos\alpha}, \qquad (358)$$

$$T_1S'' = U = \frac{H}{\cos\beta}, \qquad (359)$$

$$H = M + h,$$

according to the notation and equations of article 49.

From which,

$$S' = \frac{M}{C_1 h \cos\alpha}, \tag{360}$$

$$S'' = \frac{M}{T_1 h \cos\beta}. \tag{361}$$

Calling $C_1 = 8{,}000$ pounds, $T_1 = 10{,}000$ pounds, α being the inclination of the top chord for all parts between the points of contrary flexure, while $\beta = 0$, and β being the slope of top chord for the remainder of the beam, while $\alpha = 0$, we have, at either end,

$$\tan\beta = \frac{25 - 15}{100} = 0.1, \quad \cos\beta = 0.99503;$$

$$\tan\alpha = 0, \qquad \cos\alpha = 1.$$

At centre,

$$\tan\alpha = 0.1, \quad \cos\alpha = 0.99503;$$

$$\tan\beta = 0, \quad \cos\beta = 1.$$

At left end, (361), area of top section,

$$S'' = \frac{30000000}{10000 \times 180 \times 0.99503} = 16.750 \text{ inches.}$$

At left end, (360), area of bottom section,

$$S' = \frac{30000000}{8000 \times 180 \times 1} = 20.832 \text{ inches.}$$

At centre, (360), area of top section,

$$S' = \frac{30000000}{8000 \times 240 \times 0.99503} = 15.703 \text{ inches.}$$

At centre, (361), area of bottom section,

$$S'' = \frac{30000000}{10000 \times 240 \times 1} = 12.500 \text{ inches.}$$

At right end, (361), area of top section,

$$S'' = \frac{30000000}{10000 \times 300 \times 0.99503} = 10.050 \text{ inches.}$$

At right end, (360), area of bottom section,

$$S' = \frac{30000000}{8000 \times 300 \times 1} = 12.500 \text{ inches.}$$

The totals are:—

At left end,

$$S_1 = 37.582 \text{ inches;}$$

at centre,

$$S_c = 28.203 \text{ inches;}$$

at right end,

$$S_2 = 22.550 \text{ inches;}$$

differing somewhat, as was to be expected, from the areas computed on the supposition of equal top and bottom sections.

The deflection for each part of this girder is given by (311). See Fig. 102.

1st, AB, fixed at A; $h_1 = 180$ inches, $h = h_0 = 210$, $\frac{h_1}{h} = \frac{6}{7}$, $h_0 - h_1 = 30$, and we have

$$y = D_1 = \frac{2 \times 9000 \times 300^2}{25000000 \times 30^2}\left\{180 - 210\left(1 - 2.302585 \log\frac{7}{6}\right)\right\}$$

$$= \frac{9}{125} \times 2.3715 = 0.171 \text{ inch.}$$

4th, EF, fixed at F; $h_1 = 300$, $h = h_0 = 270$,

$h_0 - h_1 = -30$, $\frac{h_1}{h} = \frac{10}{9}$,

$$y = D_4 = \frac{2 \times 9000 \times 300^2}{25000000 \times (-30)^2}\left\{300 - 270\left(2.302585 \log\frac{10}{9} + 1\right)\right\}$$

$$= \frac{9}{125} \times 1.5528 = 0.112 \text{ inch.}$$

Now, in equation (311), $\frac{l}{h_0 - h_1}$ is constant, since the length, l, varies as the height, h. Therefore, —

2d, BC, fixed at C; $h = 210$,

$$y = D_2 = \tfrac{9}{125}(h_1 - 210 \times 2.302585 \log h_1 + 210 \times 2.302585 \log 210 - 210).$$

3d, CE, fixed at C; $h = 270$,

$$y = D_3 = \tfrac{9}{125}(h_1 - 270 \times 2.302585 \log h_1 + 270 \times 2.302585 \log 270 - 270),$$

and

$$D_1 + D_2 = D_3 + D_4.$$

After making the substitutions, and reducing, we find

$$h_1 = 236.15 \text{ inches,}$$

which is the depth of the girder at lowest point, C. Hence distance of lowest point from left end is

$$10(236.15 - 180) = 561.5 \text{ inches.}$$

$$D_2 = 0.108 \text{ inch}, \quad D_3 = 0.167 \text{ inch.}$$

$$D_1 + D_2 = D_3 + D_4 = 0.279 \text{ inch at } C.$$

EXAMPLE 2. — Take the same girder as in the preceding example, but let the load, $W = 200{,}000$ pounds, be 75 feet from the left end. Then the points of contrary flexure are, where $x = \frac{3}{8}l$, and $x = \frac{7}{8}l$, and by (337), since now $d' = \frac{3}{4}l$,

$$M_1 = M_2 = -\tfrac{1}{2} \times 200000 \times \tfrac{3}{4} \times \tfrac{1}{4} \times 1200$$
$$= -22500000 \text{ inch-pounds.}$$

At the weight, $x = d' = \frac{3}{4}l$, (338) gives

$$M = 200000 \times \tfrac{1}{4} \times \tfrac{3}{8} \times 1200 = 22500000 \text{ inch-pounds.}$$

At the centre, $x = \frac{1}{2}l$,

$$M_c = 200000 \times \tfrac{1}{4} \times \tfrac{1}{8} \times 1200 = 7500000 \text{ inch-pounds.}$$

At left end, by (361), area of top section,

$$S'' = \frac{22500000}{10000 \times 180 \times 0.99503} = 12.56 \text{ inches.}$$

At left end, by (360), area of bottom section,

$$S' = \frac{22500000}{8000 \times 180 \times 1} = 15.63 \text{ inches.}$$

At the weight, (360), area of top section,

$$S' = \frac{22500000}{8000 \times 270 \times 0.99503} = 10.47 \text{ inches.}$$

At the weight, (361), area of bottom section,

$$S'' = \frac{22500000}{10000 \times 270 \times 1} = 8.33 \text{ inches.}$$

At right end, (361), area of top section,

$$S'' = \frac{22500000}{10000 \times 300 \times 0.99503} = 7.54 \text{ inches.}$$

At right end, (360), area of bottom section,

$$S' = \frac{22500000}{8000 \times 300 \times 1} = 9.38 \text{ inches.}$$

Since equations (338) and (340) are of the first degree with respect to M and x, and since h varies uniformly with x, we

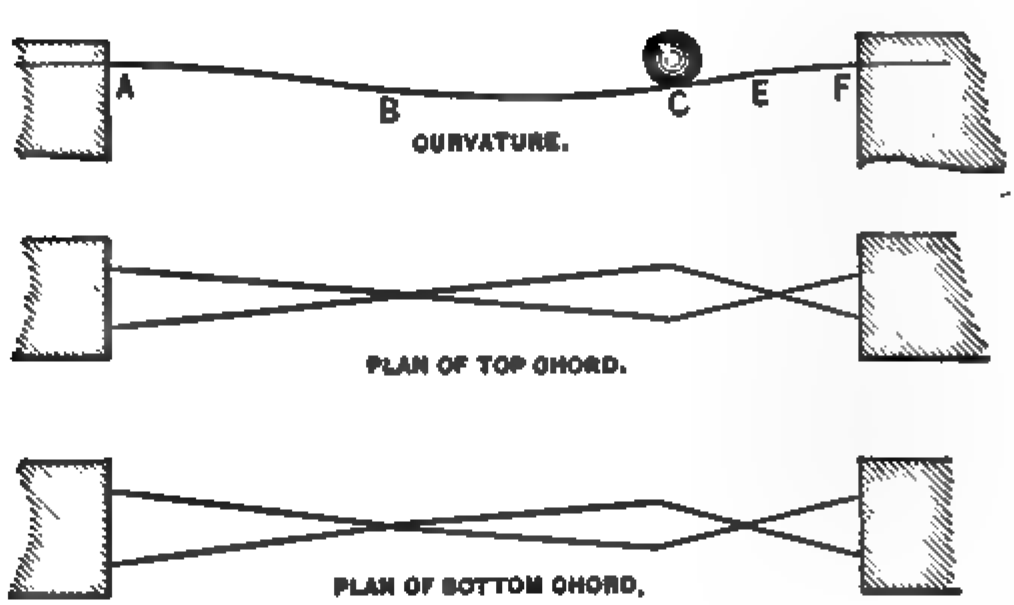


FIG. 102.

have M and S in (360) and (361), varying uniformly from the ends of the girder and from the weight to the points of contrary flexure, as shown in plan of chords, Fig. 102, where the

depth of each chord is supposed to be uniform, and the variation in size of chord attained by varying the thickness only.

The deflection for each of the four semi-beams into which the girder now becomes divided, is to be found as in the preceding example, where the load was at the centre.

We now have, for the part $AB = \frac{3}{8}l$, fixed at A, $h_1 = 180$, $h = h_0 = 225$, $h_1 \div h_0 = \frac{4}{5}$, $h_0 - h_1 = 45$; and (311) becomes

$$y = D_1 = \frac{2 \times 9000 \times 450^2}{25000000 \times 45^2}\left\{180 - 225\left(1 - 2.302585 \log\frac{5}{4}\right)\right\}$$
$$= \frac{9}{125} \times 5.206 = 0.37486 \text{ inch.}$$

For the fourth part, $EF = \frac{1}{8}l$, fixed at F; $h_1 = 300$, $h = h_0 = 285$, $h_1 \div h_0 = \frac{20}{19}$, $h_0 - h_1 = -15$;

$$y = D_4 = \frac{9}{125}\left\{300 - 285\left(1 + 2.302585 \log\frac{20}{19}\right)\right\} = \frac{9}{125} \times 0.382$$
$$= 0.0275 \text{ inch.}$$

For the second part, BC, $h = h_0 = 225$,

$$y = D_2 = \tfrac{9}{125}(h_1 - 225 \times 2.302585 \log h_1 + 225 \times 2.302585 \log 225 - 225).$$

For the third part, CE, $h = h_0 = 285$,

$$y = D_3 = \tfrac{9}{125}(h_1 - 285 \times 2.302585 \log h_1 + 285 \times 2.302585 \log 285 - 285).$$

And since $D_1 + D_2 = D_3 + D_4$, we find

$$h_1 = 234.761 \text{ inches;}$$

that is, the lowest point, C, is now at the distance

10(234.761 — 180) = 547.61 inches from the left end of the beam. Using this value of $h_{,,}$ we find D_2 and D_3, and have finally,

$$D_1 = 0.3749, \quad D_3 = 0.3622,$$

$$D_2 = 0.0148, \quad D_4 = 0.0275,$$

$$\text{Deflection at } C = 0.3897 \text{ inch} = 0.3897 \text{ inch};$$

an apparently paradoxical result, since, when the same load, $W = 200,000$ pounds, was at the centre of the girder of uniform strength, and having the same varying height and same length, $l = 100$ feet, the deflection was only 0.279 inch at the lowest point. The paradox vanishes, however, when we take into account the difference in the length of the component semi-beams for the two cases. Indeed, it may be easily shown that a girder of uniform height and strength, bearing a concentrated load, both ends being fixed, deflects least when that load is at the centre, and the four component semi-beams are of equal length.

Suppose that, in (307), we have, for—

First semi-beam,

$$x = \tfrac{1}{2}a', \text{ according to article 100};$$

second semi-beam,

$$x = \tfrac{1}{2}[l - \tfrac{1}{2}a' - \tfrac{1}{2}(l - a')] = \tfrac{1}{4}l;$$

third semi-beam,

$$= l - \tfrac{1}{2}a' - \tfrac{1}{4}l - \tfrac{1}{2}(l - a') = \tfrac{1}{4}l;$$

fourth semi-beam,

$$x = \tfrac{1}{2}(l - a').$$

Then, if u is half the sum of the four deflections, that is, if $u =$ the total deflection of the beam, we have

$$u = \frac{B_1}{2Eh}[(\tfrac{1}{2}a')^2 + 2(\tfrac{1}{4}l)^2 + \tfrac{1}{4}(l - a')^2]. \quad (362)$$

Put

$$\frac{du}{da'} = \frac{B_1}{4Eh}(2a' - l) = 0. \quad (363)$$

Therefore $a' = \frac{1}{2}l$ renders u a minimum, since $2a'$ is positive and l constant.

In a similar manner, from (311), may the position of the load be found on the beam of uniform strength and uniformly varying height, the ends being fixed, when it is required to know what position of a given load gives the least deflection.

EXAMPLE 3. — Continuous uniform load $wl = 400{,}000$ pounds upon the same girder, Fig. 102. Since the moments of the external forces are independent of the height, equation (350) applies here, giving for

$x = 0,\quad M_1 = -\frac{3}{32} \times 400000 \times 1200 = -45000000$ inch-pounds;

$x = \frac{1}{2}l,\quad M_c = \frac{1}{8} \times 400000 \times 1200 + M_1 = 15000000$ inch-pounds;

$x = l,\quad M_2 = M_1.$

By equations (360) and (361), we find —

At left end, section of top chord,

$$S'' = \frac{45000000}{10000 \times 180 \times 0.99503} = 25.125 \text{ inches.}$$

At left end, section of bottom chord,

$$S' = \frac{45000000}{8000 \times 180 \times 1} = 31.250 \text{ inches.}$$

At centre, section of top chord,

$$S' = \frac{15000000}{8000 \times 240 \times 0.99503} = 7.851 \text{ inches.}$$

At centre, section of bottom chord,

$$S'' = \frac{15000000}{10000 \times 240 \times 1} = 6.250 \text{ inches.}$$

At right end, section of top chord,

$$S'' = \frac{45000000}{10000 \times 300 \times 0.99503} = 15.075 \text{ inches.}$$

At right end, section of bottom chord,

$$S' = \frac{45000000}{8000 \times 300 \times 1} = 18.750 \text{ inches.}$$

The deflection must be the same as in example 1; viz., $D_1 + D_2 = 0.279$ inch, since the centre of gravity of each of the two loads is at the same point, and the unit strain the same.

107. Beam of Uniform Strength and Uniformly Varying Height, fixed at One End, and simply supported at the Other. — Since the position of the point of contrary flexure depends upon the moments due the external forces, which moments are independent of the height of the girder, we already have, in articles 104 and 105, the point of contrary flexure, and the moment at the fixed end, M_2, for the present case of uniformly varying height, if the load be either concentrated or uniform and continuous.

The cross-section at any point is given generally by equation (332), the deflection of each of the three component semi-beams

by (311), and the equation $D_1 = D_2 + D_3$ fixes the lowest point.

EXAMPLE 1. — Take a girder of the same varying height and same length as in the examples of article 106, Fig. 102, of wrought-iron, but now fixed at the right end and simply supported at the left; that is, let $E = 25,000,000$, $B_1 = 9,000$, $C_1 = 8,000$, $T_1 = 10,000$, $l = 1,200$ inches, $h_0 = 180$ inches = height at left end, $h_1 = 300$ inches = height at fixed end, $\tan\alpha = \frac{25 - 15}{100} = 0.1$ = tan of slope of top chord, $\cos\alpha = 0.99503$, $\tan\beta = 0$, $\cos\beta = 1$, since bottom chord is horizontal.

Let the load $W = 200,000$ pounds be at the distance $a' = \frac{3}{8}l$ from the unfixed end, the point of contrary flexure being at $x = \frac{1}{2}(l + a') = \frac{5}{8}l$. Then, from (352),

$$\begin{aligned} \text{Moment at fixed end} = M_2 &= -200000\frac{\frac{3}{8}(1 - \frac{3}{8})l}{1 + \frac{3}{8}} \\ &= -32000000 \text{ inch-pounds.} \\ \text{Moment at weight,} \quad M_{a'} &= 32000000 \text{ inch-pounds.} \\ \text{Moment at left end,} \quad M_1 &= 0. \end{aligned}$$

At the left end, (360) and (361) give the chord cross-sections = 0; but, of course, as before shown and exemplified for all such cases, the end must be enlarged to bear the shearing and crushing strains with permanent safety.

At the load, (360) gives

$$S' = 15.46 \text{ inches} = \text{section at top.}$$

At the load, (361) gives

$$S'' = 12.31 \text{ inches} = \text{section at bottom.}$$

At fixed end, (361) gives

$$S'' = 10.72 \text{ inches} = \text{section at top.}$$

At fixed end, (360) gives

$$S' = 13.33 \text{ inches} = \text{section at bottom.}$$

Applying equation (311) to the three parts of this beam, AB, BD, DE, we find the deflection, Fig. 103, —

AB, fixed at B; $h_0 = 180$,

$$y = D_1 = \tfrac{9}{125}[h_1 - 180 \times 2.302585(\log h_1 - \log 180) - 180].$$

BD, fixed at B; $h_0 = 280$,

$$y = D_2 = \tfrac{9}{125}[h_1 - 280 \times 2.302585(\log h_1 - \log 280) - 280].$$

DE, fixed at E; $h_0 = 280$, $h_1 = 300$, $h_0 - h_1 = -20$, $\dfrac{h_1}{h_0} = \dfrac{15}{14}$,

$$y = D_3 = \tfrac{9}{125}[300 - 280(2.302585 \log \tfrac{15}{14} + 1)] = \tfrac{9}{125} \times 0.681$$
$$= 0.049 \text{ inch.}$$

From the equation $D_1 = D_2 + D_3$ we have

$$h_1 = 229.731 \text{ inches,}$$

$$\therefore \quad D_1 = 0.419 \text{ inch} = \text{deflection at lowest point } B.$$

EXAMPLE 2. — Take the same girder, with the same conditions, as in example 1, except that the load is now wl = 400 000 pounds, uniformly distributed.

The moments are found by (356); thus,

$$x = 0,\ M = M_1 = 0.$$

$$x = \tfrac{1}{3}l,\ M = \frac{400000 \times 1200}{18} = 26666666.$$

$$x = l,\ M_2 = -80000000 \text{ inch-pounds.}$$

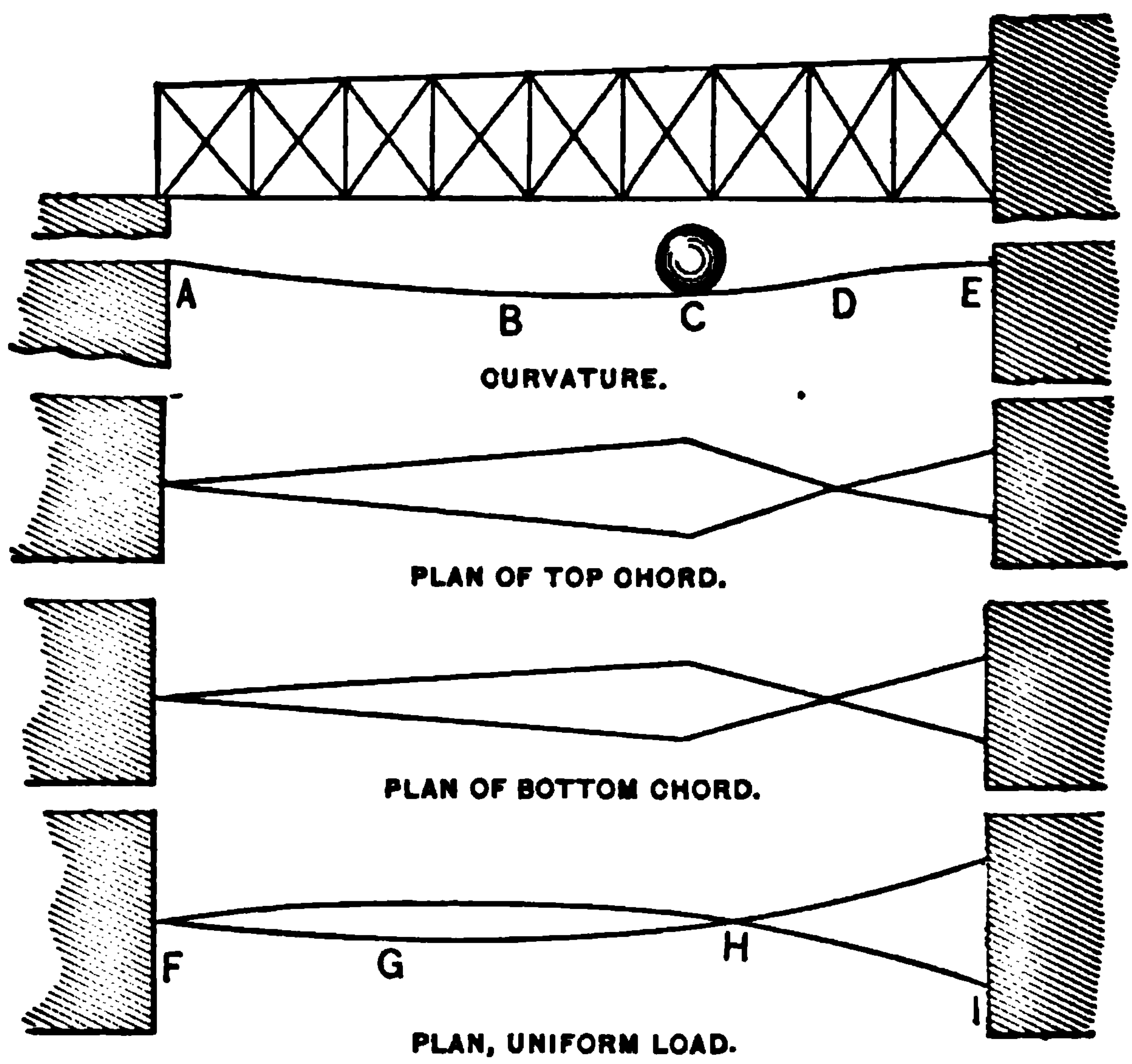


FIG. 103.

Using these moments, we find the required cross-sections, by means of equations (360) and (361), as follows:

$x = \tfrac{1}{3}l$, section of top chord, $S' = 15.23$ square inches.
$x = \tfrac{1}{3}l$, section of bottom chord, $S'' = 12.12$ square inches.
$x = l$, section of top chord, $S'' = 26.80$ square inches.
$x = l$, section of bottom chord, $S' = 33.33$ square inches.

Deflection for the three parts, by (311):—

FG, fixed at G; $h_0 = 180$ inches,

$$y = D_1 = \tfrac{9}{125}[h_1 - 180 \times 2.302585(\log h_1 - \log 180) - 180].$$

GH, fixed at G; $h_0 = 260$ inches,

$$y = D_2 = \tfrac{9}{125}[h_1 - 260 \times 2.302585(\log h_1 - \log 260) - 260].$$

HI, fixed at I; $h_0 = 260$, $h_1 = 300$, $h_0 - h_1 = -40$, $\dfrac{h_1}{h_0} = \dfrac{15}{13}$,

$$y = D_3 = \tfrac{9}{125}[300 - 260(2.302585 \log \tfrac{15}{13} + 1)].$$

From $D_1 = D_2 + D_3$, we find $h_1 = 226.547$ at the point $x = 10(226.547 - 180) = 465.47$ inches,

$$\therefore\ D_1 = 0.37 \text{ inch}$$

at G, Fig. 103.

SECTION 5.

Camber.

108. Camber is the slight upward curving or crowning that is sometimes given to a girder, in order to obviate the sagging which would otherwise result from the deflection of the same girder made without this slight arching. The effect of camber is, therefore, to keep the track line straight under the working-load, and thereby prevent that increase of stress which would otherwise be developed by the falling and rising of loads moving rapidly along a line originally straight. In no other respect

does camber augment the efficiency of the structure. Sometimes, however, a greater upward curvature than that here contemplated is given to the floor line of highway bridges, as being more pleasing to the eye; but so large a convexity, if effected in the girder itself, is always at the expense of material or of efficiency, as will appear from a comparison of the capabilities of two girders shaped like Figs. 23 and 80, of equal length and equal height between axes of chords.

It is evident that camber may be given to the floor or track line in three ways:—

1st, The girders may be made in normal shape, and the floor or track line be raised sufficiently to counteract the deflection due the total load. In this case the two chords of each girder will sag, while the cambered floor line becomes straight under load.

2d, The chord which carries the floor line may be cambered, while the other is built in normal shape. In this case the uncambered chord will sag, while the other assumes its normal shape under load.

3d, The girder may be so built, that, before the load is imposed, its proper floor line will have a deflection equal and opposite to the deflection due the total load, and that the whole girder will assume its normal shape under load.

We need examine and exemplify only the second and third cases.

109. Change of Length calculated from the Unit Strain. — If λ_1 = total contraction for the original length l_1, and λ_2 = total elongation for the original length l_2, of any strained member, we have, within the elastic limit where the amount of displacement per unit of length varies as the stress, —

For compressed member,

$$\lambda_1 = \frac{C_1 l_1}{E_c}; \tag{364}$$

for extended member,

$$\lambda_2 = \frac{T_1 l_2}{E_t}; \tag{365}$$

C_1 and T_1 being the allowed unit strains, and E_c and E_t the moduli of compressive and of tensile elasticity respectively.

The total difference between the lengths of the two chords of a girder after deflection is, therefore,

$$\lambda = \lambda_1 + \lambda_2 = \left(\frac{C_1}{E_c} + \frac{T_1}{E_t}\right)l, \tag{366}$$

provided the chords were of equal length, l, before deflection, and of uniform strength.

If an originally straight girder of equal and parallel chords take the circular form, Fig. 104, after deflection, the neutral line being midway between the chords, we must have for it,

$$\lambda_1 = \lambda_2 = \frac{C_1 l}{E_c} = \frac{T_1 l}{E_t} = \frac{\frac{1}{2}(C_1 + T_1)l}{\frac{1}{2}(E_c + E_t)} = \frac{B_1 l}{E}, \tag{367}$$

$$\therefore \lambda = \lambda_1 + \lambda_2 = \frac{C_1 l}{E_c} + \frac{T_1 l}{E_t} = \frac{2B_1 l}{E}, \tag{368}$$

if $E = \frac{1}{2}(E_c + E_t)$ = modulus of transverse elasticity, and $B_1 = \frac{1}{2}(C_1 + T_1)$ = bending unit strain.

110. Elongation and Contraction calculated from Deflection. — Let $ABCD$, Fig. 104, represent an open-built semi-girder fixed horizontally at A and C, and having its deflection, $D = NH$, greatly exaggerated in the figure; the actual lines NFM and HM being sensibly equal each to l, the original

length of the parallel chords AB and CD of the semigirder whose height is $AC = h$.

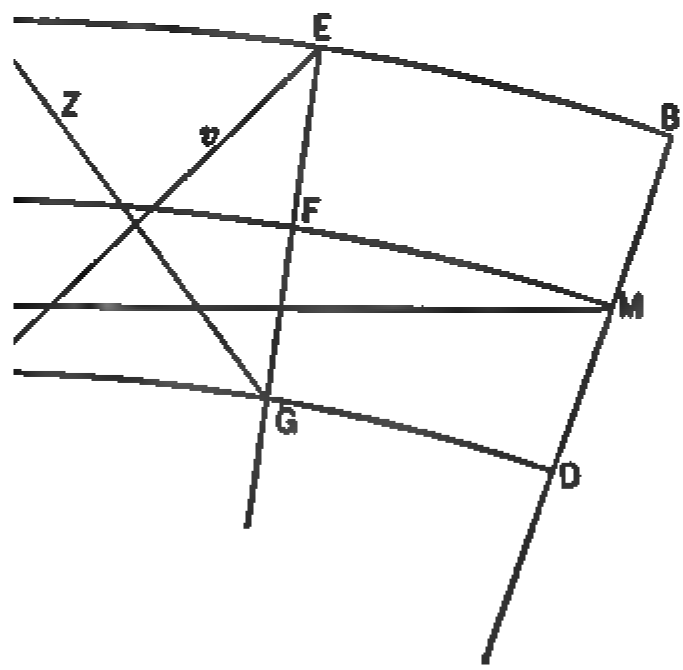


FIG. 104.

We may without appreciable error, for the present purpose, regard the deflection curve as circular.

Take, as radii of curvature, r for neutral line NM, $r + \frac{1}{2}h$ for the extended chord, $r - \frac{1}{2}h$ for the contracted chord. Then, from the geometry of the figure, we have

$$l^2 = D(2r - D),$$

$$\therefore\ r = \frac{l^2}{2D} + \tfrac{1}{2}D.$$

$$r = \frac{l^2}{2D}, \tag{369}$$

since D is very small compared with r.

Also, from the figure,

$$\frac{r}{r + \frac{1}{2}h} = \frac{l}{l + \lambda_1},$$

$$\therefore \quad l + \lambda_1 = \frac{l(r + \frac{1}{2}h)}{r}, \tag{370}$$

which is the length to be given to the chord in compression, the figure being inverted. Again,

$$\frac{r}{r - \frac{1}{2}h} = \frac{l}{l - \lambda_2},$$

$$\therefore \quad l - \lambda_2 = \frac{l(r - \frac{1}{2}h)}{r}, \tag{371}$$

the length required for the chord in tension, the figure being inverted.

Subtracting (371) from (370), we find the total difference in length required,

$$\lambda = \lambda_1 + \lambda_2 = \frac{hl}{r} = \frac{2Dh}{l}, \tag{372}$$

after eliminating r by means of (369).

It may be noted that (368) and (372) give us $D = \dfrac{B_1 l^2}{Eh}$, which is equation (308) for the semi-girder of the length l.

In (369) and (372), we must, of course, put $\frac{1}{2}l$ for l, when we apply these equations to a girder of the length l supported at both ends.

111. Suppose, now, that it is required to camber only the straight, horizontal bottom chord, to which the moving-load is to be applied. This supposition includes Classes II., IV., VII., IX., X., and XII. of article 49.

We may, by the formula proper for the given girder, find the deflection at each panel point, or apex, of the bottom chord. If we now assume that no apices of the bottom chord are to be moved horizontally, by reason of the adopted camber, we must theoretically *increase* each normal panel length, c, of this chord, in the ratio $\frac{c + \Delta c}{c} = \frac{1}{\cos\beta}$; β being the inclination to the horizon of any panel length, $c + \Delta c$ of the bottom chord when cambered, and Δc being the change of length in the bottom chord for any panel, by reason of the adopted camber. Also, $\tan\beta = \frac{D - D_1}{c}$, D_1 and D being the deflections at any two consecutive apices.

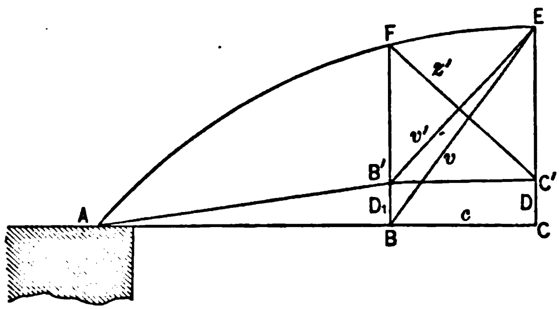


FIG. 105.

Then each vertical member, as FB, Fig. 105, coming down to a lower apex, B, must be shortened by the amount of deflection, D_1, computed for the apex; and each diagonal or brace, $BE = v$, terminating at the same apex, must be shortened in the ratio

$$\frac{v'}{v} = \left\{\frac{c^2 + (h - D_1)^2}{c^2 + h^2}\right\}^{\frac{1}{2}}, \qquad (373)$$

where h is the normal difference of level between the ends of any diagonal.

In this case the effective depth of the girder at the centre has been lessened by D.

EXAMPLE. — Let us find the changes of length required to effect camber in the bottom chord alone of a wrought-iron parabolic bowstring, where $l = 2{,}400$ inches, $h = 225$, $n = 16$ panels of $c = \frac{2400}{16} = 150$ inches each, $C_1 = 6{,}698$ pounds, $T_1 = 10{,}000$ pounds, $B_1 = 8{,}349$ pounds, and $E = 24{,}000{,}000$ after the frame has taken its permanent "set;" but, as explained in article 90, we will, for the present purpose, take $E = 16{,}000{,}000$, to provide for any sagging that might otherwise be caused by the first full load, beyond what the elasticity of the frame can recover.

By equation (319), putting $\frac{1}{2}l = 1{,}200$ for l, we have deflection at centre,

$$D = \frac{1.386295 \times 8349 \times 1200^2}{16000000 \times 225} = 4.6297 \text{ inches.}$$

Now we may, by using equation (318), find the deflection at each panel point; but it will be practically accurate, and more simple, to regard the cambered bottom chord as a parabola, having the central height $D = 4.6297$ inches, and then find, by equations (136) and (137), both the normal heights, h, and the height of each lower apex after camber is effected. Thus, (136) now becomes

$$y = \left(1 - \frac{4x^2}{2400^2}\right) \times 4.6297$$

for bottom chord, and

$$y = \left(1 - \frac{4x^2}{2400^2}\right) \times 225$$

for top chord; the origin being at the middle of the bottom

chord in its normal shape. From these equations and (373) we compute D, v', s', h, in inches.

x	h	D	ΔD	$h-D$	$h_r - D_{r+1}$	$h_{r+1} - D_r$	v'	s'
0	225.00	4.63	−0.07	220.37	220.44	216.85	266.64	263.67
150	221.48	4.56	−0.22	216.92	217.14	206.38	263.91	258.14
300	210.94	4.34	−0.36	206.60	206.96	189.09	255.60	241.36
450	193.43	3.98	−0.51	189.45	189.96	164.77	242.04	222.82
600	168.75	3.47	−0.65	165.28	165.93	133.64	223.68	200.90
750	137.11	2.82	−0.79	134.29	135.08	95.62	201.86	177.88
900	98.44	2.03	−0.95	96.41	97.36	50.71	174.76	158.34
1050	52.74	1.08	−1.08	51.66	52.74	-	-	-

Theoretically, the end panel lengths of bottom chord, where the inclination, β, is greatest, would become

$$(150^2 + 1.085^2)^{\frac{1}{2}} = 150.0039 \text{ inches.}$$

But this is practically equal to the normal length, 150 inches; hence we will not change the panel lengths of the bottom chord.

It will be perceived that the girder thus cambered becomes the parabolic crescent.

Instead of computing dimensions as above, it is evident that the elevation may be drawn accurately, on a large scale, from the central deflection D, and h and l; and then all desired lengths can be taken off as accurately as the work will be "laid out" in the shop. The camber curve may always be drawn circular for an originally straight chord.

112. Similarly, if we would camber only the straight upper horizontal chord of Classes III., IV., VIII., IX., XI., XII., of

article 49, without moving appreciably the upper apices horizontally, we must increase the normal length of each vertical

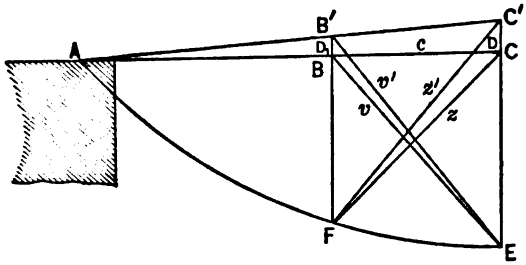


FIG. 106.

member by the deflection due at its place, and the normal length of each diagonal in the ratio

$$\frac{v'}{v} = \left\{\frac{c^2 + (h + D)^2}{c^2 + h^2}\right\}^{\frac{1}{2}}.$$

See Fig. 106, where the efficient depth of the girder has been increased at the centre by the value of the deflection D.

EXAMPLE. — Let us invert the uncambered girder of article 111, and effect the same amount of camber, $D = 4.6297$ inches, in the straight top chord alone. We have the same values of D and h as before, and readily find the required lengths of verticals and diagonals, in inches, numbering from the centre.

$$v' = [c^2 + (h_r + D_{r+1})^2]^{\frac{1}{2}} = c\left\{1 + \left(\frac{h_r + D_{r+1}}{c}\right)^2\right\}^{\frac{1}{2}}, \quad (374)$$

$$z' = [c^2 + (h_{r+1} + D_r)^2]^{\frac{1}{2}} = c\left\{1 + \left(\frac{h_{r+1} + D_r}{c}\right)^2\right\}^{\frac{1}{2}}. \quad (375)$$

x	h	D	$h + D$	$h_r + D_{r+1}$	d'	$h_{r+1} + D_r$	d''
0	225.00	4.63	229.63	229.56	274.22	226.11	271.34
150	221.48	4.56	226.04	225.82	271.09	215.50	262.57
300	210.94	4.34	215.28	214.92	262.08	197.77	248.22
450	193.43	3.98	197.41	196.90	247.53	172.73	228.77
600	168.75	3.47	172.22	171.57	227.90	140.58	205.58
750	137.11	2.82	139.93	139.14	204.59	101.26	180.98
900	98.44	2.03	100.47	99.52	180.01	54.77	159.69
1050	52.74	1.08	53.82	-	-	-	-

In like manner may we effect camber in a straight chord of any one of the classes cited in this and the preceding article. And, if it is required to preserve the normal height between chords after camber, we must change both.

113. When it is desired that the effective depth of the girder be not altered by the camber, then both chords must be displaced vertically by the amount of the deflection at the several apices, and in the opposite direction.

In articles 111 and 112 we have made no appreciable change in the length of either chord by reason of camber; and, of course, the length of each chord will be changed as the load takes out the camber.

Strictly, regarding camber as the inverse of the operation performed by deflection, we should increase the normal length of the compressed chord by λ_1, equation (364), and diminish that of the stretched chord by λ_2, equation (365); but, since the whole change of length required in either chord is very small for each panel, we shall distribute, in the present case, the whole difference, $\lambda = \lambda_1 + \lambda_2$, of length due to deflection additively among the panel lengths of the *compressed* chord. Hence camber thus produced will require no change in the normal panel lengths of the chord in tension, which, under the load, will resume its normal line, but increased in length by λ_2.

At the same time, the compressed chord loses λ_1 of its increment, and retains λ_2.

This change of length in either chord which rests upon the points of support, may, if necessary, be provided for in the same manner that provision is made for the effect of change of temperature. The length of any vertical member is not to be altered appreciably for camber in this case, since the vertical displacement of each chord is assumed to be the same for any given value of x; and the slight change in their length caused by the spreading of the verticals to fit the change in the compressed chord, is hardly measurable.

But the length of any diagonal member will be changed as follows: —

Let $ABCE$, Fig. 107, represent any one of the n normal panels of a girder, and $A'B'C'E'$ the same panel when cambered

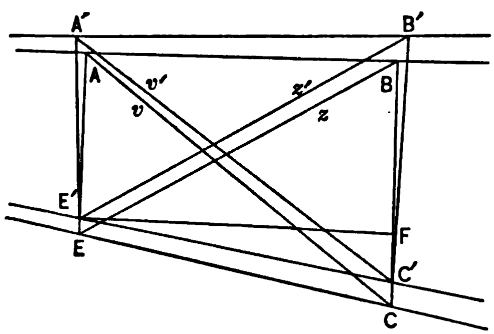


FIG. 107.

by adding $\frac{\lambda}{n}$ to $\frac{l}{n} = c$, the horizontal projection of the chord AB in compression. The panel points in both chords are displaced vertically by the deflections $D_r = CC' = BB'$, and $D_{r+1} = EE' = AA'$, appreciably; and the points A and B

are removed horizontally by the space $\frac{\lambda}{n}$. Hence practically we have

$$E'C' = EC,$$

$$A'B' = AB + \frac{\lambda}{n},$$

$$A'E' = AE,$$

$$B'C' = BC,$$

$$E'F = \frac{l}{n}, \quad FC' = \frac{l}{n}\tan\beta;$$

β being now the inclination of EC or $E'C'$ to the horizon.

$$s' = E'B' = \left\{\left(\frac{l}{n} + \frac{\lambda}{2n}\right)^2 + \left(h_r - \frac{l}{n}\tan\beta\right)^2\right\}^{\frac{1}{2}}, \qquad (376)$$

$$v' = A'C' = \left\{\left(\frac{l}{n} + \frac{\lambda}{2n}\right)^2 + \left(h_{r+1} + \frac{l}{n}\tan\beta\right)^2\right\}^{\frac{1}{2}}. \qquad (377)$$

Instead of $\frac{\lambda}{n}$, we may evidently employ equations (364) and (365) in finding the proper increment for any panel length of compressed chord.

EXAMPLE. — Take $BC = h_r = 240$ inches, $AE = h_{r+1} = 200$ inches, $E'F = \frac{l}{n} = c = 150$ inches, $\frac{\lambda}{n} = \frac{\lambda_1 + \lambda_2}{n} = 0.12$ inch, $\frac{\lambda}{2n} = 0.06$ inch, $\tan\beta = \frac{1}{2} \times \frac{240 - 200}{150} = \frac{2}{15}$. Then

$$A'B' = AB + 0.12,$$

$$s' = [(150.06)^2 + (240 - 150 \times \tfrac{2}{15})^2]^{\frac{1}{2}} = 266.304 = v'$$
$$= [(150.06)^2 + (200 + 20)^2]^{\frac{1}{2}} \text{ inches.}$$

CHAPTER VII.

PILLARS.

SECTION I.

Strength of Pillars, by Rational Formulæ.

114. Under the general term *pillars* we shall include columns, posts, struts, props, braces in compression, and, in a word, every member in a structure whose function it is to resist compressive force applied at its end, and, in general, in the line of the longitudinal axis of the member.

It is assumed that a pillar has no lateral support or pressure applied between its ends, except when, owing to an unavoidable existing lateral force (as, for example, the weight of a horizontal strut), a counter-force is applied as a balance. But a pillar may have its ends in the conditions known as round, hinged, flat, imbedded, fixed; the two ends being in the same or in different conditions. Pillars may be long or short, solid or hollow; may have a uniform or variable cross-section of any desired form.

Long pillars yield chiefly by bending and breaking across; short blocks of ordinary building material yield by being crushed without bending, properly so called. At what exact ratio of length to diameter bending first takes place in a given material, is not at present very definitely ascertained; but it will be safe to assume, in the present state of our knowledge, that bending will occur when this ratio is as low as three for such

material as wrought-iron. Experiment has shown what, perhaps, we might have inferred from a stalk of wheat,—that material is saved by using hollow instead of solid pillars to support a given load.

115. Pillars of Uniform Cross-Section.

Let l = length of pillar,

h = least diameter,

r = least radius of gyration of cross-section,

S = area of cross-section,

D = greatest deflection of pillar;

all in inches.

E = modulus of transverse elasticity,

C = crushing-strength of standard specimen of the material,

P = breaking-weight applied at the end of the pillar and in the line of its axis before deflection,

$Q = \dfrac{P}{S}$ = breaking-weight per square inch of cross-section;

all in pounds per square inch.

$I = Sr^2$ = least moment of inertia (so called) of cross-section.

M_x = moment of forces developed in any normal cross-section by the given load P.

M_1 = the end moment at the lower end when that end is fixed.

M_2 = the end moment at the top when the upper end is fixed.

Suppose the pillar vertical, Figs. 108, 109, 110, and take the origin of rectangular co-ordinates at the lowest point of the pillar's axis, which call also the axis of x; that of y being horizontal.

Then, from equations (15), (93), and (187), we have the moment at any height, x,

$$M_x = -EI\frac{d^2y}{dx^2} = \frac{M_1 - M_2}{l}x - M_1 + Py, \quad (378)$$

wherein no account is taken of the modified condition of every cross-section due to the longitudinal pressure, Q, per unit.

Now, since the full unit strength of the cross-section of the unloaded pillar is C, and the remaining unit strength of the loaded pillar's cross-section is $(C - Q)$, it follows that the expression for the moment of the internal forces developed in any cross-section must be diminished in the ratio $\frac{C-Q}{C}$.

We then have

$$M_x = -P\varepsilon^2\frac{d^2y}{dx^2} = \frac{M_1 - M_2}{l}x - M_1 + Py \qquad (379)$$

if

$$\varepsilon^2 = \frac{EI(C-Q)}{PC} = \frac{Er^2(C-Q)}{QC}.$$

There will be three cases, according as we consider neither, both, or one only, of the ends fixed.

CASE I. — If neither end can produce any moment, $M_1 = M_2 = 0$; and we have, from (379),

$$\varepsilon^2\frac{d^2y}{dx^2} = -y. \qquad (380)$$

Multiplying by $2dy$,

$$2\varepsilon^2\frac{dy\,d(dy)}{dx^2} = -2y\,dy.$$

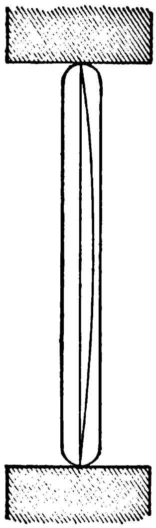
Integrating this equation, and putting a^2 for the arbitrary constant of integration,

$$\varepsilon^2\frac{dy^2}{dx^2} = a^2 - y^2,$$

from which

$$\frac{dx}{\varepsilon} = \frac{dy}{(a^2 - y^2)^{\frac{1}{2}}}.$$

Integrating again between the limits, for x, 0 and l; and for y, 0 and 0;



$$\therefore \quad l = \epsilon\left[\sin^{-1}\frac{y}{a}\right]_0^0 = \epsilon n\pi,$$

where n may be any whole number; but, in order that P may have the least value it can have, consistent with the bending of the pillar necessarily assumed in establishing equations (378) and (379), n must be equal to unity. (See Rankine's "Applied Mechanics," p. 352.)

$$\therefore \quad l^2 = \pi^2\epsilon^2 = \frac{\pi^2 Er^2(C - Q)}{QC},$$

$$Q = \frac{C}{1 + \frac{Cl^2}{\pi^2 Er^2}}, \qquad (381)$$

Fig. 108.

which is the formula for pillars with rounded ends that can generate no end moments, Fig. 108. The curved line shows the deflected axis.

Case II. — If both ends of the pillar are equally fixed, Fig. 109, so that the elastic curve at each end, after flexure, has for its tangent the original undeflected axis, then, in equation (379),

$$M_1 = M_2,$$

whence

$$P\epsilon^2\frac{d^2y}{dx^2} = M_1 - Py. \qquad (382)$$

Multiplying by $2dy$, equation (382) becomes

$$2P\epsilon^2\frac{dy\,d(dy)}{dx^2} = 2M_1dy - 2Py\,dy.$$

Integrating, we find

$$P\varepsilon^2\frac{dy^2}{dx^2} = 2M_1y - Py^2 + a, \qquad (383)$$

where a, the arbitrary constant, must vanish, since $\frac{dy}{dx} = 0$ when $y = 0$. Hence, from (383),

$$\frac{dx}{\varepsilon\sqrt{P}} = \frac{dy}{(2M_1y - Py^2)^{\frac{1}{2}}}.$$

Integrating again, with the condition that $y = 0$ when $x = 0$, there results, after cancelling $\sqrt{P}$ from the denominators,

$$\frac{x}{\varepsilon} = \sin^{-1}\left(\frac{Py - M_1}{M_1}\right) + \frac{\pi}{2}. \qquad (384)$$

Also we have $y = 0$ when $x = l$, so that (384) becomes

$$\frac{l}{\varepsilon} = \sin^{-1}(-1) + \frac{\pi}{2} = \frac{3\pi}{2} + \frac{\pi}{2} = 2\pi,$$

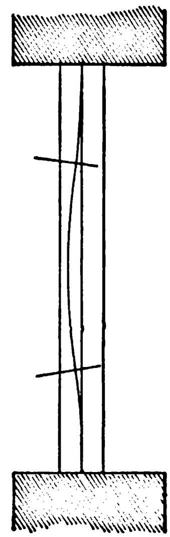


FIG. 109.

to be consistent with the permanence of l and with the least positive value of P. Therefore

$$l^2 = 4\pi^2\varepsilon^2 = \frac{4\pi^2Er^2(C - Q)}{QC},$$

$$Q = \frac{C}{1 + \frac{Cl^2}{4\pi^2Er^2}}, \qquad (385)$$

which is the formula for pillars with both ends equally and fully fixed.

CASE III. — When only one end of the pillar is fixed, Fig. 110, and the other end can cause no end moment, we have (say) $M_1 = 0$, and derive, from equation (379),

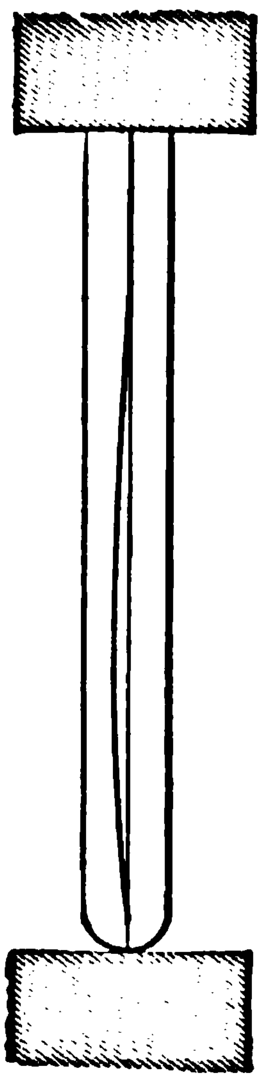


FIG. 110.

$$Pr^2\frac{d^2y}{dx} = \left(\frac{M_2}{l}x - Py\right)dx, \qquad (386)$$

an equation whose second member cannot be integrated "without specific connection between x and y," unless there is such a relation among the quantities which compose it that the member shall be wholly a function of x. (See De Morgan's "Differential and Integral Calculus," p. 208.) Not knowing the specific connection between x and y, nor what relation among its component quantities there may be to render the second member a "function of x only," I shall here assume a connection expressed by the equation of a parabola; viz.,

$$y = x + bx^2, \qquad (387)$$

where, since $y = 0$ when $x = l$, $b = -\frac{1}{l}$; and shall attempt only an approximate solution for this third case.

Putting this value of y into equation (386), it becomes

$$Pr^2\frac{d^2y}{dx^2} = mx - Pbx^2$$

if $m = \frac{M_2 - Pl}{l}$.

Integrating, with the condition that $\frac{dy}{dx} = 0$ when $x = l$, since the upper end of the pillar is now fixed,

$$\therefore\ Pr^2\frac{dy}{dx} = \tfrac{1}{2}m(x^2 - l^2) - \tfrac{1}{3}Pb(x^3 - l^3). \qquad (388)$$

Integrating again between the limits 0 and y, 0 and x,

$$\therefore \quad Pe^2y = \tfrac{1}{2}m\left(\frac{x^3}{3} - l^2x\right) - \tfrac{1}{3}Pb\left(\frac{x^4}{4} - l^3x\right). \qquad (389)$$

But $y = 0$ when $x = l$; hence, from (389),

$$m = \tfrac{3}{4}Pbl. \qquad (390)$$

If in (388) we make $\frac{dy}{dx} = 0$, we may find the value of x which renders $y = D$ a maximum. Dividing by $x - l$, and, by means of (390), eliminating m and Pb, we derive from (388), when y is a maximum,

$$x = \tfrac{1}{16}l(1 \pm \sqrt{33}) = 0.42153l,$$

since the negative value is not here admissible.

Taking $x = 0.42153l$, $y = x + bx^2$, and $b = -\frac{1}{l}$, we find from (389), after restoring e^2 and m, and reducing,

$$Q = \frac{C}{1 + \dfrac{Cl^2}{22.511Er^2}} = \frac{C}{1 + \dfrac{Cl^2}{2.28\pi^2Er^2}}, \qquad (391)$$

which is an approximate formula for pillars having but one end fixed.

The nearness and sufficiency of this approximation will be examined in article 120.

It should be observed here that equations (381), (385), and (391) are applicable to all pillars that yield by bending, of whatever uniform cross-section, and of whatever material constructed. Examples of the application of these three equations may be found in the tables of article 121.

SECTION 2.

Hodgkinson's Empirical Formulæ for the Strength of Cast-Iron and Timber Pillars.

116. The eminent English experimenter Mr. Eaton Hodgkinson deduced, from his experiments upon pillars of cast-iron and pillars of timber, formulæ which have found place in all works on applied mechanics.

Using the notation of article 115, and taking those values of the constants which have been adopted by such writers as Rankine and Humber, Mr. Hodgkinson's formulæ for the *ultimate* strength of cylindrical cast-iron pillars, where the length of each is not less than thirty times the diameter if the ends are flat, and not less than fifteen times the diameter if the ends are rounded, become, —

Solid cast-iron pillars,

$$P = A\frac{h^{3.6}}{(\frac{1}{12}l)^{1.7}}; \qquad (392)$$

hollow cast-iron pillars,

$$P = A\frac{h^{3.6} - h_1^{3.6}}{(\frac{1}{12}l)^{1.7}}; \qquad (393)$$

h_1 being the pillar's internal diameter, and A "representing the strength of a pillar 1 foot long, and 1 inch in diameter, and being a constant for a given quality of iron, but ranging in value, for different irons, from 75,000 to 112,000." The mean values of A adopted by Professor Rankine are, —

Solid pillars with rounded ends,

$$A = 14.9 \text{ tons} = 33376 \text{ pounds};$$

solid pillars with flat ends,

$$A = 44.16 \text{ tons} = 98918 \text{ pounds};$$

hollow pillars with rounded ends,

$$A = 13 \text{ tons} = 29120 \text{ pounds};$$

hollow pillars with flat ends,

$$A = 44.3 \text{ tons} = 99232 \text{ pounds}.$$

It hence results experimentally that "fixing" both ends of a pillar, Fig. 109, enables it to support about three times the load which would break it were the ends unfixed, Fig. 108, and incapable of developing moment. For a pillar fixed at one end and rounded at the other, Fig. 110, Mr. Hodgkinson found the strength to be a mean between the two strengths of the same pillar when both ends are rounded and when both ends are flat. We then have, for cast-iron pillars, —

Solid, one flat and one round end,

$$A = 66147 \text{ pounds};$$

hollow, one flat and one round end,

$$A = 64176 \text{ pounds}.$$

When the length is less than 30 or 15 times the diameter respectively, Mr. Hodgkinson first finds P by equations (392) and (393), and then corrects P by means of this supplementary formula; P_1 being the corrected value sought.

$$P_1 = \frac{PCS}{P + \frac{3CS}{4}} = \frac{CS}{1 + \frac{3CS}{4P}}, \qquad (394)$$

$$\therefore \quad Q_1 = \frac{P_1}{S} = \frac{C}{1 + \frac{3CS}{4P}} = \frac{C}{1 + \frac{3C}{4Q}}, \qquad (395)$$

which is an empirical equation identical in *form* with (381), (385), and (391), analytically established.

117. The Hodgkinson formula for the *ultimate* resistance of pillars of *oak* and of *red pine* to crushing by bending, as adopted by Professor Rankine, "Applied Mechanics," p. 365, is, with our notation, article 115,

$$Q = \frac{P}{S} = 500 C \frac{h^2}{l^2}, \tag{396}$$

a formula to be used only when $Q < C$, the crushing-strength of the material, Table II., article 60.

Applications of the Hodgkinson formulæ are given in tables of article 121.

SECTION 3.

Gordon's Empirical Formula, with Rankine's Modification.

118. We have, in article 115, $Q = \dfrac{P}{S} =$ the direct unit pressure of the load upon every cross-section of the pillar.

Now, if B_1 is the *additional* unit pressure due to bending-moment upon those fibres where the bending-moment is greatest, and if f denote the greatest intensity of unit pressure, we have

$$f = Q + B_1. \tag{397}$$

Regarding, with reference to the central moment, a loaded pillar of uniform cross-section as in the condition of a beam supported at both ends, and carrying the central weight $W = \dfrac{4PD}{l}$, since equations (15), (46), and (187) give us

$$M = PD = \tfrac{1}{4} Wl = \frac{2B_1 I}{h}, \tag{398}$$

we find

$$Wl = \frac{8B_1I}{h} = \frac{48EID}{l^2},$$

from (211);

$$\therefore \quad D = \frac{B_1l^2}{6Eh}.$$

From which, for a given value of $\frac{B_1}{E}$,

$$D \sim \frac{l^2}{h}.$$

But (398) gives

$$B_1 \sim \frac{PDh}{I} \sim \frac{PD}{Sh} \qquad (399)$$

if $I = kh^2S$, k being a constant depending upon the form of the pillar's cross-section (see Table III., article 62);

$$\therefore \quad B_1 \sim \frac{Pl^2}{Sh^2}.$$

Whence (397) becomes

$$f = \frac{P}{S}\left(1 + \frac{l^2}{ah^2}\right),$$

f and a being constants to be determined by experiment;

$$\therefore \quad \frac{P}{S} = \frac{f}{1 + \frac{l^2}{ah^2}}, \qquad (400)$$

which is of the form "proposed by Tredgold," and is now known as the "Gordon Formula," having been, after some "disuse, revived by Mr. Lewis Gordon, who determined the values" of a and f, for certain materials, from the results of Mr. Hodgkinson's experiments.

119. If, in equation (399), we put Sr^2 for I, using still the notation of article 115, we find

$$B_1 \sim \frac{PDh}{Sr^2} \sim \frac{Pl^2}{Sr^2}. \tag{401}$$

Therefore, from (397),

$$f = \frac{P}{S}\left(1 + \frac{l^2}{a_1 r^2}\right),$$

$$\frac{P}{S} = \frac{f}{1 + \frac{l^2}{a_1 r^2}}, \tag{402}$$

which is Professor Rankine's modification of the Gordon formula; r being the least radius of gyration of the cross-section.

The Gordon (400) and the Rankine (402) formulæ are identical if we make

$$\frac{a_1}{a} = \frac{h^2}{r^2}. \tag{403}$$

120. Supposing f to be constant for varying conditions of the pillar, both a and a_1 will be found to require different coefficients, according as the pillar has neither, one, or both, of its ends fixed.

Assuming that equations (400) and (402) apply to a pillar fixed in direction at both ends, Fig. 109, we see that the length, l, between the points of contrary flexure, is in the condition of a pillar not fixed at its ends, and has only the strength of a pillar of twice its length, $2l$, fixed at both ends; that is, for a pillar rounded at both ends, we have,—

Gordon's formula,

$$\frac{P}{S} = \frac{f}{1 + \frac{4l^2}{ah^2}}; \tag{404}$$

Rankine's formula,

$$\frac{P}{S} = \frac{f}{1 + \frac{4l^2}{a_1 r^2}}. \tag{405}$$

Similarly, in Fig. 109, the length, l, between either point of contrary flexure and the remoter end is in the condition of a pillar with one fixed and one rounded end, and has only the strength of a pillar $\frac{4}{3}l$ in length. We have, then, for a pillar fixed at one end and rounded at the other, —

Gordon's formula,

$$\frac{P}{S} = \frac{f}{1 + \frac{\frac{16}{9}l^2}{ah^2}}; \tag{406}$$

Rankine's formula,

$$\frac{P}{S} = \frac{f}{1 + \frac{\frac{16}{9}l^2}{a_1 r^2}}. \tag{407}$$

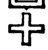
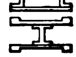
This is Mr. Hodgkinson's ingenious explanation of the variation among the strengths of these three classes of pillars, a variation which he discovered by a comparison of the results of his experiments.

If we invert the three numerical co-efficients of the fractions in the denominators of (400), (404), (406), or (402), (405), (407) (viz., 1, 4, $\frac{16}{9}$), and multiply the inverted numbers by 4, we have the relation, 4, 1, 2.25; while 4, 1, 2.28, is the relation of the corresponding constants in equations (385), (381), (391), determined analytically. We may hence infer that the degree of approximation in (391) is close to the true value. Especially, since we can seldom tell the exact amount of influence which given end bearings exert, may we regard (391) practically correct.

TABLE IV.

VALUES OF f AND a OF THE GORDON, AND OF f_1 AND a_1 OF THE RANKINE FORMULA.

A., American Bridge Company, Chicago, Ill.; K., Keystone Bridge Company, Pittsburgh, Penn.

Material.	Form of Section.	Experimenters.	Authority.	Gordon Formula.		Rankine Formula.	
				f	a	f_1	a_1
Iron, Cast . .		Hodgkinson.	Gordon.	80000	400	-	-
Iron, Cast . .		Hodgkinson.	Gordon.	80000	267	-	-
Iron, Cast .		Hodgkinson.	Gordon.	80000	133	-	-
Iron, Wrought .		Hodgkinson.	Gordon.	36000	3000	-	-
Iron, Wrought .	"	Hodgkinson.	Rankine.	36000	3000	36000	36000
Iron, Wrought .	"	Hodgkinson.	Stoney.	35840	3000	-	-
Iron, Wrought .		Hodgkinson.	Stoney.	30660	3000	-	-
Iron, Wrought .		Hodgkinson.	Stoney.	40032	3000	-	-
Iron, Wrought .		Davies.	Unwin.	42560	900	-	-
Iron, Wrought .		A. K.	Lovett.	49580	3000	42980	36000
Iron, Wrought .		A. K.	Lovett.	43725	3000	38650	36000
Iron, Wrought .		A.	Lovett.	38271	3000	37029	36000
Iron, Wrought .		K.	Lovett.	36523	3000	33531	36000
Steel, Mild . .		-	Baker.	67200	1400	-	-
Steel, Strong .	"	-	Baker.	114240	900	-	-
Steel, Mild . .		-	Baker.	67200	2480	-	-
Steel, Strong .	"	-	Baker.	114240	1600	-	-
Timber . . .	"	Hodgkinson.	Rankine.	7200	250	-	-
Oak and Fir .	"	Rondelet.	Stoney.	1.5 C of Table II.	250	-	-
Stone and Brick,	"	-	Rankine.	C of Table II.	600	-	-

SECTION 4.

Strength of Pillars computed by the Preceding Formulæ, and compared with the Strength experimentally determined.

121. The following tables, V., VI., VII., VIII., IX., X., XII., contain data derived from experiments on the strength of pillars, probably as trustworthy as any yet made and published. To these tests the appropriate formulæ, either direct or inverted, have been applied; and the values of f, C, or Q for a given pillar, computed by different formulæ, have been tabulated in the same horizontal line.

In Table XI. no experimental values are given, but the assumed values of E and C are within the limits fixed by experiments upon steel. In Table VII., when the thickness, t, of the metal is less than a fifty-fifth part of the least diameter, h, of the pillar, the computed value of Q, the breaking-weight, in pounds, per square inch, has been diminished in the ratio $\frac{55t}{h}$, as seems to be required by the tests.

For columns having rounded or hinged ends, in Table V., the formulæ for those having one flat and one round end have been used, as more in harmony with the tests than the formulæ for columns having no end moments.

It must be confessed that there are anomalies of considerable magnitude in the experiments themselves; and, of course, there appear corresponding variations from the test values in the numbers computed according to the laws of the applied formulæ.

It is to be regretted that we have not, accompanying these tests for Q, also experimental determinations of C and of E, for each pillar tabulated, but have been obliged to use probable mean values of C in all the calculations of Q, and probable mean values of E in all but Table V.

TABLE V.—WROUGHT-IRON PILLARS.

DATA, AND VALUES OF f AND f_1, FROM THOMAS D. LOVETT'S REPORT TO THE TRUSTEES OF THE CINCINNATI SOUTHERN RAILWAY, DEC. 1, 1875.

P., Phœnix column, hollow cylinder, 4 flanged segments riveted; A., American column, 2 flanged bars riveted to the flanges of an I-beam; K., Keystone column, octagonal tube, 4 flanged segments bolted; S., Square column, 2 plates and 2 channels riveted.

No.	Name.	Condition of Ends.	S Area of Cross-section.	h Least Diameter.	l Length.	r^2 Square of Least Radius of Gyration.	$\frac{E}{100000}$ E, Modulus of Transverse Elasticity.	$Q=\frac{P}{S}$ Breaking-Weight, by Experiment.	f Gordon, $Q\left\{1+\frac{l^2}{3000h^2}\right\}$	f_1 Rankine, $Q\left\{1+\frac{l^2}{36000r^2}\right\}$	C (385), $\frac{Q}{1-\frac{Ql^2}{4\pi^2Er^2}}$
			sq. ins.	ins.	ins.	sq. ins.	lbs. to in.	lbs. to in.	lbs. to in.	lbs. to in.	lbs. to in.
1	P.	Flat.	13.58	8.250	336	8.935	285	36600	57500	49400	62141
2	P.	Flat.	13.58	8.250	336	8.935	257	34800	54600	47000	61417
3	P.	Flat.	13.70	8.125	324	8.935	291	31000	47400	41100	45390
4	P.	Flat.	14.09	8.050	180	8.536	274	37500	43700	41500	43182
5	A.	Flat.	14.97	8.000	360	5.388	260	23700	39700	39500	53304
6	A.	Flat.	20.10	9.500	324	8.635	329	27800	37500	37200	37578
7	A.	Flat.	20 10	9.500	240	8.653	236	31500	37500	37300	40648
8	K.	Flat.	18.83	8.625	324	9.798	237	27800	40800	36000	40783
9	K.	Flat.	12.96	9.200	324	10.883	-	24100	34100	30600	30725
10	K.	Flat.	14.49	9.375	324	11.178	275	27500	38400	34700	36084
11	K.	Flat.	18.83	9.625	324	11.424	281	21100	29100	26500	25569
12	K.	Flat.	15.13	9.500	324	11.464	193	30000	42100	37600	46914
13	K.	Flat.	15.13	9.500	324	11.464	236	25400	36100	31900	33850
14	K.	Flat.	19.20	9.500	324	12.041	265	25000	34700	31100	31579
15	K.	Flat.	23.67	8.850	180	7.833	247	32000	36400	35700	37026
16	K.	Flat.	14.62	8.300	180	9.206	238	30000	34700	32900	33798
17	K.	Flat.	14.80	9.000	180	10.353	296	36900	41800	40100	41895
18	K.	Flat.	14.84	9.250	180	10.834	346	28800	32400	31200	30738
19	K.	Flat.	14.25	9.300	60	11.044	-	33600	34100	33900	33945
20	S.	Flat.	13.60	7.500	312	9.347	278	30000	47300	38700	41939
21	S.	Flat.	26.05	9.250	324	10.909	301	30200	39800	38300	39977
22	S.	Flat.	13.70	8.430	288	11.628	289	33200	46100	39800	41896
									$Q\left\{1+\frac{l^2}{1500h^2}\right\}$	$Q\left\{1+\frac{l^2}{18000r^2}\right\}$	(391) $\frac{Q}{1-\frac{Ql^2}{2.28\pi^2Er^2}}$
23	A.	Hinged.	12.50	8.000	240	5.479	289	26700	42700	42300	46976
24	A.	Hinged.	19.90	10.000	240	8.733	231	26500	36700	36200	39925
25	A.	Hinged.	25.05	10.750	312	10.092	304	24000	37500	31100	36276
26	A.	Hinged.	20.72	10.000	312	8.733	260	22000	36300	35600	37868
27	P.	Round.	13.89	8.125	324	8.935	271	21700	44700	35900	37289
28	K.	Hinged.	13.12	9.220	324	10.954	295	22000	40100	33700	32239
29	S.	Hinged.	13.60	10.000	309	11.000	310	25500	41700	37800	37150
	Means,	-	-	-	-	-	271⅙	-	40190	36710	39041

Mean value of E, equal to 27,1[illegible], used for Nos. 9 and 19.

TABLE VI.

Solid Rectangular Pillars of Wrought-Iron.

Flat ends well bedded; Hodgkinson's Experiments.

Data from Bindon B. Stoney's Theory of Strains in Girders and Similar Structures.

$k^2 = 12r^2$. Assume $E = 27,311,111$, and $C = 50,000$.

	S	k	l	$l \div k$	$Q = \frac{P}{S}$	Excess over Q by	
No.	Sectional Area.	Least Diameter.	Length.	Ratio of Length to Least Diameter.	Breaking-Weight, by Experiment.	Gordon Formula, $Q = \dfrac{35840}{1 + \dfrac{l^2}{3000k^2}}$	Equation (385), $Q = \dfrac{C}{1 + \dfrac{Cl^2}{4\pi^2 Er^2}}$
	sq. ins.	ins.	ins.		lbs. per sq. in.	lbs. per sq. in.	lbs. per sq. in.
30	1.0465	1.0230	7.5	7.331	48682	−13473	− 134
31	1.0465	1.0230	15.0	14.663	34554	− 1105	+10103
32	1.0475	1.0230	30.0	29.326	25327	+ 2527	+ 8489
33	2.9880	0.9960	30.0	30.121	29655	− 2137	+ 3570
34	2.2970	0.7630	30.0	39.319	27767	− 4115	− 890
35	1.0486	1.0240	60.0	58.594	17268	− 555	− 88
36	4.5900	1.5300	90.0	58.524	19987	− 3343	− 2896
37	1.5011	0.5026	30.0	59.689	16853	− 470	− 90
38	5.8166	0.9960	60.0	60.241	17698	− 1479	− 1138
39	2.9950	0.9950	60.0	60.303	18067	− 1865	− 1530
40	2.3090	0.7670	60.0	78.227	12969	− 1179	− 1619
41	4.5300	1.5100	120.0	79.470	10165	+ 1377	+ 914
42	1.0490	1.0240	90.0	87.891	9753	+ 273	− 317
43	2.9915	0.9955	90.0	90.407	9912	− 289	− 900
44	5.8307	0.9950	90.0	90.452	9280	+ 336	− 276
45	1.4980	0.5070	60.0	118.343	5653	+ 670	+ 33
46	1.5110	0.5070	60.0	118.343	5604	+ 719	+ 82
47	2.9750	0.9950	120.0	120.603	4280	+ 1848	+ 1218
48	2.3060	0.7660	120 0	156.658	3379	+ 525	+ 32
49	1.4980	0.5023	90.0	179.176	2410	+ 653	+ 240
50	1.4980	0.5030	120.0	238.659	816	+ 978	+ 714

The substance of section 1 of this chapter, together with some of these tables, appeared first in a contribution by the author to Van Nostrand's "Eclectic Engineering Magazine" for December, 1879, New York.

It is confidently expected that the new United-States Government testing-machine, already in use at the Arsenal in Watertown, Mass., will contribute a set of values for the constants to be used in tension, compression, cross-breaking, and torsion, and for pillars, much more in agreement with the capabilities of the actual members of structures than any values of these constants (if, indeed, they shall turn out to be constants at all) hitherto determined.

TABLE VII. — RECTANGULAR TUBULAR PILLARS OF WROUGHT-IRON. THIN.

Ends flat and well bedded. Hodgkinson's Experiments.

DATA FROM BINDON B. STONEY'S "THEORY OF STRAINS IN GIRDERS AND SIMILAR STRUCTURES."

Square of Least Radius of Gyration $= r^2 = \frac{h^2(h + 3b)}{12(h + b)} = \frac{h^2}{6}$ when $h = b$.

Assume $E = 27,311,111$. $\therefore\ C = 28,657$, $f = 30,660$, computed mean values.

No.	S Sectional Area.	b Breadth.	Q = Breaking-Weight, in Lbs., per Sq. Inch, By Experiment.	Gordon Formula, $Q = \frac{30660}{1 + \frac{l^2}{3000h^2}}$	Equation (385), $Q = \frac{C}{1 + \frac{Cl^2}{4\pi^2 E r^2}}$
	sq. ins.	ins.			
51	0.504	4.10	10980	9646	10197
52	1.020	4.10	19260	19093	20394
53	1.484	4.25	25171	24215	25426
54	2.395	4.25	21585	24225	25426
55	1.885	8.10	23169	18742	20152
56	1.532	8.17	15201	13041	14024
57	6.890	8.40	20981	24225	23835
58	7.396	8.90	26913	25269	26318
59	8.347	8.90	25000	25269	26408
60	2.395	4.25	23201	26698	26757
61	1.532	8.17	16215	14361	14657
62	1.885	8.10	24111	20964	21384
63	3.550	8.10	19732	24678	23930
64	2.070	8.10	13276	11641	11183
65	0.504	4.10	11513	11526	11159
66	3.550	8.10	23100	24762	23972
67	4.926	8.37	20364	26304	25437
68	7.737	8.37	25716	28696	27750
69	8.466	8.40	26675	28696	27730
70	1.520	4.25	23584	18755	27777
71	2.070	8.10	13301	11981	11440
72	1.885	4.10	21880	23780	22718
73	3.550	8.10	23208	26466	24567
74	0.504	4.10	19400	19109	11441
75	1.520	4.25	27417	30167	28434
76	1.532	8.17	15921	16508	13569
77	30.000	18.00	30464	30469	28598
78	1.889	8.10	20010	24370	[illegible]
79	1.550	8.10	24153	20414	24736

TABLE VIII.

Hollow Cylindrical Pillars of Wrought-Iron.

Ends flat and well bedded. Hodgkinson's Experiments.

Data from Bindon B. Stoney's "Theory of Strains in Girders and Similar Structures." $k^2 = 8r^2$.

No.	S Sectional Area.	k Diameter.	$l \div k$ Ratio of Length to Diameter.	$l \div r$ Ratio of Length to Radius of Gyration.	$k \div t$ Ratio of Diameter to Thickness of Metal.	Q = Breaking-Wt., in Lbs., per Sq. Inch, By Experiment.	Gordon Formula, $Q = \dfrac{40050}{1 + \dfrac{l^2}{3000k^2}}$	Equation (385), $Q = \dfrac{42290}{1 + \dfrac{Cl^2}{4\pi^2 Er^2}}$
	sq. ins.	ins.						
80	0.444	1.495	80.00	226.274	15.00	14673	12782	12873
81	0.610	1.964	60.00	172.816	18.80	23206	18204	18127
82	1.435	2.340	51.28	145.042	10.80	22179	21342	21810
83	1.605	2.350	51.00	144.250	9.70	21572	21451	21926
84	0.804	2.490	47.80	135.199	23.27	29798	22735	23289
85	0.444	1.495	40.00	113.137	15.00	31180	26120	26914
86	1.350	3.000	40.00	112.877	20.00	27671	26120	26959
87	0.610	1.964	30.50	86.267	18.80	33299	30571	31745
88	1.414	3.035	29.60	83.721	18.00	29789	30997	32212
89	1.707	4.050	29.60	83.721	29.00	27657	30997	32212
90	1.900	4.060	29.60	83.721	26.10	26263	30997	32212
91	1.371	2.335	25.70	72.690	11.40	29998	32823	34219
92	1.472	2.350	25.50	72.125	10.60	29330	33024	34321
93	0.804	2.490	24.10	68.165	23.27	35100	33554	35025
94	1.613	4.052	22.20	62.791	30.90	33331	34399	35961
95	2.879	4.000	22.20	62.791	16.50	26046	34399	35961
96	2.897	4.000	22.20	62.791	16.00	26503	34399	35961
97	2.837	4.000	22.20	62.791	16.50	27816	34399	35961
98	0.804	2.490	21.00	59.397	23.27	36489	34917	36536
99	0.444	1.495	20.00	56.569	15.00	34220	35338	37004
100	1.800	6.180	19.40	54.871	65.00	33375	35586	37280
101	2.540	6.360	18.90	53.334	49.00	35985	35789	37524
102	0.610	1.964	15.30	43.275	18.80	36980	37151	39028
103	2.895	3.995	15.00	42.426	16.30	30024	37256	39145
104	2.848	3.995	15.00	42.426	16.50	34453	37256	39145
105	2.547	6.366	14.10	39.881	48.90	41664	37561	39487
106	1.407	2.343	12.80	36.204	11.10	38214	37976	39953
107	1.435	2.335	12.80	36.204	11.40	36639	37976	39953
108	1.435	2.335	12.80	36.204	11.40	35389	37976	39953
109	1.651	2.383	12.50	35.355	9.70	33107	38067	40055
110	1.358	2.343	12.30	34.790	11.60	39569	38127	40030
111	1.554	2.373	12.20	34.507	10.27	36906	38157	40155
112	1.799	6.175	9.70	28.075	61.10	38355	38832	40853
113	1.414	3.000	9.30	26.305	19.60	37392	38928	41023
114	2.845	4.000	7.00	19.799	16.00	47844	39406	41182
115	2.850	4.026	6.95	19.657	16.00	48567	39415	41573
116	1.799	6.125	4.90	13.859	62.50	41361	39732	41931

Assume $E = 24{,}000{,}000$. $\therefore\ C = 42{,}290,\ f = 40{,}050$, Means.

TABLE IX.

SOLID CYLINDRICAL PILLARS OF CAST-IRON.

Ends flat and well bedded. Hodgkinson's Experiments.

DATA, AND PER CENT OF VARIATION FROM Q, BY THE HODGKINSON AND GORDON FORMULÆ, TAKEN FROM WILLIAM E. MERRILL'S "IRON TRUSS BRIDGES FOR RAILROADS."

$k^2 = 16r^2$, $E = 12{,}215{,}000$, $C = 109{,}801$, $f = 80{,}000$.

	$l \div k$	k	Q	Variation from Q, per cent, by		
No.	Ratio of Length to Diameter.	Diameter.	Breaking-Weight, by Experiment.	Hodgkinson's Formulæ, Equations (392), (394).	Gordon's Formula, Equation (400), $Q = \dfrac{80000}{1 + \dfrac{l^2}{400k^2}}$	Equation (385).
		ins.	lbs. per sq. in.			
117	4	0.520	107674	− 6	−29.000	− 3.641
118	8	0.500	88964	− 3	−21.000	+ 0.085
119	10	0.777	67502	+13	− 4.000	+19.227
120	13	0.768	55959	+13	− 0.001	+21.444
121	15	0.500	57321	+ 2	−11.000	+ 5.267
122	15	0.785	50182	+11	+ $\frac{1}{6000}$	+20.243
123	15	1.000	51248	+ 9	+ 7.000	+17.741
124	20	0.500	45485	− 1	−13.000	− 1.758
125	20	0.775	45596	− 1	−10.000	− 1.998
126	20	1.022	38770	+12	+ 5.000	+15.257
127	24	0.500	36644	+ 2	−11.000	− 3.891
128	26	0.777	32860	+ $\frac{1}{7}$	− 9.000	− 3.503
129	30	0.510	33111	−10	−25.000	−22.497
130	30	1.010	25350	+ 5	− 3.000	+ 1.231
131	39	0.770	18921	− 8	−13.000	−11.284
132	39	1.560	15153	+ 6	+11.000	+10.777
133	40	0.510	18749	− 2	−13.000	−14.241
134	47	1.290	12291	− 3	+ $\frac{1}{8}$	− 1.261
135	61	0.500	8464	+ 6	− 7.000	−10.881
136	61	0.997	7990	+ $\frac{1}{3}$	− 2.000	− 5.581
137	79	0.770	5274	+ 2	− 8.000	−12.286
138	119	0.510	2384	+19	− 7.000	−12.416

Equation (392), as used in Table IX., is

$$P = 98922\frac{k^{3.55}}{(\frac{1}{12}l)^{1.7}}.$$

See "Iron Truss Bridges for Railroads," p. 26.

For $E = 12{,}215{,}000$, see Stoney's "Theory of Strains," p. 180.

TABLE X.

Solid Cylindrical Pillars of Cast-Iron.

Ends rounded. Hodgkinson's Experiments.

Data, and Per Cent of Variation from Q, by the Hodgkinson and Gordon Formulæ, taken from William E. Merrill's "Iron Truss Bridges for Railroads."

$h^2 = 16r^2$, $E = 15{,}268{,}750$, $C = 109{,}801$, $f = 80{,}000$.

No.	$l \div h$ Ratio of Length to Diameter.	h Diameter.	Q Breaking-Weight, by Experiment.	Variation from Q, per cent, by Hodgkinson's Formulæ, Equation (392), $P = 33379\frac{h^{3.76}}{(\frac{1}{12}l)^{1.7}}$	Gordon's Formula, $Q = \frac{80000}{1 + \frac{l^2}{100h^2}}$	Equation, $Q = \frac{C}{1 + \frac{Cl^2}{\pi^2 Er^2}}$
		ins.	lbs. per sq. in.			
139	8	0.500	76939	−25	−34	−18.269
140	10	0.770	49280	− 7	−18	+ 2.877
141	13	0.760	38590	−15	−25	− 4.203
142	15	0.497	27124	+ $\frac{1}{2}$	−11	+11.735
143	15	0.990	25660	+10	− 6	+18.110
144	20	0.760	20331	−13	−21	− 4.633
145	20	1.010	19642	− 9	−19	− 1.288
146	20	1.520	17928	+ 3	−10	+ 8.149
147	23	1.290	13187	+ 5	− 7	+16.175
148	26	0.767	14289	−23	−29	−13.472
149	30	0.500	9697	−13	−19	− 1.464
150	30	0.990	7931	+ 9	− 2	+20.477
151	31	1.940	7717	+13	− 3	+16.599
152	31	1.960	8051	+14	− 3	+11.763
153	34	1.765	6360	+ 5	−10	+19.262
154	34	1.780	7058	+ 6	−10	+ 7.467
155	39	0.770	5854	− 5	−17	+ 0.137
156	39	1.535	5755	+ $\frac{1}{2}$	−16	+ 1.859
157	40	1.520	5985	− $\frac{1}{8}$	−14	− 6.650
158	47	1.290	4367	− $\frac{1}{8}$	−18	− 6.000
159	47	1.295	4149	− $\frac{1}{2}$	−18	− 1.060
160	61	0.500	2745	− 5	−23	− 9.872
161	61	0.990	2471	+ 8	−16	+ 0.121
162	79	0.770	1675	+ 2	−24	−11.105
163	121	0.500	728	+10	−25	−12.083

TABLE XI.—SOLID STEEL PILLARS. FIXED ENDS.

Computed Breaking-Weights.

No.	Kind.	$l \div h$	Q = Breaking-Weight, in Lbs., per Square Inch.				
			Baker's Formulæ.		$E = 29000000$ $C = 100000$ Equation (385).	$E = 30000000$ $C = 200000$ Equation (385).	$E = 36000000$ $C = 300000$ Equation (385).
		Ratio of Length to Diameter.	Mild. $Q = \dfrac{67200}{1 + \dfrac{l^2}{1400h^2}}$	Strong. $Q = \dfrac{114240}{1 + \dfrac{l^2}{900h^2}}$	$Q = \dfrac{100000}{1 + \dfrac{l^2}{715.55h^2}}$	$Q = \dfrac{200000}{1 + \dfrac{l^2}{370.11h^2}}$	$Q = \dfrac{300000}{1 + \dfrac{l^2}{296.09h^2}}$
164	Round	10	62720	102816	87739	157457	217802
165	Round	20	52267	79089	64143	96117	127608
166	Round	30	40904	57120	44291	58280	74264
167	Round	40	31359	41127	30901	37573	46847
168	Round	50	24123	30240	22253	25791	31768
169	Round	60	18816	22848	16581	18645	22799
170	Round	70	14933	17727	12742	14046	17095
171	Round	80	12061	14084	10056	10934	13266
172	Round	90	9903	11424	8117	8739	10580
173	Round	100	8253	9433	6678	7138	8627
		Ratio of Length to Least Diameter.	$Q = \dfrac{67200}{1 + \dfrac{l^2}{2480h^2}}$	$Q = \dfrac{114240}{1 + \dfrac{l^2}{1600h^2}}$	$Q = \dfrac{100000}{1 + \dfrac{l^2}{954.07h^2}}$	$Q = \dfrac{200000}{1 + \dfrac{l^2}{493.48h^2}}$	$Q = \dfrac{300000}{1 + \dfrac{l^2}{394.787h^2}}$
174	Rectangular . .	10	63271	107520	90512	166301	239368
175	Rectangular . .	20	57867	91392	70459	110462	149017
176	Rectangular . .	30	49307	73113	51458	70827	91469
177	Rectangular . .	40	40847	57111	37355	47144	59154
178	Rectangular . .	50	33465	44582	27759	32970	40751
179	Rectangular . .	60	27411	35150	20950	24111	29525
180	Rectangular . .	70	22582	28121	16298	18303	24620
181	Rectangular . .	80	18768	22848	12973	14317	17355
182	Rectangular . .	90	15752	18844	10538	11459	13881
183	Rectangular . .	100	13354	15757	8710	9406	11344

TABLE XII.

Solid Square Pillars of Pine.

Data from Bindon B. Stoney's "Theory of Strains in Girders and Similar Structures."

$h^2 = 12r^2$. Take $E = 1460000$, $C = f = 5000$.

No.	$l \div h$ Ratio of Length to Least Diameter.	Q = Breaking-Weight, in Lbs., per Square Inch.						
		Rondelet's Proportionals. Flat Ends.	Brereton's Tests. Ends in Ordinary Manner.	Gordon Formula, $Q = \frac{5000}{1 + \frac{l^2}{250h^2}}$	Hodgkinson's Formula, $Q = \frac{500Cl^2}{h^2}$	Equation (381). No End Moment.	Equation (391). One End fully fixed.	Equation (385). Both Ends fully fixed.
184	1	5000	-	-	5000	-	-	-
185	12	4167	-	3176	5000	3126	3959	4349
186	24	2500	-	1513	4940	1471	2437	3126
187	36	1667	-	809	1989	782	1485	2135
188	48	833	-	489	1085	462	960	1471
189	60	417	-	325	693	313	660	1076
190	72	209	-	230	483	221	478	642
191	10	-	1867	3571	5000	3530	4228	4529
192	20	-	1789	1923	5000	1876	2889	3530
193	30	-	1400	1087	2777	1053	1891	2581
194	40	-	1244	676	1563	653	1273	1875

CHAPTER VIII.

PROPORTIONS AND WEIGHTS OF ALL THE MEMBERS OF A BRIDGE EXCEPTING THE GIRDERS PROPER.

122. The Floor.

Let l = length of floor, in feet.
q = breadth of floor, in feet.
t = thickness of floor, in feet.
u = weight of one cubic foot of the material, in pounds.

$\therefore$ Volume of floor = lqt cubic feet
= $0.012lqt$ thousand feet, board measure.

$$F = \text{weight of floor} = ulqt \text{ pounds.} \tag{408}$$

123. The Joists, Longitudinal.

$l \div n$ = length of joist in each panel, in feet.
d = depth of joist, in inches.
b = thickness of joist, in inches.
n = number of equal panels.
g = distance between centres of joists, in feet.
$q \div g$ = number of joists in any panel, each of the two outside ones having the thickness $\frac{1}{2}b$, and being counted as one-half a joist.
$nq \div g$ = whole number of joists in the n panels.
L = panel weight of uniform load, in tons.
u_1 = weight of one cubic foot of the material, in pounds.

Weight upon the joists of one panel $= \frac{ulqt}{n} + 2000L$ pounds,

Uniformly distributed load on one joist $= \frac{ultg}{n} + \frac{2000gL}{q}$ pounds.

Add weight of joist itself $= \frac{bdlu_1}{144n}$ pounds.

Total uniform load for each joist is, therefore,

$$\frac{ultg}{n} + \frac{2000gL}{q} + \frac{bdlu_1}{144n} = \frac{l}{n}w,$$

where w is the number of pounds per linear foot to be supported by one joist.

Now, by equation (52), we have for the external forces, greatest moment at centre,

$$M = \tfrac{1}{8}w\left(\frac{l}{n}\right)^2 = \tfrac{1}{8}\frac{lw}{n} \times \frac{l}{n} = \frac{ul^2tg}{8n^2} + \frac{250glL}{nq} + \frac{bdl^2u_1}{1152n^2} \text{ foot-pounds};$$

and for the internal forces of a rectangular beam, equation (160), the moment of resistance is

$$R = \tfrac{1}{6}Bbd^2 \text{ inch-pounds} = \tfrac{1}{72}Bbd^2 \text{ foot-pounds.}$$

Introducing f, the factor of safety, and equating M and $R \div f$, we find

$$\frac{Bbd^2}{72f} = \frac{ul^2tg}{8n^2} + \frac{250glL}{nq} + \frac{bdl^2u_1}{1152n^2}. \qquad (409)$$

Taking the value of B from Table II., and assigning a value to b or d, we may find, from (409), the required depth or thickness of each joist.

If we neglect the weight of the joist itself, which omission the factor of safety may well warrant, the last term in (409) vanishes, and we have at once

$$\text{Thickness of joist} = b = \frac{9fgl}{n^2qd^2B}(uqlt + 2000nL).$$

$$\text{Depth of joist} \quad = d = \left\{\frac{9fgl}{n^2qbB}(uqlt + 2000nL)\right\}^{\frac{1}{2}}.$$

$$J = \text{weight of } (nq + g) \text{ joists} = \frac{bdlqu_1}{144g} \text{ pounds.} \quad (410)$$

In a similar manner may the dimensions and weight of any other joist or beam or stringer be found; that is, by equating the greatest moment due the external forces acting on the beam, to the greatest allowable moment due the internal forces resisting.

124. The Wrought-Iron I Floor Beams, Transverse, supporting the Joists, Floor, and Load.

Let d_2 = depth of beam, in inches.

d_1 = depth of web, in inches.

$d_2 - d_1$ = depth of two flanges, in inches.

b_2 = breadth of one flange.

$b_2 - b_1$ = thickness of web.

q_1 = entire length of beam, in feet.

S = cross-section of beam, in square inches.

$n - 1$ = number of beams in bridge.

m = weight of one cubic inch of wrought-iron, in pounds.

$D = \dfrac{F + J + 2000nL}{n}$ = uniform load on any beam, in pounds.

Then, by equation (52),

$$\text{Moment of external forces} = M = \tfrac{3}{2}Dq_1 \text{ inch-pounds.}$$

And, from equation (161),

$$\text{Moment of internal forces} = R = \frac{B(b_2d_2^3 - b_1d_1^3)}{6d_2} \text{ inch-pounds.}$$

Whence, introducing f as the factor of safety,

$$\tfrac{3}{2}Dq_1 = \frac{B(b_2d_2^3 - b_1d_1^3)}{6d_2f},$$

$$\therefore \quad \frac{b_2d_2^3 - b_1d_1^3}{d_2} = \frac{9Dq_1f}{B} \qquad (411)$$

Let us take now the dimensions of the cross-section of a well-proportioned I-beam, as, for instance, $d_2 = 15$, $d_1 = 12\tfrac{3}{4}$, $b_2 = 5\tfrac{3}{8}$, $b_1 = 4\tfrac{3}{4}$, and express the relation

$$d_2 = \tfrac{20}{17}d_1 = \tfrac{120}{43}b_2 = \tfrac{60}{19}b_1,$$

$$\therefore \quad d_1 = \tfrac{17}{20}d_2, \quad b_2 = \tfrac{43}{120}d_2, \quad b_1 = \tfrac{19}{60}d_2.$$

Therefore (411) becomes

$$[\tfrac{43}{120} - \tfrac{19}{60}(\tfrac{17}{20})^3]d_2^3 = \frac{9Dq_1f}{B},$$

$$\therefore \quad d_2 = 3.80122\left(\frac{Dq_1f}{B}\right)^{\frac{1}{3}}. \qquad (412)$$

$$\text{Area of section} = S = b_2d_2 - b_1d_1 = \tfrac{107}{1200}d_2^2$$

$$= 1.28839\left(\frac{Dq_1f}{B}\right)^{\frac{2}{3}}. \qquad (413)$$

$$P = \text{weight of floor beams} = 12q_1m(n-1)S$$

$$= 15.46068mq_1(n-1)\left(\frac{Dq_1f}{B}\right)^{\frac{2}{3}}. \qquad (414)$$

If the beam actually used has a form of cross-section varying materially from that here assumed, the co-efficient of (412) must be made to conform thereto.

We may compensate for the omission of the beam's own weight from the formula, first, by selecting from the manufacturer's list of beams that one whose depth agrees most nearly with our computed depth *above* it; and second, by using, in the calculation, the entire length of beam, instead of the net length between bearings.

Having thus employed the formula to determine the depth of beam required for the given load, the weight may be taken from the manufacturer's tables. Indeed, the manufacturer's tables of strength may be used without this calculation, whenever they are *known to be trustworthy*, by selecting the depth of beam corresponding to the required length and "safe load."

125. The System of Lateral Support. — This system includes whatever arrangement of struts, ties, and braces is employed to prevent a lateral bending of the girders, and their rotation about their points of support.

The arrangement must manifestly vary with the form and height of girder; a high girder with straight chords allowing a complete horizontal trussing overhead and under the floor, while arched top chords allow only a partial head-bracing, and low girders for "through" bridges can only be laterally braced from below.

In all cases, the horizontal systems, top and bottom, should be rigidly connected with the girders, whether angle braces are employed or not. For high girders with straight chords, there are generally used, a strut at every pair of opposite top joints, $n + 1$ in number, and a pair of diagonal ties at the top and bottom of each panel, $4n$ in number.

The proportions of these members may be computed in the same manner as the proportions are found for a girder uniformly loaded, using the assumed pressure of wind against the

side of the bridge and load as the uniform horizontally (or otherwise) acting load.

For girders admitting full head-bracing, we thus compute:

q = length of horizontal strut, in feet.

$\sqrt{q^2 + \left(\frac{l}{n}\right)^2}$ = length of horizontal diagonal, in feet.

S_1 = cross-section of each strut, in square inches, Assumed or computed.
S_2 = cross-section of each diagonal, in square inches,

m = weight of one cubic inch of wrought-iron.

$$U = \text{weight of horizontal struts} \quad = 12qm(n + 1)S_1. \qquad (415)$$

$$X = \text{weight of horizontal diagonals} = 48mnS_2\sqrt{q^2 + \frac{l^2}{n^2}}. \qquad (416)$$

126. Finally, there should be added whatever weight of wood or iron is not included in the foregoing specifications, but is employed in the actual completion and equipment of the structure. Call this weight p pounds to the panel; then we have

$$Y = \text{weight of residue} = np \text{ pounds.} \qquad (417)$$

127. Take K = weight of bridge exclusive of the girders, in pounds; then

$$K = F + J + P + U + X + Y \text{ pounds.} \qquad (418)$$

And if G = weight of girders, in pounds,

$$\text{Weight of bridge} = 2000nW = K + G \text{ pounds.} \qquad (419)$$

(*a*) *To find the moment at each joint due the entire weight* n(W + L), *and thence the horizontal strain in chords by equation* (95).

$$H = M \div h = \text{moment divided by height.}$$

Equation (65) applies here if for W we put $W + L$, and we have

$$M_a = \frac{(W + L)l}{2n}(n - 1) \times 1,$$

$$\therefore\ H_a = \frac{(W + L)l}{2nh}(n - 1) \times 1 = \text{strain on } Aa, a_1b_1;$$

$$M_b = \frac{(W + L)l}{2n}(n - 2) \times 2,$$

$$\therefore\ H_b = \frac{(W + L)l}{2nh}(n - 2) \times 2 = \text{strain on } ab, b_1c_1;$$

$$M_c = \frac{(W + L)l}{2n}(n - 3) \times 3,$$

$$\therefore\ H_c = \frac{(W + L)l}{2nh}(n - 3) \times 3 = \text{strain on } bc, c_1d_1;$$

$$M_d = \frac{(W + L)l}{2n}(n - 4) \times 4,$$

$$\therefore\ H_d = \frac{(W + L)l}{2nh}(n - 4) \times 4 = \text{strain on } cd, d_1e_1;$$

. . . .

$$M_h = \frac{(W + L)l}{2n}[n - (n - 1)](n - 1),$$

$$\therefore\ H_h = \frac{(W + L)l}{2nh}[n - (n - 1)](n - 1) = \text{strain on } hB, g_1h_1;$$

where H is the greatest horizontal strain, in tons, at the successive joints; the strain on each chord being assumed to act at the centre or axis of the chord, whose depth is small compared with h.

(*b*) *To find the compression on verticals, and the tension on diagonals, due to permanent load,* nW, *alone.*

From equation (65), dividing by h, and from the formulæ for Class IX., article 49,

$$H_A = 0;$$

$$H_a = \frac{Wl}{2nh}(n - 1) \times 1,$$

$$\therefore \quad \Delta H = H_a - H_A = \frac{Wl}{2nh}(n - 1) = \text{hor. component of } Aa_1;$$

$$H_b = \frac{Wl}{2nh}(n - 2) \times 2,$$

$$\therefore \quad H_b - H_a = \frac{Wl}{2nh}(n - 3) = \text{hor. component of } ab_1;$$

$$H_c = \frac{Wl}{2nh}(n - 3) \times 3,$$

$$\therefore \quad H_c - H_b = \frac{Wl}{2nh}(n - 5) = \text{hor. component of } bc_1;$$

. . . .

$$H_k = \frac{Wl}{2nh}[n - (n - 1)](n - 1),$$

$$\therefore \quad H_B - H_k = \frac{Wl}{2nh}\Big[-(n - 1)\Big] = \text{hor. component of } h_1B.$$

Therefore, from the triangle of forces, equations (3), the vertical components are

$$Z = \Delta H \tan\phi = \pm \Delta H \frac{nh}{l}; \qquad (420)$$

$$\therefore \quad Z_A = \tfrac{1}{2}W(n - 1) = \text{compression on } AC \text{ or } BD,$$

$$Z_a = \tfrac{1}{2}W(n - 3) = \text{compression on } aa_1 \text{ or } hh_1,$$

$$Z_b = \tfrac{1}{2}W(n - 5) = \text{compression on } bb_1 \text{ or } gg_1,$$

. . . .

$$Z_B = \tfrac{1}{2}W(n - 1) = \text{compression on } BD \text{ or } AC.$$

And the strain Y along any diagonal is

$$\Delta H \div \cos\phi,$$

or

$$Z \div \sin\phi = \frac{Z\sqrt{l^2 + n^2h^2}}{nh}; \qquad (421)$$

$$\therefore\ Y_A = \frac{W}{2\sin\phi}(n-1) = \text{tension on } Aa_1 \text{ or } Bh_1,$$

$$Y_a = \frac{W}{2\sin\phi}(n-3) = \text{tension on } ab_1 \text{ or } hg_1,$$

$$Y_b = \frac{W}{2\sin\phi}(n-5) = \text{tension on } bc_1 \text{ or } gf_1,$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$Y_B = \frac{W}{2\sin\phi}\Big[-(n-1)\Big] = \text{tension on } Bh_1 \text{ or } Aa_1.$$

(*c*) *Maximum strain on verticals and diagonals from moving-load,* nL, *alone.*

To find this strain Z_L, we subtract equation (64) from (68), divide remainder by h for greatest difference of horizontal strains at adjacent joints, and multiply the quotient by $\tan\phi$; thus, after putting L for W, the difference between (68) and (64) is

$$\frac{Ll}{2n^2} \times r(r+1) = \text{maximum } \Delta Hh \text{ (say)},$$

$$\therefore\ Z_L = \frac{\tan\phi}{h} \times \frac{Ll}{2n^2} \times r(r+1) = \frac{L}{n} \times \frac{r(r+1)}{2}, \qquad (422)$$

where r is the number of apex loads on the girder as the moving-load advances, and Z_L is the compression on the $(r+1)^{\text{th}}$ vertical;

$$\therefore \quad Z_b = \frac{L}{n} \times 1 = \text{compression on } bb_1, \qquad r = 1;$$

$$Z_c = \frac{L}{n} \times 3 = \text{compression on } cc_1, \qquad r = 2;$$

$$Z_d = \frac{L}{n} \times 6 = \text{compression on } dd_1, \qquad r = 3;$$

$$Z_e = \frac{L}{n} \times 10 = \text{compression on } ee_1, \qquad r = 4;$$

. . . .

$$Z_B = \frac{L}{n} \times \frac{(n-1)n}{2} = \text{compression on } BD, \quad r = n - 1.$$

The greatest strain on diagonals due to moving-load, nL, is

$$Y = Z_L \div \sin\phi; \qquad (423)$$

$$\therefore \quad Y_b = \frac{L}{n \sin\phi} \times 1 = \text{tension on } a_1b, \qquad r = 1;$$

$$Y_c = \frac{L}{n \sin\phi} \times 3 = \text{tension on } b_1c, \qquad r = 2;$$

$$Y_d = \frac{L}{n \sin\phi} \times 6 = \text{tension on } c_1d, \qquad r = 3;$$

$$Y_e = \frac{L}{n \sin\phi} \times 10 = \text{tension on } d_1e, \qquad r = 4;$$

. . . .

$$Y_B = \frac{L}{n \sin\phi} \times \frac{(n-1)n}{2} = \text{tension on } h_1B, \quad r = n - 1.$$

(*d*) Combining the strains due nW and nL, and, for convenience, writing N for $\frac{(W + L)l}{2nh}$, we find, for any number of panels: —

MAXIMA STRAINS IN PRATT TRUSS.

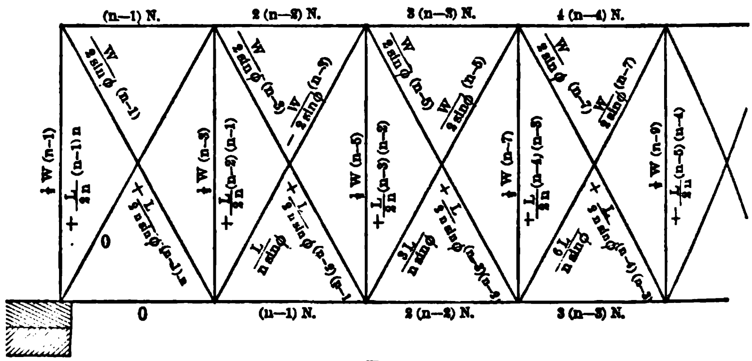


FIG. 112.

UNIFORM DEAD AND LIVE LOADS.

Loads applied at lower joints: —
W = panel weight of dead load.
L = panel weight of live load.
l = length of truss from centre to centre of end pins.
h = height of truss from centre to centre of pins.
n = number of panels.

$$\sin\phi = \frac{nh}{\sqrt{l^2 + n^2h^2}}, \quad N = \frac{(W + L)l}{2nh} = \frac{W + L}{2\tan\phi}.$$

129. Weight of the Structure determined. — (*a*) *To find the weight of the top chord.*

Suppose $Q \div f$ to be the greatest allowable pressure to the square inch of section of top chord, and Q to be of the same denomination with W and L; and suppose f to be a number called the factor of safety. Q is known as the breaking-weight

of a column of the given material, having the length of one panel, and the cross-section of the top chord for any given panel.

Let m = weight of one cubic inch of the material, in pounds.

We then have, for each one of the equal panel lengths of the top chord,

$$\text{Area of section} = \frac{fH}{Q} \text{ square inches,}$$

$$\text{Volume of one panel length} = \frac{12flH}{Qn} \text{ cubic inches,}$$

$$\text{Weight of one panel length} = \frac{12mfl}{Qn}H \text{ pounds,}$$

$$\text{Weight of top chord} = \frac{12mfl}{Qn}\Sigma H \text{ pounds.}$$

From (*a*) of the preceding article we find

$$\Sigma H = \frac{(W+L)l}{2nh}\left\{\begin{array}{l} n[1+2+3+4+\ldots(n-1)\text{ terms}] \\ -[1^2+2^2+3^2+4^2+\ldots(n-1)\text{ terms}] \\ +\frac{1}{4}n^2 \text{ for } n \textit{ even}, \quad +\frac{1}{4}(n^2-1) \text{ for } n \textit{ odd}. \end{array}\right\}$$

$$= \frac{(W+L)l}{2nh} \times \frac{n}{12}(2n^2+3n-2),\ n \text{ even};$$

$$= \frac{(W+L)l}{2nh} \times \frac{1}{12}(2n^3+3n^2-2n-3),\ n \text{ odd}.$$

Substituting these values for ΣH, we have

$$\left.\begin{array}{l} \text{Weight of top chord} = \dfrac{mfl^2(W+L)}{2Qnh}(2n^2+3n-2) \\ \qquad\qquad\qquad\qquad\qquad (n \text{ even}), \\ \qquad\qquad\qquad\quad = \dfrac{mfl^2(W+L)}{2Qn^2h}(2n^3+3n^2-2n-3) \\ \qquad\qquad\qquad\qquad\qquad (n \text{ odd}). \end{array}\right\} \quad (424)$$

(*b*) Similarly, if $T \div f =$ the greatest allowable tensile strain, we find

$$\text{Weight of bottom chord} = \frac{12mfl}{Tn}\Sigma H \text{ pounds.}$$

$$\Sigma H = \frac{(W+L)l}{2nh}\left\{\begin{array}{r} + 2(n-1) \text{ for two end panels,} \\ n[1+2+3+4+\ldots(n-1) \text{ terms}] \\ -[1^2+2^2+3^2+\ldots(n-1) \text{ terms}] \\ -\frac{1}{4}n^2 \text{ for } n \text{ even}, \quad -\frac{1}{4}(n^2-1) \text{ for } n \text{ odd.} \end{array}\right\}$$

$$= \frac{(W+L)l}{2nh} \times \tfrac{1}{12}(2n^3 - 3n^2 + 22n - 24) \text{ when } n \text{ is even,}$$

$$= \frac{(W+L)l}{2nh} \times \tfrac{1}{12}(2n^3 - 3n^2 + 22n - 21) \text{ when } n \text{ is odd.}$$

$\therefore$ Weight of bottom chord

$$\left.\begin{array}{l} = \dfrac{mfl^2(W+L)}{2Tn^2h}(2n^3 - 3n^2 + 22n - 24) \\ \qquad (n \text{ even}), \\ = \dfrac{mfl^2(W+L)}{2Tn^2h}(2n^3 - 3n^2 + 22n - 21) \\ \qquad (n \text{ odd}). \end{array}\right\} \quad (425)$$

(*c*) In finding the weight of the verticals, let $Q_1 \div f$ be the allowed working unit strain in compression; then

$$\text{Area of section} = \frac{f(Z_W + Z_L)}{Q_1} \text{ square inches,}$$

$$\text{Volume of one strut} = \frac{12fh(Z_W + Z_L)}{Q_1} \text{ cubic inches,}$$

$$\text{Weight of one strut} = \frac{12mfh(Z_W + Z_L)}{Q_1} \text{ pounds,}$$

$$\text{Weight of all verticals} = \frac{12mfh(\Sigma Z_W + \Sigma Z_L)}{Q_1} \text{ pounds.}$$

Hence, from the strain sheet, Fig. 112, using the proper limits of summation, we derive, when n is even,

$$\Sigma Z_W = 2 \times \frac{W}{2}\{n(\tfrac{1}{2}n) - [1 + 3 + 5 + 7 + \ldots (\tfrac{1}{2}n) \text{ terms}]\}$$

$$= \frac{Wn^2}{4},$$

$$\Sigma Z_L = 2 \times \frac{L}{2n}\Big\{n^2(\tfrac{1}{2}n) - n[1 + 3 + 5 + 7 \ldots (\tfrac{1}{2}n) \text{ terms}]$$

$$+ 2\left[1 + 3 + 6 + 10 + \ldots \left(\frac{n}{2} - 1\right)\text{terms}\right]$$

$$+ \tfrac{1}{2}\left(n - \frac{n}{2}\right)\left[n - \left(\frac{n}{2} + 1\right)\right] \text{ for middle strut}\Big\}$$

$$= \frac{L}{n} \times \tfrac{1}{24}(7n^3 + 3n^2 - 10n).$$

But when n is odd, we thus sum,

$$\Sigma Z_W = 2 \times \frac{W}{2}\left\{n\frac{n-1}{2} - \left(1 + 3 + 5 + 7 + \ldots \frac{n-1}{2} \text{ terms}\right)\right\}$$

$$= \frac{W(n^2 - 1)}{4}.$$

$$\Sigma Z_L = 2 \times \frac{L}{2n}\Big\{n^2\frac{n-1}{2} - n\left(1 + 3 + 5 + 7 + \ldots \frac{n-1}{2} \text{ terms}\right)$$

$$+ 2\left(1 + 3 + 6 + 10 + \ldots \frac{n-3}{2} \text{ terms}\right)\Big\}$$

$$= \frac{L}{n} \times \tfrac{1}{24}(7n^3 - 3n^2 - 7n + 3).$$

Wherefore

$$\left.\begin{aligned} \text{Weight of verticals} &= \frac{3mfhWn^2}{Q_1} + \frac{mfhL}{2Q_1}(7n^2 + 3n - 10) \\ &\qquad (n \text{ even}), \\ &= \frac{3mfhW(n^2 - 1)}{Q_1} \\ &\qquad + \frac{mfhL}{2Q_1n}(7n^3 - 3n^2 - 7n + 3) \\ &\qquad (n \text{ odd}). \end{aligned}\right\} \quad (426)$$

(*d*) In determining the weight of the diagonals in terms of the unknown weight of the structure, nW, we shall disregard the effect of the permanent weight, nW, upon the strains developed in the *counter* diagonals by the moving-load, nL.

By so doing, the value of W comes out a little greater than strict theory requires; but in general practice the "counters" are inserted somewhat in excess of theoretical demands.

When, however, W shall have been thus determined, the strains upon all the members are to be computed according to the strain sheet, Fig. 112.

Strain on any diagonal due to L is Y_L.

$$\text{Area of cross-section} = \frac{fY_L}{T} \text{ square inches,}$$

$$\text{Volume of one diagonal} = \frac{12fh}{T\sin\phi} Y_L \text{ cubic inches,}$$

$$\text{Weight of one diagonal} = \frac{12mfh}{T\sin\phi} Y_L \text{ pounds.}$$

From Fig. 112,

$$\Sigma Y_L = 2 \times \frac{L}{n\sin\phi}[1 + 3 + 6 + 10 + \ldots (n-1) \text{ terms}]$$

$$= \frac{2L}{n\sin\phi} \times \frac{n(n^2-1)}{6},$$

therefore weight of diagonals due uniform moving-load, nL, alone is

$$\frac{4mfhL}{T\sin^2\phi}(n^2 - 1). \tag{427}$$

The weight of the diagonals due to the dead load, nW, is manifestly to be derived from the weight of the verticals due dead load if for h we put $h \div \sin^2\phi$, and for Q_1 we put T.

$$\left.\begin{aligned}\therefore\quad \text{Weight of diagonals} &= \frac{4mfhL(n^2-1)}{T\sin^2\phi} + \frac{3mfhWn^2}{T\sin^2\phi} \\ &\qquad (n \text{ even}), \\ &= \frac{4mfhL(n^2-1)}{T\sin^2\phi} + \frac{3mfhW(n^2-1)}{T\sin^2\phi} \\ &\qquad (n \text{ odd}).\end{aligned}\right\} \quad (428)$$

(*e*) Taking the sum of the weights thus found, we have, when n is *even*, total weight of girder, in pounds,

$$\begin{aligned}G = \frac{mfl^2(W+L)}{2nh}&\left(\frac{2n^2+3n-2}{Q} + \frac{2n^3-3n^2+22n-24}{Tn}\right) \\ &+ 3mfhn^2W\left(\frac{1}{Q_1} + \frac{1}{T\sin^2\phi}\right) \\ &+ mfhL\left\{\frac{7n^2+3n-10}{2Q_1} + \frac{4(n^2-1)}{T\sin^2\phi}\right\}. \quad (429)\end{aligned}$$

But when n is *odd*, total weight of girder, in pounds, is

$$\begin{aligned}G = \frac{mfl^2(W+L)}{2n^2h}&\left(\frac{2n^3+3n^2-2n-3}{Q} + \frac{2n^3-3n^2+22n-21}{T}\right) \\ &+ 3mfh(n^2-1)W\left(\frac{1}{Q_1} + \frac{1}{T\sin^2\phi}\right) \\ &+ mfhL\left\{\frac{7n^3-3n^2-7n+3}{2Q_1n} + \frac{4(n^2-1)}{T\sin^2\phi}\right\}. \quad (430)\end{aligned}$$

EXAMPLE I.—Wrought-iron girder of 6 equal panels. Take $n = 6$, $l = 60$ feet, $h = 10$ feet, $f = 4$, $m = \frac{5}{18}$ pound, $L = 8$ tons, $T = 24$ tons, $Q = 16$ tons, $Q_1 = 12$ tons, $\tan\phi = 1$, $\sin\phi = \frac{1}{2}\sqrt{2} = 0.70711$. Therefore, from (429),

$$G = 483.333\,W + 4267 \text{ pounds},$$

equal to $2000nW$ if nW is the girder's own weight in tons.

$$\begin{aligned}\therefore\quad &\text{Panel weight of girder} = W \;\, = 0.3704775 \text{ ton}, \\ &\text{Total weight of girder} = nW = 2.2228650 \text{ tons}.\end{aligned}$$

130. But if the structure is a bridge having two equal girders whose combined weight is G, and an additional permanent weight of K pounds, then the weight of the bridge is

$$2000nW = K + G \text{ pounds,}$$

as shown by equation (419).

Continuing the first example of article 129, we compute K as follows: —

For the floor, we have $l = 60$ feet $=$ length.

Take $q = 18$ feet $=$ breadth.

$t = \frac{2.5}{12}$ feet $=$ thickness.

$u = 54$ pounds $=$ weight of one cubic foot of oak.

From (408),

$$\text{Weight of floor} = 54 \times 60 \times 18 \times \frac{2.5}{12} = 12150 \text{ pounds} = F.$$

The joists: —

$$l \div n = 10 \text{ feet} = \text{panel length of joist.}$$

Take $b = 3$ inches $=$ thickness.

$g = 2$ feet $=$ space between centres.

$q \div g = 9 =$ number of joists in each panel.

$nq \div g = 54 =$ number of joists in bridge.

$u_1 = 54 = u.$

$B = 10{,}600.$

$f = 9.$

Then, by article 123, we have

$$d = \left\{\frac{9 \times 9 \times 60 \times 2}{6^2 \times 18 \times 3 \times 10600}\left(54 \times 18 \times 60 \times \frac{2.5}{12} + 2000 \times 6 \times 8\right)\right\}^{1}$$
$$= 7.1424 \text{ inches.}$$

Call $d = 8$ inches,

$$\therefore \quad \text{Weight of joists} = \frac{3 \times 8 \times 60 \times 18 \times 54}{144 \times 2} = 4860 \text{ pounds} = J.$$

For the iron I-beams, we have, from article 124,

$$D = \frac{F + J + 2000nL}{n} = 18835 \text{ pounds}.$$

Take $q_1 = 19$ feet $=$ entire length of beam.
$f = 4 =$ factor of safety.
$B = 52{,}567$, from Table II.

Whence, by equation (412),

$$\text{Required depth of beam} = d_2 = 3.80122\left(\frac{18835 \times 19 \times 4}{52567}\right)^{\frac{1}{3}}$$
$$= 11.435 \text{ inches}.$$

Call $d_2 = 12$ inches; then, by (413),

$$\text{Area of section} = S = 1.28839\left(\frac{18835 \times 19 \times 4}{52567}\right)^{\frac{2}{3}} \times \left(\frac{12}{11.435}\right)^2$$
$$= 12.84 \text{ square inches},$$

since similar sections are to each other as the squares of their like dimensions.

Now this cross-section, 12.84, agrees very nearly with that of the "12-inch light I-beam" of the Union Iron Mills, Pittsburgh, Penn., whose weight is 42 pounds to the foot, and area $= 42 \times \frac{3}{10} = 12.6$ square inches.

Using this beam, we then have

$$\text{Weight of 5 floor beams} = 5 \times 19 \times 42 = 3990 \text{ pounds} = P.$$

Use full head trussing; the struts to be composed of two T-bars, each $5\frac{1}{2}$ pounds to the foot, latticed with $1\frac{1}{4} \times \frac{1}{4}$ inch

braces, at 45 degrees, the whole weighing $12\frac{1}{2}$ pounds to the running foot; length = 18 feet.

$$\text{Weight of } (n+1) \text{ horizontal struts} = 7 \times 18 \times 12\tfrac{1}{2}$$
$$= 1575 \text{ pounds} = U.$$

Let the horizontal diagonal ties be $1\frac{1}{8}$ inches in diameter, weighing 3.359 pounds to the foot. Then

$$\text{Weight of 24 horizontal ties} = 24 \times 3.359\sqrt{10^2 + 18^2}$$
$$= 1660 \text{ pounds} = X.$$

Call the residue 100 pounds to the panel; that is, in all = 600 pounds = Y.

$$\therefore\ K = F + J + P + U + X + Y = 24835 \text{ pounds},$$
$$G = \text{weight of girders} = 4267 + 483.333\,W,$$
$$K + G = \text{weight of bridge} = 29102 + 483.333\,W$$
$$= 12000\,W \text{ pounds};$$
$$\therefore\ \text{Panel weight of bridge} = W = 2.526947 \text{ tons},$$
$$\text{Total weight of bridge} = nW = 15.161682 \text{ tons}.$$
$$\text{Panel weight of dead load on each girder} = 1.26347 \text{ tons},$$
$$\text{Panel weight of live load on each girder} = 4 \text{ tons}.$$
$$\tfrac{1}{2}(W + L) = 5.26347 \text{ tons} = \text{total panel weight for one girder}.$$

Putting this value, 5.26347 tons, for $W + L$, in the expression for N, article 128, (d), we find

$$N = \frac{5.26347l}{2nh} = \frac{5.26347 \times 60}{2 \times 6 \times 10} = 2.63174 \text{ tons}.$$

And from the strain sheet, Fig. 112, the greatest chord strains are

$$H_1 = 2.63174 \times 5 \times 1 = 13.15870 \text{ tons},$$
$$H_2 = 2.63174 \times 4 \times 2 = 21.05392 \text{ tons},$$
$$H_3 = 2.63174 \times 3 \times 3 = 23.68566 \text{ tons}.$$

Putting $\frac{1}{2}W = 1.26347$ for W, and $\frac{1}{2}L = 4$ for L, the strain sheet gives, for each of two girders:—

Greatest compression on verticals:

$$\begin{aligned}
Z_1 &= 0.63174 \times 5 + 0.33333 \times 5 \times 6 = 13.15870 \text{ tons,}\\
Z_2 &= 0.63174 \times 3 + 0.33333 \times 4 \times 5 = 8.56188 \text{ tons,}\\
Z_3 &= 0.63174 \times 1 + 0.33333 \times 3 \times 4 = 4.63174 \text{ tons,}\\
Z_4 &= \qquad\qquad\qquad\; 0.33333 \times 2 \times 3 = 2.00000 \text{ tons.}
\end{aligned}$$

Also, for the diagonals:

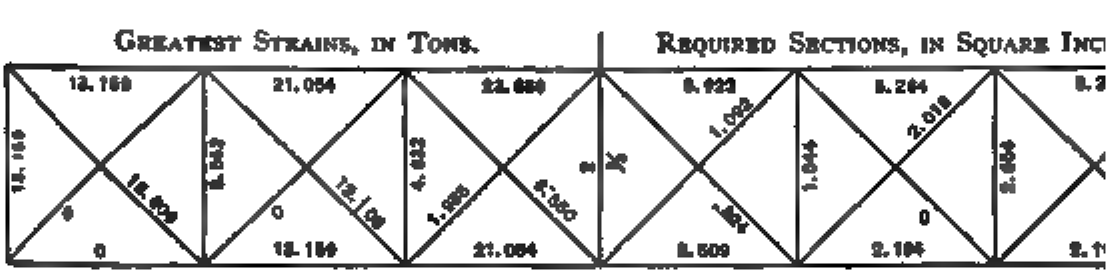
$$\frac{L}{n \sin\phi} \text{ becomes } \frac{4}{6 \sin\phi} = 0.94281 \text{ ton,}$$

$$\frac{W}{2 \sin\phi} \text{ becomes } \frac{0.63174}{\sin\phi} = 0.89342 \text{ ton.}$$

And, from Fig. 112:—

Greatest strain on diagonals:

$$\begin{aligned}
&Y_1, \text{ counter, } = 0 \qquad\qquad\qquad\;\; - 0.89342 \times 5 < 0 \quad \text{to}\\
&Y_2, \text{ counter, } = 0.94281 \times 1 - 0.89342 \times 3 < 0 \quad \text{to}\\
&Y_3, \text{ counter, } = 0.94281 \times 3 - 0.89342 \times 1 = 1.93501 \text{ to}\\
&Y_4, \text{ main, } = 0.94281 \times 6 + 0.89342 \times 1 = 6.55028 \text{ to}\\
&Y_5, \text{ main, } = 0.94281 \times 10 + 0.89342 \times 3 = 12.10836 \text{ to}\\
&Y_6, \text{ main, } = 0.94281 \times 15 + 0.89342 \times 5 = 18.60925 \text{ to}
\end{aligned}$$



LOAD APPLIED AT BOTTOM JOINTS. DIAGONALS IN TENSION.—CLASS IX.

FIG. 113.

131. Now, it is very evident that the bridge would de entirely upon the floor system for stability in a longitu direction if we should omit the bottom chords and the co ties, which are marked as receiving no strain in the *end* p

It is therefore usual to insert these members in the end panels, and also, in this style of girder, to stiffen the bottom chords by cross-bracing in the end panels, so that each bottom chord may there act as a strut.

Some builders place counters in all panels, even where the assumed behavior of the given load does not require them. By so doing they provide for *concentrated* loads greater than the assumed uniform apex load, as well as enhance the symmetry of the structure.

In short-span bridges, as in the present example, some of the vertical struts require a greater cross-section than the actual downward pressure upon them would indicate; for, besides this pressure along the axis of the strut, it should be able to resist probable lateral blows from the travel of the road, even though every strut be protected from ordinary collision with hubs.

To provide for the increase of bridge weight from these sources, above the weight computed from the given load nL, we have added the last term, Y, of K, which term should be large enough to cover every thing not otherwise included.

No absolutely definite rule can be given for the size of these parts, but the smallest counter ties should be so large as not to look *wiry*, say not less than one inch in cross-section; the bottom chord in the first panel may equal in size that of the second panel; and the size of any vertical strut should enable it to resist such lateral shocks as are probable in the situation.

132. Thus far the panel weights, W and L, have been assumed to be applied at the lower joints only. In the nature of the case, however, it is plain that the weight of the top chords and the system of head-bracing, as also the weight of the girder diagonals, can only reach the bottom joints through the vertical struts. But as the weight of these members is small compared with the whole weight of the bridge, and as

the calculation is a little more simple when W is applied at one point instead of two, it is usual to make the above assumption.

It is proper, however, in this place, to indicate the changes to be made in the strain sheet, Fig. 112, by changes in the distribution of the loads.

1st, Suppose the panel weight, L, of live load, and the panel weight of the floor system, and half the panel weight of the girders, to be applied at each lower joint, and the other half of the girders' weight, and the system of head-trussing, to be uniformly distributed at the upper joints.

We have $G =$ weight of girders, in pounds.

$G \div 2n =$ one-half panel weight of girders, in pounds.

Take $A =$ panel weight of head-bracing,

$$\therefore \quad \text{Load at each lower joint} = L + W - A - \frac{G}{2n},$$

$$\text{Load at each upper joint} = A + \frac{G}{2n} \text{ pounds.}$$

The strain sheet, Fig. 112, applies to this case if to the compression on each vertical we add $A + \dfrac{G}{2n}$ pounds, but to each end post $\frac{1}{2}\left(A + \dfrac{G}{2n}\right)$. And the additional weight of the verticals due to this change of loading is

$$\frac{12mfh}{Q_1}\left(A + \frac{G}{2n}\right)n \text{ pounds,}$$

which is to be added to the weight of verticals in equation (426), and, of course, to the second member of (429) and (430), and thence a new expression for G be found.

2d, Suppose we have a "deck" bridge, and that both W and L are applied at the upper joints.

Then to each vertical compression given in strain sheet, Fig. 112, we must add $W + L$; and the additional weight of the verticals is, with $\frac{1}{2}(W + L)$ on each end post,

$$\frac{12mfh}{Q_1}(W + L)n \text{ pounds,}$$

to be placed in the second member of (429) and (430), *provided* the bridge has its points of support at the bottom, as in the figure; but if the points of support are at the ends of the upper chords, then no end posts are required, and their weight may be deducted from the second member of (429) and (430), and

$$\frac{12mfh}{Q_1}(W + L)(n - 1)$$

be added.

3d, In case of the deck bridge, if we suppose half the weight of the girders, and the weight, nA_{11}, of the bottom horizontal bracing, to be applied uniformly at the bottom joints, while the remainder of the loading is applied at the upper joints, we must then add to the pressure on each vertical, Fig. 112,

$$L + W - A_1 - \frac{G}{2n} \text{ pounds,}$$

instead of $W + L$. And the additional weight of girders from this source is

$$\frac{12mfh}{Q_1}\left(L + W - A_1 - \frac{G}{2n}\right)n \text{ pounds,}$$

or

$$\frac{12mfh}{Q_1}\left(L + W - A_1 - \frac{G}{2n}\right)(n - 1) \text{ pounds,}$$

minus the weight of the end posts, according as the girders are supported at bottom or at top.

133. The deck bridge requires, especially when its points of support are at the bottom, a thorough system of lateral sway-bracing, which may be made by inserting diagonals between each top chord and the bottom chord of the opposite girder at the panel joints, in addition to the horizontal systems already provided for.

The proper size of these diagonals can be determined by calculation when the applied external forces are given, so as to conform to the magnitude, situation, and uses of the structure.

Their weight is to be included in the value of K, the constant part of the bridge weight.

134. To determine the best number of panels, n, and the best height, h, of girder, for a bridge of given span, l, and given moving panel load, L, we may find, by means of equations (419), (429), and (430), an expression for W, the panel weight of bridge, in terms of n and h; then, putting $\left(\frac{dW}{dn}\right) = 0$, and $\left(\frac{dW}{dh}\right) = 0$, we shall have two simultaneous equations which will yield those values of n and h that will render W a minimum. But in practice it will be found more convenient, since n is always an integer, and the two simultaneous equations are of a high degree, to find W in terms of h alone for several different values of n, presumably including the best, and then to find from $\frac{dW}{dh} = 0$, for each value of n, the value of h which renders W least. It is evident that the values of n and h which *simultaneously* render W least are the values sought. For the present purpose, we must, of course, retain n and h, or their equivalents, wherever they occur in both K and G. Let us, therefore, re-examine the several terms of K and G, and put them into suitable form for general application.

The value of F, the weight of floor, (408), is independent of n and h, and requires no change.

If for joists we call

$$d = b^2, \tag{431}$$

we shall have a good ratio of breadth, b, to depth, d; and, in (410), $bd = b^3$, and

$$b = \left\{ \frac{9fgl}{n^2qB}(ulqt + 2000nL) \right\}^{\frac{1}{5}}, \tag{432}$$

$$\therefore \quad J = \frac{lqu_1}{144g}\left\{ \frac{9fgl}{n^2qB}(ulqt + 2000nL) \right\}^{\frac{3}{5}}, \tag{433}$$

which is the weight of the joists, in pounds.

Restoring the value of D, we write, for (414),

$$P = 15.46068mq_1(n - 1)\left\{ \frac{(F + J + 2000nL)q_1f}{nB} \right\}^{\frac{2}{3}} \text{ pounds}, \tag{434}$$

equal to weight of $(n - 1)$ wrought-iron I-beams having the proportions assumed in deriving equation (412).

135. If we take into account the greatest probable pressure of wind horizontally against the side of each open girder and its moving-load, or against the entire side of each wholly covered structure, we find the strains due to wind, in the chords and entire lateral system, by making the proper changes in the strain sheet, Fig. 112.

For any through bridge of Class IX., let the uniform wind pressure to be resisted by the top or bottom lateral system be $W_1 = \frac{1}{2}hw\frac{l}{n}$ tons per panel; w being the horizontal pressure of wind per square foot, in tons. And for the bottom lateral system, which alone is affected by the wind pressure against the moving-load, let the uniform moving wind pressure per panel be $L_1 = sw\frac{l}{n}$ tons; s being the height of train or other moving-load, in feet.

From (424) we derive the additional weight of top cl due to wind pressure by substituting $2W_1 = \frac{hwl}{n}$ for $(W$ - since, in order to provide for the wind coming either wa must increase each chord for increased compression, an putting q for h, and formulating thus,

$$\left.\begin{array}{c}\text{Weight of top chords}\\ \text{due to wind}\end{array}\right\} = \frac{mfl^3hw}{2Qn^2q}(2n^2 + 3n - 2) \quad (n \text{ even}),$$

$$= \frac{mfl^3hw}{2Qn^3q}(2n^3 + 3n^2 - 2n - 3) \quad (n \text{ odd}).$$

Similarly, from (425), putting $(2W_1 + 2L_1) = \frac{wl}{n}(h$ - for $(W + L)$, and q for h,

$$\left.\begin{array}{c}\text{Weight of bottom}\\ \text{chords due to wind}\end{array}\right\} = \frac{mfl^3w(h+2t)}{2Tn^3q}(2n^3 - 3n^2 + 22n - 24) \quad (n \text{ even}),$$

$$= \frac{mfl^3w(h+2t)}{2Tn^3q}(2n^3 - 3n^2 + 22n - 21) \quad (n \text{ odd}).$$

And, from (426), we derive the weight of the horizontal s between the top chords by putting $W_1 = \frac{1}{2}hw\frac{l}{n}$ for W, o f q for h, Q_2 for Q_1, and adding,

$$\frac{12mfq}{Q_2}W_1n,$$

by reason of the load being applied to the compressed c as explained in article 132.

$$\left.\begin{array}{l}\text{Weight of top horizontal}\\ \quad\text{struts due to wind}\end{array}\right\} \left.\begin{array}{l} = \dfrac{3mfqwlh}{Q_2}(\tfrac{1}{2}n + 2) \\ \qquad (n \text{ even}), \\ = \dfrac{3mfqwlh}{Q_2}\left(\dfrac{n^2 - 1}{2n} + 2\right) \\ \qquad (n \text{ odd}). \end{array}\right\} = U. \quad (437)$$

The floor beams which carry the moving load generally act as the horizontal struts between the loaded chords; and they are usually so large, in comparison with the struts actually required to resist the wind pressure, that we may with little error make no further allowance for these beams acting as horizontal struts than that already suggested in article 124.

But, if it is required, we can find the additional metal to compensate the floor beams for this end pressure by treating each beam as a pillar whose least diameter is its depth, since the longitudinal joists or stringers prevent deflection sideways.

Thus, q_1 being the length, d the depth, of the wrought-iron I floor beams, and S the cross-section due to the total effect of wind pressure, P, in tons, applied longitudinally at the end of a beam, we have, from equation (400),

$$S = \frac{P\left(1 + \dfrac{(12q_1)^2}{ad^2}\right)}{f_1} \text{ square inches,}$$

to be added to section of each beam, in order to neutralize effect of wind upon the loaded horizontal system of struts.

$$\Sigma S = \frac{1 + \dfrac{(12q_1)^2}{ad^2}}{f_1} f \Sigma P$$

equals total additional section of I-beams; f being the factor of safety.

Now, in this case, ΣP takes the place of ΣZ_W and ΣZ_L, found by summing the vertical strains, Fig. 112, and used in equation (426), provided we put W_1 for W, L_1 for L. For, adding $n(W_1 + L_1)$, since the load is applied on the windward side in the direction of the wind's motion, and subtracting the pressures then upon the end struts, since no struts or I-beams are used *on* the abutments, will not alter ΣP.

$$\left.\begin{array}{l} \therefore \text{ Weight of I-beams} \\ \text{due to wind, from} \\ (426), \end{array}\right\} \left.\begin{array}{l} = \dfrac{3mfq_1W_1n^2}{Q_3} + \dfrac{mfq_1L_1}{2Q_3}(7n^2 + 3n - 10) \\ \qquad\qquad (n \text{ even}), \\ = \dfrac{3mfq_1W_1(n^2 - 1)}{Q_3} + \dfrac{mfq_1L_1}{2Q_3n}(7n^3 - 3n^2 - 7n + 3) \\ \qquad\qquad (n \text{ odd}), \end{array}\right\} \quad (438)$$

where

$$Q_3 = \frac{f_1}{1 + \dfrac{(12q_1)^2}{ad^2}}, \quad W_1 = \frac{wlh}{2n}, \quad L_1 = \frac{wle}{n}, \quad m = \frac{5}{18}.$$

l = length, in feet, between centres of end pins.
f_1 = numerator of Gordon formula (400).
n = number of panels.
a = constant. (See Table IV.)
h = height of girders, in feet, between centres of chords.
q_1 = entire length of floor beam, in feet.
d = depth of beam, in inches.
e = height of train or moving wind-resisting surface.
w = pressure of wind per square foot, in tons.

136. In finding the diagonals of the horizontal systems, top and bottom, due to wind pressure applied on either side, we must plainly make all the diagonals *mains*, and the two in any one panel each equal to the original main tie in that panel.

Using the strain sheet, Fig. 112, as a horizontal system now, putting W_1 for W, L_1 for L, q for h, $\sin\phi_1 = \dfrac{nq}{\sqrt{l^2 + n^2q^2}}$ for $\sin\phi$, Y_1 for Y, the strain in any horizontal diagonal tie due

to wind, we have, for the horizontal system between the loaded chords,

$$\left.\begin{array}{l}\text{Sum of horizontal}\\ \text{diagonal strains}\\ \text{due to wind}\end{array}\right\} = \Sigma Y_1 = \frac{4W_1}{2\sin\phi_1}\left\{\frac{n^2}{2} - \left(1+3+5+7+\ldots\frac{n}{2}\text{ terms}\right)\right\}$$
$$+\frac{4L_1}{2n\sin\phi_1}\left\{\frac{n^3}{2} - n\left(1+3+5+7+\ldots\frac{n}{2}\text{ terms}\right)\right.$$
$$\left.+2\left[1+3+6+10+\ldots\left(\frac{n}{2}-1\right)\text{ terms}\right]\right\}$$
$$=\frac{W_1 n^2}{2\sin\phi_1}+\frac{L_1}{12\sin\phi_1}(7n^2-4) \qquad (n\text{ even}), \tag{439}$$

$$=\Sigma Y_1 = \frac{4W_1}{2\sin\phi_1}\left\{\frac{n-1}{2}n - \left(1+3+5+7+\ldots\frac{n-1}{2}\text{ terms}\right)\right\}$$
$$+\frac{4L_1}{2n\sin\phi_1}\left\{\frac{n-1}{2}n^2 - \left(1+3+5+7+\ldots\frac{n-1}{2}\text{ terms}\right)\right.$$
$$+2\left(1+3+6+10\ldots\frac{n-3}{2}\text{ terms}\right)$$
$$+\text{ the }\left(\frac{n-1}{2}\right)^{\text{th}}\text{ term of the series }(1+3+6+10+\ldots)$$
$$\left.\text{for the middle panel}\right\}$$
$$=\frac{W_1(n^2-1)}{2\sin\phi_1}+\frac{7L_1(n^2-1)}{12\sin\phi_1} \qquad (n\text{ odd}). \tag{440}$$

Therefore, for horizontal system uniting loaded chords,

$$\left.\begin{array}{l}\text{Weight of horizontal}\\ \text{diagonals due to}\\ \text{wind pressure}\end{array}\right\} = \Sigma Y_1 \times \frac{12mfq}{T\sin\phi_1}$$
$$=\frac{mfq}{T\sin^2\phi_1}[6W_1n^2 + L_1(7n^2-4)] \qquad (n\text{ even}),$$
$$=\frac{mfq(n^2-1)}{T\sin^2\phi_1}(6W_1+7L_1) \qquad (n\text{ odd}). \tag{441}$$

137. It may be noted here, that, however cor
cient the horizontal systems are made, they wil
maintain the stability of the bridge under the
if the posts and horizontal struts at the ends of
not sufficient to resist the lateral pressure transr
from these horizontal systems. That is to say,
work of the bridge must be, with regard to t
incapable of lateral motion, whether of translati
distortion.

The required stability may be secured by mak
large end posts fast to the abutments for light a
tures, and by attaching these end posts to rigid
means of diagonal braces. But as all this ex
over the ordinary panel weight rests directly
ments, it does not enter into the formulæ for str
uniform panel pressures, W, L; W_{1}, L_{1}.

This excess of weight, however, has an influe
values of n and h; and, calling the excess E_w, p
proceed to formulate its value, and find the
stability.

138. To find the additional strains and weig
members of a bridge of two girders of Class I
resist a given wind pressure, let Fig. 114 represe
of the end frame of a through bridge of this class
its full moving-load.

Then, according to our previous notation,
zontal pressure at A is

$$P_2 = \tfrac{1}{2}nW_1 = \tfrac{1}{2}wlh;$$

and at B,

$$P_3 = \tfrac{1}{2}n(W_1 + L_1) = \tfrac{1}{2}wl(h + 2e)$$

The vertical pressure on each abutment is $\tfrac{1}{2}n(W$

Now, supposing these ends of iron rest upon a plane stone surface, and calling the "co-efficient of friction" for iron upon stone $\frac{1}{2}$ (see any good treatise on elementary mechanics), we must have, according to the received law of friction,

$$P_2 + P_3 < \tfrac{1}{2}n(W + L), \qquad (444)$$

which is the condition that prevents lateral translation along a plane stone surface, BE, Fig. 114.

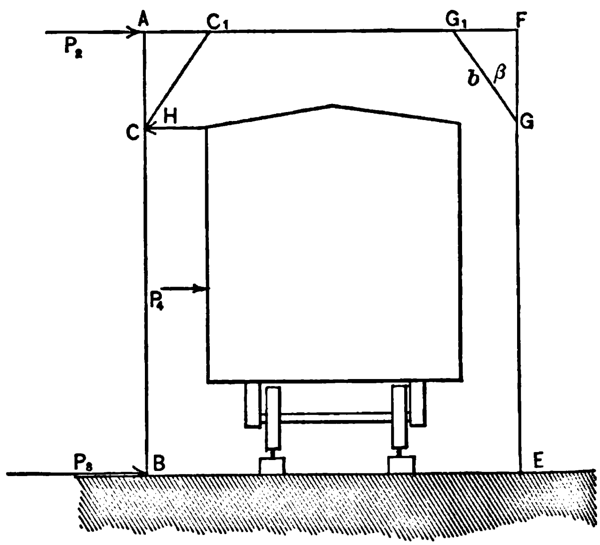


FIG. 114.

For stability against overturning, —

1st, Without live load. Take the moments about E; we need, since $AF = q$, and $FE = h$, as above,

$$P_2h < \tfrac{1}{2}nWq; \qquad (445)$$

or, if each end of the girders is tied to the abutment with a force, t,

$$P_2h < \tfrac{1}{4}nWq + qt, \qquad (446)$$

the condition that prevents rotation of unloaded bridge about the points of support.

2d, With live load resting upon a beam attached to the girders at the ends B and E, we require the condition

$$P_2h < \frac{n}{4}(W + L)q, \qquad (447)$$

girders not tied down; or

$$P_2h < \frac{n}{4}(W + L)q + qt \qquad (448)$$

when they are tied with the force t.

But, if the end of the live load rests directly on the abutment, and is not connected with the girders, the condition of stability is

$$P_2h < \tfrac{1}{4}q[Wn + L(n - 1)], \qquad (449)$$

or

$$P_2h < \tfrac{1}{4}q[Wn + L(n - 1)] + qt. \qquad (450)$$

3d, For the stability of the load itself against turning on its own points of support, we must have

$$P_4e < L_2g; \qquad (451)$$

e being the height of moving-load, g the gauge or breadth of base, and P_4 the wind pressure acting upon the part L_2 of this load.

The strains developed in the frame $BAFE$ will be greatest when the end posts cannot move laterally at the bottom nor

are truly fixed, so that, when under wind pressure, the tangents to their elastic curves, at the bases, will be vertical.

Let b = length of brace, $GG_{,,}$ in feet, and β = the angle it makes with the vertical post; then —

Shearing-strain at any cross-section of BC or GE is

$$S = \tfrac{1}{2}P_2. \tag{452}$$

Moment at any point, x, above B or E, is equal to

$$\tfrac{1}{2}P_2 x = M = \tfrac{1}{2}P_2(h - b\cos\beta), \text{ at } C \text{ or } G. \tag{453}$$

Taking moments about A for the post,

$$\tfrac{1}{2}P_2 h = Hb\cos\beta,$$

$$\therefore\ H = \frac{P_2 h}{2b\cos\beta}, \tag{454}$$

which is the horizontal component of the brace strain D.

$$\therefore\ D = \frac{H}{\sin\beta} = \frac{P_2 h}{2b\sin\beta\cos\beta} \tag{455}$$

in tension or compression.

Moment at any point between C_1 and $G_{,,}$

$$M = \tfrac{1}{2}P_2 h. \tag{456}$$

To enable each end post to resist this additional moment, (453), it would require $R = Bacd$ as the moment of the internal stresses on the added material, if it be added to the outsides of the post, at the distance $\tfrac{1}{2}d$, in inches, from the neutral axis; B being the ultimate bending unit strength of section, a = width of post, in inches, and c = the uniform thickness of additional iron on each side due to the greatest moment at C or G.

Hence

$$\tfrac{12}{2}P_2(h - b\cos\beta) = \frac{Bacd}{f},$$

$$\left.\begin{array}{l}\text{Whole thickness of added}\\ \quad\text{iron}\end{array}\right\} = 2c = \frac{12P_2(h - b\cos\beta)f}{aBd} \text{ inches.}$$

$$\left.\begin{array}{l}\text{Cross-section of added}\\ \quad\text{iron}\end{array}\right\} = 2ac = \frac{12P_2(h - b\cos\beta)f}{Bd} \text{ square inches.}$$

$$\left.\begin{array}{l}\text{Volume of 4 posts due}\\ \quad\text{to wind}\end{array}\right\} = \frac{4 \times 12^2P_2(h - b\cos\beta)fh}{Bd} \text{ cubic inches.}$$

$$\left.\begin{array}{l}\text{Weight to be added to}\\ \quad\text{4 posts at end due}\\ \quad\text{wind bending,}\end{array}\right\} = \frac{12^2mfwlh^2(h - b\cos\beta)}{Bd} \text{ pounds.} \qquad (457)$$

Similarly, from (456), for the two top horizontal end struts, of length q feet, and depth d_1 inches,

$$\left.\begin{array}{l}\text{Weight to be added to 2 end struts to}\\ \quad\text{resist bending from wind force}\end{array}\right\} \left.\begin{array}{l} = \dfrac{2 \times 12^2mfP_2qh}{Bd_1} \\ = \dfrac{12^2mfwlh^2q}{2Bd_1} \text{ pounds.}\end{array}\right\} \qquad (458)$$

If d_2 is the least diameter, in inches, of a brace of length b feet, with fixed ends, to resist the longitudinal pressure D, (455), then, by the Gordon formula, (400), we have

$$\text{Cross-section of brace, } S = \frac{fD\left\{1 + \dfrac{(12b)^2}{3000d_2^2}\right\}}{f_1} \text{ square inches;}$$

f, as before, being the factor of safety, and f_1 the numerator of Gordon formula.

$$\therefore \text{ Volume of 4 braces} = 48bS = \frac{48fD\left\{1 + \dfrac{(12b)^2}{3000d_2^2}\right\}b}{f_1} \text{ cubic inches.}$$

$$\left.\begin{aligned}\text{Weight of 4 braces} &= \frac{24mfP_2h\left\{1+\dfrac{(12b)^2}{3000d_2^2}\right\}}{f_1\sin\beta\cos\beta}\\ &= \frac{6mfwlh^2\left\{1+\dfrac{(12b)^2}{3000d_2^2}\right\}}{f_1\sin\beta\cos\beta}\text{ pounds.}\end{aligned}\right\}\quad(459)$$

$$E_w = 12^2mflh^2w\left\{\frac{h-b\cos\beta}{Bd}+\frac{q}{2Bd_1}+\frac{1+\dfrac{(12b)^2}{3000d_2^2}}{24f_1\sin\beta\cos\beta}\right\},\quad(460)$$

which is the excess of weight, in pounds, of wrought-iron, on the two abutments, due to wind pressure, and not affecting the uniform panel weight of bridge, W.

139. In the preceding investigation, we have assumed that the entire effort of the wind to distort the rectangular cross-section of the bridge is to be resisted by the two end frames alone.

Instead of this provision, however, we may fix firmly each horizontal strut of the unsupported lateral system throughout the bridge to the ends of the posts abutting upon it, and thus transfer the whole wind pressure to that lateral system which is between the chords resting upon the supports. The same transfer would also be accomplished should we connect the posts rigidly to the other, or supported, lateral struts. This procedure would enable us to dispense with the horizontal diagonals of the unsupported system but for the necessity of retaining them to keep the chords they connect from deflecting horizontally.

In this case, of course, the horizontal diagonals and struts of the supported system will have twice as great horizontal pressure to resist as in the former case, and (438) and (441) must be multiplied by 2.

The horizontal struts of the unsupported system must be

able to resist, in a vertical direction, the bending-moment found by (456), when for P_2 we put W_{11}, or its value $\frac{1}{2}wh\frac{l}{n}$, giving

$$M = \frac{1}{4}wh^2\frac{l}{n} = \frac{Bacd_2}{12f} \qquad (461)$$

for each strut; d_2 being the depth of horizontal strut, in inches, a the width, and c the thickness of each of two plates of iron added at the distance $\frac{1}{2}d_2$ from the neutral axis of the strut. $B \div f$ = allowed bending unit strain, in tons, per square inch, since w is in tons. Then the weight of all these horizontal struts due to the bending-moment, (461), must be the same as in (458) if we put d_2 for d_{11} and regard the two extreme struts as one, since each sustains but half a panel pressure.

At the same time, these horizontal struts must resist, in the direction of their least diameters, the bending-moment due to the longitudinal strain brought upon them by the attached horizontal diagonals in adjusting the bridge.

Now these horizontal diagonals, between unsupported chords, may be of uniform size, having a cross-section S, (say) of not less than about 1 square inch. Then, if the allowed unit strain upon them is $T \div f$, and if their inclination to the plane of the girder is ϕ_1, we have the longitudinal pressure of two diagonals, from adjustment, to be provided for, equal to

$$P_4 = \frac{2TS}{f}\sin\phi_1 = \frac{2TS}{f}\sqrt{\frac{1}{1 + \frac{l^2}{n^2q^2}}}. \qquad (462)$$

And if $\frac{Q_2}{f} = \frac{f_1}{f\left(1 + \frac{(12q)^2}{3000d_3^2}\right)}$ = the allowed pressure per square inch upon a strut, Q_2, P_4, f_1, and T being of the same denom-

ination, and f_1 = numerator of Gordon formula, f = factor of safety, then

$$P_a \div \frac{Q_2}{f} = \frac{2TS\sin\phi_1}{Q_2} = S_1, \qquad (463)$$

which is the cross-section of the strut due to the end pressure P_a.

Hence, from (461) and (463), —

Total section of a horizontal strut between the unsupported chords is, in square inches,

$$2ac + S_1 = \frac{6wfh^2l}{nBd_2} + \frac{2TS\sin\phi_1}{Q_2}. \qquad (464)$$

And the weight of n horizontal struts between the unsupported chords, to resist the adjustment and distortion strains, is

$$12mnq(2ac + S_1) = \frac{72mfwlh^2q}{Bd_2} + \frac{24TS\sin\phi_1 mqn}{Q_2}, \qquad (465)$$

in pounds, where n is used instead of $(n + 1)$, since the two extreme struts suffer only the strain due to any one of the others.

For the additional iron required in the posts to resist distortion by the wind, we have, from (453), by putting $\frac{whl}{2n}$ for P_n and taking the moment at the centre of post where $x = \frac{1}{2}h$,

$$M = \frac{12wlh^2}{2 \times 4n} = \frac{Bacd}{f} \text{ inch-tons} \qquad (466)$$

as the bending-moment allowed at the weakest part of the post, each end post having but $\frac{1}{2}M$ instead of M. Therefore

$$\left.\begin{array}{l}\text{Whole thickness of added iron for 1}\\ \quad\text{post}\end{array}\right\} = 2c = \frac{3wflh^2}{anBd} \text{ inches.}$$

$$\left.\begin{array}{l}\text{Cross-section to be added to each}\\ \quad\text{post}\end{array}\right\} = 2ac = \frac{3wflh^2}{nBd}\ \text{square inches.}$$

$$\left.\begin{array}{l}\text{Weight to be added to } 2n \text{ posts to}\\ \quad\text{resist distortion of rectangular}\\ \quad\text{cross-section of bridge}\end{array}\right\} = \frac{12^2 mfwlh^3}{2Bd}\ \text{pounds.} \qquad (467)$$

Finally, the weight of $2n$ wrought-iron braces for this case also is given by (459); and the cross-section of one brace is

$$S = \frac{fD\left\{1 + \frac{(12b)^2}{3000d_2^2}\right\}}{f_1} = \frac{wflh^2\left\{1 + \frac{(12b)^2}{3000d_2^2}\right\}}{4nf_1 b \sin\beta\cos\beta}, \quad (468)$$

since D in (455) now becomes

$$\frac{wh^2l}{4nb\sin\beta\cos\beta}.$$

140. We will now exemplify the method of article 138, which provides, in the end frames alone, the means of resisting the distorting influence of the wind.

EXAMPLE. — To find the best number of panels, n, and the best height, h, for the two wrought-iron girders of a highway "through" bridge of 100 feet span $= l$, and 18 feet wide between centres of chords $= q$, single system of Class IX., Pratt Truss, under a uniform rolling load of 1 ton $=$ 2,000 pounds per running foot, in addition to the weight of bridge. Also, to find the weight, nW, of the bridge corresponding to the best values of n and h, using 4 as the factor of safety for iron, and 10 for wood, and taking account of wind pressure.

Let us compute for $n = 5, 6, 7, 8, 9, 10, 11, 12$, in succession, as explained in article 134, retaining h and W in all the expressions for weight.

1st, The floor of pine, called 50 pounds per cubic foot.

Thickness $t = \frac{2.5}{12}$ foot.
Width $q' \quad = 17.5$ feet.
Length $l \quad = 100$ feet.

$$\text{Weight of floor } F = \frac{2.5}{12} \times 17.5 \times 100 \times 50 = 18229 \text{ pounds.}$$

2d, The joists of pine at 50 pounds per cubic foot.
$g = 2$ feet between centres.
$B = 7{,}000$ pounds per square inch $=$ ultimate resistance to cross-breaking.
$f = 10$, factor of safety for pine.
$l \div n =$ panel length of joist, in feet.
$d = b^2 =$ depth of joist, in inches, by (431).
Then, by (432), we have

$$\left.\begin{matrix}\text{Thickness of}\\ \text{a joist, } b\end{matrix}\right\} = \left\{\frac{9 \times 10 \times 2 \times 100}{n^2 \times 17.5 \times 7000}(18229 + 200000)\right\}^{\frac{1}{5}} = \frac{7.96544}{n^{\frac{2}{5}}} \text{ ins.};$$

and, from (433),

$$\text{Weight of joists, } J = \frac{100 \times 17.5 \times 50}{144 \times 2}\left(\frac{7.96544^3}{n^{\frac{6}{5}}}\right) = \frac{153548}{n^{1.2}} \text{ pounds.}$$

3d, The wrought-iron I-beams, $n - 1$ in number, supporting the joists, floor, and moving-load $L = \frac{l}{n} = \frac{100}{n}$ tons per panel.

Take $\qquad B = 50{,}000$ pounds, Table II.
Length of beam $q_1 = 18.5$ feet.

$$\text{Depth } d = 3.80122\left\{\left(\frac{153548}{n^{1.2}} + 218229\right)\frac{18.5 \times 4}{50000n}\right\}^{\frac{1}{3}}$$

$$= 0.4331885\left(\frac{J + 218229}{n}\right)^{\frac{1}{3}} \text{ inches,}$$

from (412), using the proportions assumed in finding that equation.

By (434),

$$\text{Weight of I-beams, } P = 15.46068(n-1) \times \frac{5}{18}$$

$$\times\, 18.5\left(\frac{J + 218229}{n} \times \frac{18.5 \times 4}{50000}\right)^{\frac{2}{3}}$$

$$= 1.031824(n-1)\left(\frac{J + 218229}{n}\right)^{\frac{2}{3}} \text{ pounds.}$$

4th, The horizontal struts of the top lateral system of this "through" bridge.

In this example of a highway bridge, let us assume, as actual pressure of wind per square foot, the large value 75 pounds; also that the two open girders offer a resisting surface equivalent to $\frac{3}{5}$ of the surface presented if the bridge were covered, that is, equal to $\frac{3}{5}hl$. Then the whole wind force to be resisted is $\frac{3}{5} \times 75hl = 45hl$ pounds.

Wind force per running foot $= 45h$ pounds.

Wind force per square foot $= w = \frac{45}{2000} = 0.0225$ ton.

Although this wind force is actually applied to both girders, we shall regard it as distributed equally to the panel points of the two windward chords, no account being here taken of the action of wind on passing carriages.

Suppose the top horizontal struts to be I-beams, the square of whose least radius of gyration is $r^2 = 0.5$ inch, which corresponds to a six-inch beam of ordinary make. Then, using equation (385), and calling $C = 40,000$, $E = 27,300,000$, we have, in (437),

$$Q_2 = \frac{20}{1 + \dfrac{40000 \times 210^2}{4\pi^2 \times 27300000 \times 0.5}} = 4.680056 \text{ tons;}$$

and (437) gives

$$\left.\begin{array}{l}\text{Weight of top}\\ \quad\text{horizontal}\\ \quad\text{struts due}\\ \quad\text{to wind}\end{array}\right\} = U = \frac{3 \times 5 \times 4 \times 18 \times 0.0225 \times 100h}{18 \times 4.680056}\left(\frac{n}{2} + 2\right)$$

$$= 28.845581h\left(\frac{n}{2} + 2\right) \ (n \text{ even}),$$

$$= 28.845581h\left(\frac{n^2 - 1}{2n} + 2\right) \ (n \text{ odd}).$$

5th, The horizontal diagonals, top and bottom. From (441), where now $L_1 = 0$, since we take no account here of wind against live load on this highway bridge, we have, making $T = 24$ tons, $q = 18$ feet, $m = \frac{6}{18}$ for wrought-iron (as above), $W_1 = \frac{hwl}{2n}$,

$$\left.\begin{array}{l}\text{Weight of horizon-}\\ \quad\text{tal diagonals, top}\\ \quad\text{and bottom,}\end{array}\right\} = X = \frac{2 \times 5 \times 4 \times 18 \times 6}{18 \times 24 \sin^2 \phi_1} W_1 \left\{\begin{array}{l} n^2 \ (n \text{ even}),\\ (n^2 - 1)\,(n \text{ odd}),\end{array}\right.$$

where $$\frac{1}{\sin^2 \phi_1} = 1 + \frac{l^2}{n^2 q^2} = 1 + \frac{10000}{18^2 n^2}.$$

6th, Let the residual weight, Y, be 1,000 pounds for all values of n.

7th, The additional weight of iron needed in the I-beams, by reason of their acting as horizontal struts for the wind pressure on lower chord, is found by (438), after computing d as already formulated for the floor beams. Here $W_1 = \frac{hwl}{2n}$ tons; $L_1 = 0$; $f_1 = 18$ tons; $q_1 = 18.5$ feet; $a = 750$, since the ends are not fixed. Hence, from (438), where $Q_3 = \dfrac{18}{1 + \dfrac{222^2}{750d^2}}$,

$$\left.\begin{array}{r}\text{Weight to be added}\\ \text{to floor beams}\\ \text{due to wind}\end{array}\right\} = P' = \frac{3 \times 5 \times 4 \times 18.5 \times 0.0225 \times 100\left(1 + \frac{222^2}{750 d^2}\right)h}{18 \times 18 \times 2} \left\{\begin{array}{l}\times n \ (n \text{ even}),\\ \times \frac{n^2 - 1}{n} \ (n \text{ odd});\end{array}\right.$$

$$P' = 3.8541666h\left(1 + \frac{65.712}{d^2}\right)\left\{\begin{array}{l}\times n \ (n \text{ even}),\\ \times \frac{n^2 - 1}{n} \ (n \text{ odd}).\end{array}\right.$$

Collecting the terms of K thus found, we have, in terms of h,—

WEIGHTS OF THE COMPONENTS OF K, IN POUNDS.

n	5	6	7	8
Floor . . .	18229.0000	18229.0000	18229.0000	18229.0000
Joists . . .	22258.0000	17884.0000	14864.0000	12663.0000
I-Beams . .	5459.0000	5969.0000	6408.0000	6796.0000
Do. wind . .	$23.3984h$	$30.1245h$	$35.3701h$	$42.3086h$
Hor. struts .	$126.9216h$	$144.2291h$	$156.5914h$	$173.0749h$
Hor. diags. .	$120.6662h$	$125.3704h$	$125.7336h$	$133.4025h$
Residual . .	1000.0000	1000.0000	1000.0000	1000.0000
K . .	46946.0000 $+270.9862h$	43082.0000 $+299.7240h$	40501.0000 $+317.6951h$	38688.0000 $+348.7860h$

n	9	10	11	12
Floor . . .	18229.0000	18229.0000	18229.0000	18229.0000
Joists . . .	10994.0000	9688.0000	8641.0000	7784.0000
I-Beams . .	7146.0000	7465.0000	7760.0000	8035.0000
Do. wind . .	$48.1181h$	$55.3312h$	$61.6226h$	$69.1289h$
Hor. struts .	$185.8950h$	$201.9207h$	$215.0294h$	$230.7665h$
Hor diags. .	$138.1040h$	$147.2220h$	$154.0325h$	$163.9358h$
Residual . .	1000.0000	1000.0000	1000.0000	1000.0000
K . .	37369.0000 $+372.1171h$	36382.0000 $+404.4739h$	35630.0000 $+430.6845h$	35048.0000 $+463.8312h$

8th, The top chords, of 2 channels and 2 plates of wrought-iron.

In each panel let the ratio of chord's length to least diameter be 15.

Then, in (424),

$$Q = \frac{18}{1 + \frac{15^2}{3000}} = 16.7442 \text{ tons},$$

by (400).

$$L = \frac{l}{n} = \frac{100}{n} \text{ tons.}$$

$$\left.\begin{array}{l}\text{Weight of top chords due to ver-}\\ \quad\text{tical pressures, in pounds,}\end{array}\right\} = \frac{5 \times 4 \times 100^2}{2 \times 18 \times 16.7442h}\left(W + \frac{100}{n}\right)$$

$$\left\{\begin{array}{l} \times \dfrac{2n^2 + 3n - 2}{n} \ (n \text{ even}), \\ \times \dfrac{2n^3 + 3n^2 - 2n - 3}{n^2} \ (n \text{ odd}). \end{array}\right.$$

And, from (435),

$$\left.\begin{array}{l}\text{Weight of top chords due}\\ \quad\text{to wind, in pounds,}\end{array}\right\} = \frac{5 \times 4 \times 100^3 \times 0.0225h}{2 \times 18 \times 16.7442 \times 18}$$

$$\left\{\begin{array}{l} \times \dfrac{2n^2 + 3n - 2}{n^2} \ (n \text{ even}), \\ \times \dfrac{2n^3 + 3n^2 - 2n - 3}{n^3} \ (n \text{ odd}). \end{array}\right.$$

9th, The bottom chords, of flat links or I-bars.

From (425),

$$\left.\begin{array}{l}\text{Weight of bottom chords due to}\\ \quad\text{vertical forces, in pounds,}\end{array}\right\} = \frac{5 \times 4 \times 100^2}{2 \times 18 \times 24h}\left(W + \frac{100}{n}\right)$$

$$\left\{\begin{array}{l} \times \dfrac{2n^3 - 3n^2 + 22n - 24}{n^2} \ (n \text{ even}), \\ \times \dfrac{2n^3 - 3n^2 + 22n - 21}{n^2} \ (n \text{ odd}). \end{array}\right.$$

And, from (436), e being zero,

$$\left.\begin{array}{l}\text{Weight of bottom chords}\\ \text{due wind, in pounds,}\end{array}\right\} = \frac{5 \times 4 \times 100^3 \times 0.0225h}{2 \times 18 \times 24 \times 18}\left\{\begin{array}{l}\times \dfrac{2n^3 - 3n^2 + 22n - 24}{n^3} \ (n \text{ even}),\\ \times \dfrac{2n^3 - 3n^2 + 22n - 21}{n^3} \ (n \text{ odd}).\end{array}\right.$$

10th, The verticals. Take ratio of length to least diameter 30; then, in (426),

$$Q_1 = \frac{18}{1 + \frac{30^2}{750}} = 8.\dot{1}\dot{8}$$

if the ends are not fixed, and we have

Weight of verticals, in pounds,

$$= \frac{3 \times 5 \times 4}{18 \times 8.1818} Whn^2 + \frac{5 \times 4 \times 100h}{2 \times 18 \times 8.1818}\left(\frac{7n^2 + 3n - 10}{n}\right) \quad (n \text{ even}),$$

$$= \frac{3 \times 5 \times 4}{18 \times 8.1818} Wh(n^2 - 1) + \frac{5 \times 4 \times 100h}{2 \times 18 \times 8.1818}\left(\frac{7n^3 - 3n^2 - 7n + 3}{n^2}\right) \quad (n \text{ odd}).$$

11th, The girder diagonals, by (428).

$$\frac{1}{\sin^2\phi} = 1 + \frac{l^2}{n^2h^2}.$$

$$\left.\begin{array}{l}\text{Weight of girder}\\ \text{diagonals, in}\\ \text{pounds,}\end{array}\right\} = \frac{4 \times 5 \times 4 \times 100h}{18 \times 24 \sin^2\phi}\left(\frac{n^2 - 1}{n}\right) + \frac{3 \times 5 \times 4h}{18 \times 24 \sin^2\phi} Wn^2 \quad (n \text{ even}),$$

$$= \frac{4 \times 5 \times 4 \times 100h}{18 \times 24 \sin^2\phi}\left(\frac{n^2 - 1}{n}\right) + \frac{3 \times 5 \times 4h}{18 \times 24 \sin^2\phi} W(n^2 - 1) \quad (n \text{ odd}).$$

Computing for the different values of n, collecting, and arranging, we have, including the values of K above, —

WEIGHTS IN POUNDS, W IN TONS, h IN FEET.

Live load $= nL =$ 100 tons, $l =$ 100 feet.

n				$\frac{W}{h}$		Wh		$\frac{1}{h}$		h^2		h
5	Top chords	Load	4140.742		-		82815		-		-	
		Wind	-		-		-		-		103.5185	
	Bottom chords	Load	2444.444		-		48889		-		-	
		Wind	-		-		-		-		61.1111	
	Verticals		-		9.77778		-		-		208.5926	
	Diagonals		1333.333		3.33333		35555		-		88.8889	
	K		-		-		-		46946		270.9862	
	$2000nW =$		7918.519		+13.11111		+167259		+46946		+733.0973	
6	Top chords	Load	4866.256		-		81104		-		-	
		Wind	-		-		-		-		101.3803	
	Bottom chords	Load	2777.778		-		46296		-		-	
		Wind	-		-		-		-		57.8704	
	Verticals		-		14.66667		-		-		294.2387	
	Diagonals		1388.889		5.00000		30007		-		108.0247	
	K		-		-		-		43082		299.7240	
	$2000nW =$		9032 923		+19.66667		+157407		+43082		+861.2381	
7	Top chords	Load	5525.323		-		78933		-		-	
		Wind	-		-		-		-		98.6665	
	Bottom chords	Load	3174.604		-		45351		-		-	
		Wind	-		-		-		-		56.6894	
	Verticals		-		19.55556		-		-		305.9713	
	Diagonals		1360.544		6.66666		25915		-		126.9841	
	K		-		-		-		40501		317.6951	
	$2000nW =$		10060.471		+26.22222		+150199		+40501		+906.0064	
8	Top chords	Load	6221.067		-		77763		-		-	
		Wind	-		-		-		-		97.2042	
	Bottom chords	Load	3559.027		-		44488		-		-	
		Wind	-		-		-		-		55.6098	
	Verticals		-		26.07408		-		-		392.1296	
	Diagonals		1388.889		8.88889		22787		-		145.8333	
	K		-		-		-		38688		348.7860	
	$2000nW =$		11168.983		+34.96297		+145038		+38688		+1039.5629	

WEIGHTS IN POUNDS, W IN TONS, h IN FEET.

Live Load $= nL = 100$ tons, $l = 100$ feet.

n 9	Top chords	Load	6881.576	$\frac{W}{h}$	–	Wh	76462	$\frac{1}{h}$	–	h^2	–	h
		Wind	–		–		–		–		95.5774	
	Bottom chords	Load	3978.051		–		44201		–		–	
		Wind	–		–		–		–		55.2507	
	Verticals		–		32.59260		–		–		402 3777	
	Diagonals		1371.742		11.11111		20322		–		164.6090	
	K		–		–		–		37369		372.1171	
	$2000nW =$		12231.369		+43.70371		+140985		+37369		+1089 9319	
10	Top chords	Load	7564.819		–		75648		–		–	
		Wind	–		–		–		–		94.5602	
	Bottom chords	Load	4388.889		–		43889		–		–	
		Wind	–		–		–		–		54.8611	
	Verticals		–		40.74075		–		–		488.8889	
	Diagonals		1388.889		13.88889		18333		–		183 3333	
	K		–		–		–		36382		404.4739	
	$2000nW =$		13342.597		+54.62964		+137870		+36382		+1226.1174	
11	Top chords	Load	8226.202		–		74784		–		–	
		Wind	–		–		–		–		93.4796	
	Bottom chords	Load	4820.936		–		43827		–		–	
		Wind	–		–		–		–		54 7834	
	Verticals		–		48.88888		–		–		498.3164	
	Diagonals		1377.410		16 66667		16696		–		202,0202	
	K		–		–		–		35630		430.6845	
	$2000nW =$		14424.548		+65.55555		+135307		+35630		+1279.2841	
12	Top chords	Load	8903.037		–		74192		–		–	
		Wind	–		–		–		–		92.7400	
	Bottom chords	Load	5246.912		–		43724		–		–	
		Wind	–		–		–		–		54.6553	
	Verticals		–		58.66667		–		–		585.0823	
	Diagonals		1388.889		20.00000		15325		–		220.6790	
	K		–		–		–		35048		463.8312	
	$2000nW =$		15538.838		+78.66667		+133241		+35048		+1416.9878	

Multiplying each of these eight equations by $\frac{h}{20000}$, we find the uniform panel weight of bridge, W, in terms of h; thus:

$$n = 5, \quad W = \frac{8.36295 + 2.3473h + 0.0366548 7h^2}{-0.3959259 + 0.5h - 0.000655555h^2}.$$

$$n = 6, \quad W = \frac{7.87035 + 2.1541h + 0.04306191h^2}{-0.4516461 + 0.6h - 0.000983333h^2}.$$

$$n = 7, \quad W = \frac{7.50995 + 2.02505h + 0.04530032h^2}{-0.50302355 + 0.7h - 0.001311111h^2}.$$

$$n = 8, \quad W = \frac{7.2519 + 1.9344h + 0.05197814h^2}{-0.55844915 + 0.8h - 0.0017481485h^2}.$$

$$n = 9, \quad W = \frac{7.04925 + 1.86845h + 0.05449659h^2}{-0.6115684 + 0.9h - 0.002185185h^2}.$$

$$n = 10, \quad W = \frac{6.8935 + 1.8191h + 0.06130587h^2}{-0.6671298 + h - 0.002731482h^2}.$$

$$n = 11, \quad W = \frac{6.76535 + 1.7815h + 0.06396421h^2}{-0.7212274 + 1.1h - 0.0032777777h^2}.$$

$$n = 12, \quad W = \frac{6.66205 + 1.7524h + 0.07084939h^2}{-0.7769419 + 1.2h - 0.003933333h^2}.$$

In differentiating these and similar expressions for W, it will be convenient to have a typical form or mode of operation. Let

$$W = \frac{a + bh + ch^2}{a_1 + b_1h + c_1h^2} \tag{469}$$

be a type of these equations; then, after putting $\frac{dW}{dh} = 0$, and reducing, we have the equation

$$0 = ab_1 - a_1b + (ac_1 - a_1c)(2h) + (bc_1 - b_1c)h^2, \tag{470}$$

from which h is easily found: and there is no need, in these cases, of taking the second differential to ascertain whether the positive value of h, to be found from (470), renders W a maximum or a minimum; for the substitution of a member a little less or a little greater than the positive value of h so found will at once serve to verify the work, and show W to be a minimum in (469) when h takes the value given by (470).

Taking the case where $n = 9$, and using logarithms, we may solve thus:—

$n = 9$.	Logs.	Log Co-efficients.	Co-efficients.	Equation (470).
$a = 7.049250000$	0.8481429	$\log ab_1 = 0.8023854$	$ab_1 = 6.3443300$	-
$a_1 = -0.611568400$	9.7864451	$\log a_1b = 0.0579266$	$-a_1b = 1.1426900$	7.4870200
$b = 1.868450000$	0.2714815	$\log bc_1 = 7.6109697$	$bc_1 = -0.0040829$	-
$b_1 = 0.900000000$	9.9542425	$\log b_1c = 8.6906118$	$-b_1c = -0.0490469$	$= 0.0531298h^2$
$c = 0.054496590$	8.7363693	$\log ca_1 = 8.5228144$	$-ca_1 = +0.0333284$	-
$c_1 = -0.002185185$	7.3304882	$\log c_1a = 8.1876311$	$c_1a = -0.0154039$	$-0.0179245(2h)$

	$0.0531298h^2$	$-\ 0.0179245(2h)$	$=$	7.4870200
log co-efficients,	8.7253382	8.2534471		0.8743090
log quotients,		9.5281089		2.1489708
		$h^2 - 0.33737(2h)$	$=$	140.9194000
2 log 0.33737,		9.0562178	$= \log$	$+0.1138000$
		2.1493213	$= \log$	141.0332000
$\log (141.0332)^{\frac{1}{2}}$		1.0746607	$= \log$	± 11.8757400
$\frac{1}{2}$ co-efficient of h,				$+0.3373700$

When W is a minimum, $h = 12.21311$ feet.

$\log h$,	1.0868263	$a =$	7.049250	
$\log bh$,	1.3583078	$bh =$	22.819590	
$\log h^2$,	2.1736526			
$\log ch^2$,	0.9100219	$ch^2 =$	8.128720	37.997560 1.5797557 = log num.
$\log c_1h^2$,	9.5131408	$c_1h^2 =$	$-\ 0.325942$	
		$a_1 =$	$-\ 0.611568$	
		$b_1h = 0.9h =$	10.991799	10.054289 1.0023513 = log denom.

$nW = 34.013151$ tons. $W = 3.779239$ 0.5774044 = log W.

Computing h and Wn for the other values of n, we find them, when Wn is a minimum, as follows: —

Span = l = 100 Feet. Uniform Live Load = nL = 100 Tons.

No. of Panels, n.	Panel Length, $l \div n$ feet.	Best Height in Feet, h.	Ratio of Length to Height, $l \div h$.	Inclination of Diagonals to Horizon, ϕ.	Minimum Bridge Weight, nW Tons.	Ratio of Dead to Live Load.	Ratio of Dead to Total Load.
5	20	16.50041	6.0604	39° 31′ 26″	37.177990	0.37178	0.27102
6	$16\frac{2}{3}$	14.71481	6.7959	41° 26′ 30″	35.930016	0.35930	0.26433
7	$14\frac{2}{7}$	13.89555	7.1966	44° 12′ 24″	34.642979	0.34643	0.25730
8	$12\frac{1}{2}$	12.74032	7.8491	45° 32′ 44″	34.509872	0.34510	0.25656
9	$11\frac{1}{9}$	12.21311	8.1879	47° 42′ 18″	34.013151	0.34013	0.25380
10	10	11.39809	8.7734	48° 44′ 18″	34.302300	0.34302	0.25541
11	$9\frac{1}{11}$	11.02062	9.0739	50° 28′ 51″	34.156870	0.34157	0.25460
12	$8\frac{1}{3}$	10.40797	9.6080	51° 19′ 1″	34.635012	0.34635	0.25725

Hence 34.013151 tons is the least of these least weights, or the *minimum minimorum*.

By observing the first differences of the values of nW, we may perceive, that, in addition to the fact that the odd number $n = 9$ gives the lowest value of nW, the odd number $n = 11$ gives a lower value of nW than either of the even numbers, 10 and 12, adjacent to it, and that $n = 7$ renders nW nearly as small as $n = 8$. We may, therefore, almost infer from these eight cases, that, when near the best value of n, an odd number of panels is preferable to an even number. And this conclusion harmonizes with the fact, that, in case of an odd number of panels, there is *no* weight *applied* at the centre of span as there is when n is even. We may further observe that the difference between the greatest and least values of nW in these eight cases is only 3.164839 tons, provided the best values of h are used; but, if other values of h are employed, nW departs more widely from its least value.

Also in the present case, when nW is least, the inclination, ϕ, of the girder diagonals to the horizon is about $2\frac{3}{4}$ degrees above 45 degrees; and from this point ϕ increases or decreases with n if the best value is given to h.

The best ratio of length to height of girder for this span and load is 8.1879; and, near the best simultaneous values of n and h, we have approximately

$$h = \frac{l}{n}. \tag{471}$$

12th, Let us now find the value of E_w, equation (460), the quantity of wrought-iron to be added to the end framework to resist wind force tending to produce distortion, assuming that the bridge is so fixed to the abutments that neither sliding nor overturning can take place.

In equation (460), take $b = 4$ feet = length of brace, $d_2 = 6$ inches, $\beta = 45$ degrees = inclination of brace to post.

$$\therefore \quad \sin\beta = \cos\beta = 0.70711, \quad b\cos\beta = 2.82844 \text{ feet.}$$

Take $d = 12$ inches, width of end post to resist bending.

$d_1 = 12$ inches, depth of end horizontal strut.

$f_1 = 18$ tons.

$B = 25$ tons.

$q = 18$ feet.

$m = \frac{5}{18}$ pound.

$f = 4$.

$l = 100$.

$w = 0.0225$ ton.

Then, computing E_w for the eight values of h already found, we obtain, from (460) and from the table just given, the following results: —

W in Tons, h in Feet.

No. of Panels, n.	5	6	7	8
Height, h	16.500	14.715	13.896	12.740
n times panel weight, nW . .	37.178	35.927	34.643	34.510
Added iron, E_w tons	3.935	2.898	2.489	1.980
Weight of bridge, $nW + E_w$.	41.113	38.825	37.132	36.490
Weight of wood	20.244	18.057	16.547	15.446
Weight of iron	20.869	20.768	20.585	21.044
Cost of iron, at $150	$3130 35	$3115 20	$3087 75	$3156 60
Cost of wood, at $15	303 66	270 86	248 21	231 69
Cost of bridge.	3434 01	3386 06	3335 96	3388 29
Excess over least.	98 05	50 10	-	52 33
No. of Panels, n.	**9**	**10**	**11**	**12**
Height, h	12.213	11.398	11.021	10.408
n times panel weight, nW . .	34.013	34.302	34.157	34.635
Added iron, E_w tons	1.773	1.480	1.357	1.170
Weight of bridge, $nW + E_w$.	35.786	35.782	35.514	35.805
Weight of wood	14.612	13.959	13.435	13.007
Weight of iron	21.174	21.823	22.079	22.798
Cost of iron, at $150	$3176 10	$3273 45	$3311 85	$3419 70
Cost of wood, at $15	219 18	209 39	201 53	195 11
Cost of bridge	3395 28	3482 84	3513 38	3614 81
Excess over least.	59 32	146 88	177 42	278 85

Here we see that $n = 11$ and $h = 11.021$ are the conditions yielding least total weight of bridge, while the whole cost is a minimum if $n = 7$, $h = 13.896$, and $(l \div n) = 14\frac{2}{7}$.

Notice that both of these minima of weight and cost correspond to an odd number of panels, and that the excess of cost above the lowest would in all cases more than compensate the

manufacturer for having the best simultaneous values of n and h determined by calculation, as above.

If, however, there would be sufficient head-room, we may, for this span and load, adopt either 8 or 9 panels, giving more iron, less wood, and less total weight, with a small increase of cost. In each of these 8 cases it will be seen the bridge weight is a little more than one-third the uniform moving-load, 100 tons $= nL$, and that the total dead load is slightly greater than one-fourth the sum of dead and live loads.

141. To exemplify the Method of Article 139, which provides, at Every Post, the Means of resisting the Distorting Influence of the Wind. — Taking the example of article 140, and calling the top horizontal diagonals 1 inch in diameter (that is, 0.7854 square inch cross-section), and weighing 2.654 pounds to the foot, we have

$$\left.\begin{array}{l}\text{Weight of } 2n \text{ horizontal top}\\ \quad\text{diagonals}\end{array}\right\} = 2n \times 2.654\sqrt{18^2 + \frac{100^2}{n^2}}$$

$$= 5.308\sqrt{324n^2 + 10000} \text{ pounds,}$$

$$\left.\begin{array}{l}\text{Weight of bottom horizontal}\\ \quad\text{diagonals}\end{array}\right\} = X, \left\{\begin{array}{l}\text{as found in article 140, for both}\\ \quad\text{top and bottom.}\end{array}\right.$$

Strain on a top horizontal strut, from $\frac{24}{4} = 6$ tons per square inch on two top diagonals, is equal to

$$2 \times 6 \times 0.7854 \sin\phi_1 = 9.4248 \sin\phi_1 \text{ tons.}$$

Now we already have the breaking inch strain on top horizontal struts $= 4.680056$ tons, and

$$\sin\phi_1 = \sqrt{\frac{1}{1 + \frac{l^2}{n^2q^2}}} = nq\sqrt{\frac{1}{n^2q^2 + l^2}} = \sqrt{\frac{1}{1 + \frac{10000}{324n^2}}}.$$

Therefore, in square inches,

$$\left.\begin{array}{l}\text{Cross-section of a}\\ \quad\text{top strut to re-}\\ \quad\text{sist initial strain}\\ \quad\text{on diagonals}\end{array}\right\} = \frac{9.4248}{1.170014}\sin\phi_1 = 8.0553\sqrt{\frac{1}{1+\frac{10000}{324n^2}}} = S_1.$$

And, from (461),

$$\left.\begin{array}{l}\text{Cross-section of a top}\\ \quad\text{strut to resist distort-}\\ \quad\text{ing force of wind}\end{array}\right\} = 2ac = \frac{6wflh^2}{Bd_2n} = 0.30\frac{6}{7}\frac{h^2}{n} \text{ square inches}$$

if $w = 0.0225$ ton, $f = 4$, $l = 100$, $B = 25$ tons, and $d_2 = 7$ inches.

From (465), since $12mqn = 12 \times \frac{5}{18} \times 18n = 60n$,

$$\left.\begin{array}{l}\text{Weight of } n \text{ top hori-}\\ \quad\text{zontal struts}\end{array}\right\} = 18.5143h^2 + 483.318n^2\sqrt{\frac{1}{n^2+30.8642}} \text{ pounds}$$

$$= 1851.428\frac{h}{n} + 483.318n^2\sqrt{\frac{1}{n^2+30.8642}} \text{ pounds,}$$

approximately, by reason of (471), to avoid the second power of h, for convenience.

Weight to be added to floor beams due to wind = two times P',

as already given.

In (467) take $d = 8$ inches; then

$$\left.\begin{array}{l}\text{Weight to be added to all posts}\\ \quad\text{to resist distortion}\end{array}\right\} = \frac{12^2 \times 5 \times 4 \times 0.0225 \times 100}{2 \times 18 \times 25 \times 8}h^3$$

$$= \frac{9000h}{n^2} \text{ pounds,}$$

by (471).

In the previous case the quantity of iron of uniform thickness to be added to each post is that due to the greatest moment given by equation (453). It is plain from that equation that the added iron may vary in thickness from C, Fig. 114, where it should be greatest, to the bottom, where it may be nothing. Or, without increasing the thickness of the iron, the post may be made broader at top than at bottom, and thus resist the bending-moment whenever this broadening is not accompanied by too great reduction of the thickness of the iron composing the post. In the present case $x = \frac{1}{2}h$.

Finally, from (459), calling $d_1 = 4$ inches,

$$\left.\begin{matrix}\text{Weight of all}\\ \text{braces}\end{matrix}\right\} = \frac{6 \times 5 \times 4 \times 0.0225 \times 100}{18 \times 18 \times 0.70711^2}\left(1 + \frac{48^2}{3000 \times 4^2}\right)h^2$$

$$= 1.74667h^2$$

$$= 174.667\frac{h}{n} \text{ pounds,}$$

by (471).

Computing for 8 values of n, we find, —

Weights of the Components of K, in Pounds.

n.	5	6	7	8
Floor	18229.0000	18229.0000	18229.0000	18229.0000
Joists	22258.0000	17884.0000	14864.0000	12663.0000
I floor beams	5459.0000	5969.0000	6408.0000	6796.0000
Do. wind	$46.7968h$	$60.2490h$	$70.7402h$	$84.6172h$
Horizontal top struts	1617.0000	2128.0000	2650.0000	3176.0000
	$370.2857h$	$308.5714h$	$264.4898h$	$231.4286h$
Horizontal diagonals	714.0000	781.0000	854.0000	931.0000
	$120.6662h$	$125.3704h$	$125.7336h$	$133.4035h$
Braces	$34.9333h$	$29.1111h$	$24.9524h$	$21.8333h$
Residual	1000.0000	1000.0000	1000.0000	1000.0000
K	$572.6820h$	$523.3019h$	$485.9160h$	$471.2816h$
	+49277	+45991	+44005	+42795

Weights of the Components of K, in Pounds. — *Concluded.*

n.	9	10	11	12
Floor	18229.0000	18229.0000	18229.0000	18229.0000
Joists	10994.0000	9688.0000	8641.0000	7784.0000
I floor beams	7146.0000	7465.0000	7760.0000	8035.0000
Do. wind	$96.2362h$	$110.6624h$	$123.2452h$	$138.2578h$
Horizontal top struts	3701.0000	4225.0000	4746.0000	5263.0000
	$205.7143h$	$185.1429h$	$168.3117h$	$154.2857h$
Horizontal diagonals	1011.0000	1093.0000	1177.0000	1263.0000
	$138.1040h$	$147.2220h$	$154.0325h$	$163.9358h$
Braces	$19.4074h$	$17.4667h$	$15.8788h$	$14.5555h$
Residual	1000.0000	1000.0000	1000.0000	1000.0000
K	$459.4619h$	$460.4940h$	$461.4682h$	$471.0348h$
	+42081	+41700	+41553	+41574

The strain throughout each top chord due to the initial strain, $6 \times 0.7854 = 4.7124$ tons on each diagonal between top chords, is

$$4.7124 \cos \phi_1 \text{ tons,}$$

and the allowed inch strain here is

$$\frac{16.7442}{4} = 4.18605 \text{ tons.}$$

Therefore the additional cross-section of iron for both top chords due to initial strain on top diagonals is, in square inches,

$$\frac{2 \times 4.7124}{4.18605} \cos \phi_1 = 2.25148 \cos \phi_1 = \frac{225.148}{\sqrt{324n^2 + 10000}}.$$

$$\left.\begin{array}{l}\text{Additional weight}\\ \text{for top chords,}\\ \text{pounds, due}\\ \text{initial strain on}\\ \text{top diagonals}\end{array}\right\} = \frac{12 \times 100 \times 5 \times 225.148}{18\sqrt{324n^2 + 100^2}} = \frac{75049.333}{\sqrt{324n^2 + 10000}}.$$

The effect of wind on the bottom chords in this case will be twice what it was in the example of article 140, and may be taken from the table therein given.

Also, the weights of the girder diagonals will be the *same* as given in that article.

We may expect a heavier bridge this time than was found in the last example, by reason of the initial strain now assumed on the top diagonals, and the smaller values of d for the top struts, the posts, and the braces, in comparison with the values used in the two end frames to resist wind.

Computing weights for the different values of n, and collecting results, we have, —

$l = 100$ Foot-Weights in Pounds, W and L in Tons, h in Feet, $nL = 100$ Tons.

n			$\frac{W}{h}$		Wh		$\frac{1}{h}$		h^2		h
5	Top chords, Load . .	4140.742		-		82815		-		-	
	Top chords, Initial st.,	-		-		-		558		-	
	Bottom chords, Load .	2444.444		-		48889		-		-	
	Bottom chords, Wind .	-		-		-		-		122.2222	
	Verticals, total . . .	-		9.77778		-		-		568.5926	
	Girder diagonals . . .	1333.333		3.33333		35555		-		88.8889	
	K . .	-		-		-		49277		572.6820	
	$2000n\,W =$	7918.519		+13.11111		+167259		+49835		+1352.3857	
6	Top chords, Load . .	4866.256		-		81104		-		-	
	Top chords, Initial st.,	-		-		-		510		-	
	Bottom chords, Load .	2777.778		-		46296		-		-	
	Bottom chords, Wind .	-		-		-		-		115.7408	
	Verticals, total . . .	-		14.66667		-		-		544.2387	
	Girder diagonals . . .	1388.889		5.00000		30007		-		108.0247	
	K . .	-		-		-		45991		583.3019	
	$2000n\,W =$	9032.923		+19.66667		+157407		+46501		+1291.3061	
7	Top chords, Load . .	5525.323		-		78933		-		-	
	Top chords, Initial st.,	-		-		-		467		-	
	Bottom chords, Load .	3174.604		-		45351		-		-	
	Bottom chords, Wind .	-		-		-		-		113.3788	
	Verticals, total . . .	-		19.55556		-		-		489.6447	
	Girder diagonals . . .	1360.544		6.66666		25915		-		126.9841	
	K . .	-		-		-		44005		485.9160	
	$2000n\,W =$	10060.471		+26.22222		+150199		+44472		+1215.9236	

$l = 100$ Foot-Weights in Pounds, W and L in Tons, h in Feet, $nL = 100$ Tons.

n			$\frac{W}{h}$	Wh	$\frac{1}{h}$	h^2	h
8	Top chords { Load . .	6221.067	-	77763	-	-	
	Top chords { Initial st.,	-	-	-	418	-	
	Bottom chords { Load .	3559.027	-	44488	-	-	
	Bottom chords { Wind .	-	-	-	-	111.2196	
	Verticals, total . . .	-	26.07408	-	-	532.7546	
	Girder diagonals . . .	1388.889	8.88889	22787	-	145.8333	
	K . .	-	-	-	42795	471.2816	
	$2000nW =$	11168.983	+34.96297	+145038	+43213	+1261.0891	
9	Top chords { Load . .	6881.576	-	76462	-	-	
	Top chords { Initial st.,	-	-	-	394	-	
	Bottom chords { Load .	3978.051	-	44201	-	-	
	Bottom chords { Wind .	-	-	-	-	110.5014	
	Verticals, total . . .	-	32.59260	-	-	513.4888	
	Girder diagonals . . .	1371.742	11.11111	20322	-	164.6090	
	K . .	-	-	-	42081	459.4619	
	$2000nW =$	12231.369	+43.70371	+140985	+42475	+1248.0611	
10	Top chords { Load . .	7564.819	-	75648	-	-	
	Top chords { Initial st.,	-	-	-	364	-	
	Bottom chords { Load .	4388.889	-	43889	-	-	
	Bottom chords { Wind .	-	-	-	-	109.7222	
	Verticals, total . . .	-	40.74075	-	-	578.8889	
	Girder diagonals . . .	1388.889	13.88889	18333	-	183.3333	
	K . .	-	-	-	41700	460.4940	
	$2000nW =$	13342.597	+54.62964	+137870	+42064	+1332.4384	
11	Top chords { Load . .	8226.202	-	74784	-	-	
	Top chords { Initial st.,	-	-	-	338	-	
	Bottom chords { Load .	4820.936	-	43827	-	-	
	Bottom chords { Wind .	-	-	-	-	109.5668	
	Verticals, total . . .	-	48.88888	-	-	572.6966	
	Girder diagonals . . .	1377.410	16.66667	16696	-	202.0202	
	K . .	-	-	-	41553	461.4682	
	$2000nW =$	14424.548	+65.55555	+135307	+41891	+1345.7518	
12	Top chords { Load . .	8903.037	-	74192	-	-	
	Top chords { Initial st.,	-	-	-	315	-	
	Bottom chords { Load .	5246.912	-	43724	-	-	
	Bottom chords { Wind .	-	-	-	-	109.3106	
	Verticals, total . . .	-	58.66667	-	-	647.5823	
	Girder diagonals . . .	1388.889	20.00000	15325	-	220.6730	
	K . .	-	-	-	41574	471.0348	
	$2000nW =$	15538.838	+78.66667	+133241	+41889	+1448.6067	

Multiplying each of these eight equations by $\frac{h}{20000}$, we find the uniform panel weight of bridge, W, in terms of h; thus:

$$n = 5, \quad W = \frac{8.36295 + 2.49175h + 0.06761929h^2}{-0.3959259 + 0.5h - 0.00065555h^2}.$$

$$n = 6, \quad W = \frac{7.87035 + 2.32505h + 0.06456531h^2}{-0.4516462 + 0.6h - 0.0009833333h^2}.$$

$$n = 7, \quad W = \frac{7.50995 + 2.2236h + 0.06079618h^2}{-0.5030235 + 0.7h - 0.00131111111h^2}.$$

$$n = 8, \quad W = \frac{7.2519 + 2.16065h + 0.063054446h^2}{-0.5584492 + 0.8h - 0.001748148h^2}.$$

$$n = 9, \quad W = \frac{7.04925 + 2.12375h + 0.062240306h^2}{-0.61156845 + 0.9h - 0.0021851186h^2}.$$

$$n = 10, \quad W = \frac{6.8935 + 2.1032h + 0.06662192h^2}{-0.66712985 + h - 0.0027314482h^2}.$$

$$n = 11, \quad W = \frac{6.76535 + 2.09455h + 0.067287559h^2}{-0.7212274 + 1.1h - 0.003277777h^2}.$$

$$n = 12, \quad W = \frac{6.66205 + 2.09445h + 0.072430034h^2}{-0.7769419 + 1.2h - 0.00393333h^2}.$$

Differentiating these equations, and putting $\frac{dW}{dh} = 0$, we find results as here tabulated; h corresponding to the least value of nW.

Span $l = 100$ Feet, Uniform Live Load $= nL = 100$ Tons.

Number of Panels, n.	5	6	7	8
Height in feet, h	12.69087	12.40999	12.30556	11.79210
Weight of bridge, tons, nW. .	43.51195	40.91832	38.99626	38.45946
Panel length, $l \div n$	20.00000	$16\frac{2}{3}$	$14\frac{2}{7}$	$12\frac{1}{2}$
. $l \div h$	7.87970	8.05800	8.12640	8.48020
Slope of diagonals, ϕ	32° 23′ 50″	36° 40′ 17″	40° 44′ 28″	43° 19′ 51″
Ratio of dead to live load . . .	0.43510	0.40920	0.39000	0.38460
Ratio of dead to total load . .	0.30320	0.29040	0.28050	0.27780
Weight of bridge per lin. ft., lbs.,	870.00000	818.00000	770.00000	769.00000
Weight of wood, tons	20.22400	18.05700	16.54700	15.44600
Weight of iron, tons	23.28800	22.86100	22.44900	23.01300
Cost of iron, at $150	$3493 20	$3429 15	$3367 35	$3451 95
Cost of wood, at $15	303 36	270 86	248 20	231 69
Cost of bridge	3796 56	3700 01	3615 55	3683 64
Excess over least	181 01	84 46	0	68 09
Cost per linear foot	37 97	37 00	36 16	36 84

Number of Panels, n.	9	10	11	12
Height in feet, h	11.59226	11.06714	10.82859	10.37666
Weight of bridge, tons, nW. .	37.83525	38.08053	38.04843	38.60209
Panel length, $l \div n$	$11\frac{1}{9}$	10.00000	$9\frac{1}{11}$	$8\frac{1}{3}$
$l \div h$	8.62640	9.03580	9.23480	9.63700
Slope of diagonals, ϕ	46° 12′ 51″	47° 54′ 0″	49° 59′ 8″	51° 13′ 57″
Ratio of dead to live load . . .	0.37840	0.38080	0.38050	0.38600
Ratio of dead to total load . .	0.27450	0.27580	0.27560	0.27850
Weight of bridge per lin. ft., lbs.,	757.00000	762.00000	761.00000	772.00000
Weight of wood, tons	14.61200	13.95900	13.43500	13.00700
Weight of iron, tons	23.22300	24.12200	24.61300	25.59500
Cost of iron, at $150	$3483 45	$3618 30	$3691 95	$3839 25
Cost of wood, at $15	219 18	209 38	201 53	195 11
Cost of bridge	3702 63	3827 68	3893 48	4034 36
Excess over least	87 08	212 13	277 93	418 81
Cost per linear foot	37 03	38 28	38 93	40 34

Here, again, we find least weight, $nW = 37.83525$ tons, answering to the odd number of panels, 9, and the height, $h = 11.59226$ feet; while the inclination of diagonals to horizon, ϕ, is about $1\frac{1}{4}$ degrees above 45 degrees.

The least cost, at the rates here assumed, corresponds to 7 panels; it being understood that we have once or twice employed the approximation involved in (471).

142. Again, by the method of article 139, take the same example, except that the uniform live load is now 2 tons to the linear foot, instead of 1 ton, as in article 141.

1st, The floor, as before, weighs

$$F = \frac{2.5}{12} \times 17.5 \times 100 \times 50 = 18229 \text{ pounds.}$$

2d, By (432),

$$\text{Thickness of a joist, } b = \left\{\frac{9 \times 10 \times 2 \times 100}{n^2 \times 17.5 \times 7000}(18229 + 400000)\right\}^{\frac{1}{5}}$$

$$= \frac{9.07215}{n^{\frac{2}{5}}} \text{ inches.}$$

And, from (433),

$$\text{Weight of joists, } J = \frac{100 \times 17.5 \times 50}{144 \times 2}\left(\frac{9.07215^3}{n^{1.2}}\right) = \frac{226854}{n^{1.2}} \text{ pounds.}$$

3d, Depth of I floor beams, from (412), as in article 140,

$$d = 3.80122\left\{\left(\frac{226854}{n^{1.2}} + 418229\right)\frac{18.5 \times 4}{50000n}\right\}^{\frac{1}{3}}$$

$$= 0.4331885\left(\frac{J + 418229}{n}\right)^{\frac{1}{3}} \text{ inches.}$$

By (434),

Weight of I-beams,

$$P = 15.46068(n - 1) \times \frac{5}{18} \times 18.5\left(\frac{J + 418229}{n} \times \frac{18.5 \times 4}{50000}\right)^{\frac{2}{3}}$$

$$= 1.031824(n - 1)\left(\frac{J + 418229}{n}\right)^{\frac{2}{3}} \text{ pounds.}$$

4th, Take top horizontal diagonals, each $1\frac{1}{8}$ inches in diameter. Cross-section = 0.99402 square inches; weight = 3.359 pounds per foot. Then

Weight of $2n$ top horizontal diagonals

$$= 2n \times 3.359\sqrt{18^2 + \frac{100^2}{n^2}} = 6.718\sqrt{324n^2 + 10000} \text{ pounds.}$$

$$\left.\begin{array}{l}\text{Weight of bottom}\\ \quad\text{horizontal diag-}\\ \quad\text{onals}\end{array}\right\} = X = \frac{2 \times 5 \times 4 \times 18 \times 6}{18 \times 24 \sin^2 \phi_1} W_1 \begin{cases} n^2 \ (n \text{ even}),\\ (n^2 - 1)\ (n \text{ odd}),\end{cases}$$

as in article 140, for both top and bottom.

$$W_1 = \frac{hwl}{2n}, \quad \frac{1}{\sin^2 \phi_1} = 1 + \frac{10000}{18^2 n^2}.$$

5th, Strain on each top horizontal strut from $\frac{24}{4} = 6$ tons per square inch on two top diagonals $= 2 \times 6 \times 0.99402 \sin \phi_1 = 11.92824 \sin \phi_1$ tons; allowed inch strain on strut $= \dfrac{4.680056}{4} = 1.170014$ tons. Therefore

$$\left.\begin{array}{l}\text{Cross-section of a}\\ \quad\text{top strut to}\\ \quad\text{resist initial}\\ \quad\text{strain on diag-}\\ \quad\text{onals}\end{array}\right\} = \frac{11.92824}{1.170014} \sin \phi_1 = 10.19495 \sqrt{\frac{1}{1 + \frac{10000}{324n^2}}} = S_1.$$

From (461),

$$\left.\begin{array}{l}\text{Cross-section of a top strut}\\ \quad\text{to resist distorting force}\\ \quad\text{of wind}\end{array}\right\} = 2ac = \frac{6wflh^2}{Bd_2 n} = 0.3089\frac{h^2}{n} \text{ square inch.}$$

From (465),

$$\left.\begin{array}{l}\text{Weight of } n \text{ top hori-}\\ \text{zontal struts}\end{array}\right\} = 18.5143h^2 + 611.697n^2\sqrt{\frac{1}{n^2 + 30.8642}} \text{ pounds}$$

$$= 1851.428\frac{h}{n} + 611.697n^2\sqrt{\frac{1}{n^2 + 30.8642}} \text{ pounds,}$$

by reason of (471).

6th, Weight to be added to floor-beams, due to wind, $= 2 \times P'$ in article 140, changing d.

7th, Weight to be added to all posts to resist distortion by wind $= \frac{9000}{n^2}h$ pounds, as before.

8th, Weight of all braces $= 174.667\frac{h}{n}$ pounds, as before.

Computing for 8 values of n, we find, —

Weights of Components of K, in Pounds. $l = 100$ Feet, $nL = 200$ Tons.

n.	5	6	7	8
Floor	18229.0000	18229.0000	18229.0000	18229.0000
Joists	32884.0000	26422.0000	21960.0000	18708.0000
I floor beams	8303.0000	9102.0000	9790.0000	10398.0000
Do. wind	43.4410h	55.4297h	64.5618h	76.6677h
Horizontal top struts	2046.0000	2693.0000	3354.0000	4019.0000
	370.2857h	308.5714h	264.4898h	231.4286h
Horizontal diagonals	904.0000	989.0000	1081.0000	1178.0000
	120.6662h	125.3704h	125.7336h	133.4035h
Braces	34.9333h	29.1111h	24.9524h	21.8333h
Residual	1200.0000	1200.0000	1200.0000	1200.0000
K	569.3262h	518.4826h	479.7376h	463.3331h
	+63566	+58635	+55614	+53732

Weights of Components of K, in Pounds. $l = 100$ Feet, $nL = 200$ Tons.

n.	9	10	11	12
Floor	18229.0000	18229.0000	18229.0000	18229.0000
Joists	16243.0000	14313.0000	12767.0000	11501.0000
I floor beams . . .	10944.0000	11443.0000	11903.0000	12331.0000
Do. wind	$86.6164h$	$98.9896h$	$109.6167h$	$122.3137h$
Horizontal top struts	4684.0000	5347.0000	6006.0000	6661.0000
	$205.7143h$	$185.1429h$	$168.3117h$	$154.2857h$
Horizontal diagonals	1279.0000	1383.0000	1490.0000	1599.0000
	$138.1040h$	$147.2220h$	$154.0325h$	$163.9358h$
Braces	$19.4074h$	$17.4667h$	$15.8788h$	$14.5555h$
Residual	1200.0000	1200.0000	1200.0000	1200.0000
K	$449.8421h$	$448.8212h$	$447.8397h$	$455.0907h$
	$+52579$	$+51915$	$+51595$	$+51521$

9th, Taking $Q = 16.7442$ tons, as in article 140, $L = \frac{2l}{n} = \frac{200}{n}$ tons, we now have

$$\left.\begin{array}{l}\text{Weight of top chords due to verti-}\\ \quad\text{cal pressures, in pounds}\end{array}\right\} = \frac{5 \times 4 \times 100^2}{2 \times 18 \times 16.7442h}\left(W + \frac{200}{n}\right)$$

$$\left\{\begin{array}{l} \times \dfrac{2n^2 + 3n - 2}{n} \ (n \text{ even}), \\[2ex] \times \dfrac{2n^3 + 3n^2 - 2n - 3}{n^2} \ (n \text{ odd}). \end{array}\right.$$

Strain throughout each top chord due to initial strain of $\frac{24}{4} \times 0.99402 = 5.96412$ tons, along each diagonal between top chords, is

$$5.96412 \cos \phi_1 \text{ tons.}$$

$$\text{Allowed inch pressure on top chords} = \frac{16.7442}{4} = 4.18605 \text{ tons.}$$

Additional cross-section of iron for both top chords due to initial strain on top diagonals $\Bigg\} = \frac{2\ \ldots}{\ldots}$

$= \frac{\ldots}{\sqrt{3\ldots}}$

Additional weight for top chords due initial strain on top diagonals, pounds $\Bigg\} = \frac{12 \times 100 \times 5 \ldots}{18\sqrt{324n^2 + \ldots}}$

10th, From (425),

Weight of bottom chords due vertical forces, pounds $\Big\} = \frac{5 \times \ldots}{2 \times \ldots}$

$\left\{ \begin{array}{l} \times \frac{2n^3 - \ldots}{\ldots} \\ \times \frac{2n^3 - \ldots}{\ldots} \end{array} \right.$

From (436), s being zero, multiplying

Weight of bottom chords due wind, in pounds $\Big\} = \frac{5 \times \ldots}{\ldots}$

$\left\{ \begin{array}{l} \times \frac{2n^3 - \ldots}{\ldots} \\ \times \frac{2n^3 - \ldots}{\ldots} \end{array} \right.$

11th, From (426), Q_1 being 8.1818

Weight of verticals, in pounds,

$$= \frac{3 \times 5 \times 4}{18 \times 8.1818} Whn^2 + \frac{5 \times 4 \times 200\ldots}{2 \times 18 \times 8.18\ldots}$$
(n even)

$$= \frac{3 \times 5 \times 4}{18 \times 8.1818} Wh(n^2 - 1) + \frac{5 \times 4 \ldots}{2 \times 18 \ldots}$$
(n odd).

$$\text{Weight of verticals due wind} = 9000\frac{h}{n^2}$$

if $d = 8$ inches.

12th, From (428), $\frac{1}{\sin^2\phi}$ being equal to $1 + \frac{l^2}{n^2h^2}$,

$$\left.\begin{array}{c}\text{Weight of girder}\\ \text{diagonals}\end{array}\right\} = \frac{4\times5\times200\times4h}{18\times24\sin^2\phi}\left(\frac{n^2-1}{n}\right)+\frac{3\times5\times4h}{18\times24\sin^2\phi}Wn^2$$
$$(n \text{ even}),$$
$$= \frac{4\times5\times200\times4h}{18\times24\sin^2\phi}\left(\frac{n^2-1}{n}\right)+\frac{3\times5\times4h}{18\times24\sin^2\phi}W(n^2-1)$$
$$(n \text{ odd}).$$

We therefore have, —

Weights in Pounds, W in Tons, h in Feet, $nL = 200$ Tons.

n			$\frac{W}{h}$		Wh		$\frac{1}{h}$		h^2		h
5	Top chords { Load . .	4140.742		-		165630		-		-	
	Top chords { Initial st.,	-		-		-		706		-	
	Bottom chords { Load .	2444.444		-		97777		-		-	
	Bottom chords { Wind .	-		-		-		-		122.2222	
	Verticals	-		9.77778		-		-		777.1852	
	Girder diagonals . . .	1333.333		3.33333		71111		-		177.7778	
	K . .	-		-		-		63566		569.3262	
	$2000nW =$	7918.519		+13.11111		+334518		+64272		+1646.5114	
6	Top chords { Load . .	4866.256		-		162208		-		-	
	Top chords { Initial st.,	-		-		-		645		-	
	Bottom chords { Load .	2777.777		-		92592		-		-	
	Bottom chords { Wind .	-		-		-		-		115.7408	
	Verticals	-		14.66667		-		-		838.4774	
	Girder diagonals . . .	1388.889		5.00000		60014		-		216.0494	
	K . .	-		-		-		58635		518.4826	
	$2000nW =$	9032.922		+19.66667		+314814		+59280		+1688.7502	
7	Top chords { Load . .	5525.323		-		157866		-		-	
	Top chords { Initial st.,	-		-		-		590		-	
	Bottom chords { Load .	3174.604		-		90702		-		-	
	Bottom chords { Wind .	-		-		-		-		113.3788	
	Verticals	-		19.55556		-		-		795.6160	
	Girder diagonals . . .	1360.544		6.66666		51830		-		253.9682	
	K . .	-		-		-		55614		479.7376	
	$2000nW =$	10060.471		+26.22222		+300398		+56204		+1642.7006	

Weights in Pounds, W in Tons, h in Feet, $nL = 200$ Tons.

n	Member		$\frac{W}{h}$		Wh		$\frac{1}{h}$		h^2		h
8	Top chords { Load . .	6221.067		-		155526		-		-	
	Top chords { Initial st.,	-		-		-		542		-	
	Bottom chords { Load .	3559.027		-		88976		-		-	
	Bottom chords { Wind .	-		-		-		-		111.2196	
	Verticals	-		26.07408		-		-		924.8842	
	Girder diagonals . . .	1388.889		8.88889		45574		-		291.6666	
	K . .	-		-		-		53732		463.3331	
	$2000n\,W =$	11168.983		+34.96297		+290076		+54274		+1791.1035	
9	Top chords { Load . .	6881.576		-		152924		-		-	
	Top chords { Initial st.,	-		-		-		499		-	
	Bottom chords { Load .	3978.051		-		88402		-		-	
	Bottom chords { Wind	-		-		-		-		110.5014	
	Verticals	-		32.59260		-		-		915.8665	
	Girder diagonals . . .	1371.742		11.11111		40644		-		329.2180	
	K . .	-		-		-		52579		449.8421	
	$2000n\,W =$	12231.369		+43.70371		+281970		+53078		+1805.4280	
10	Top chords { Load .	7564.819		-		151296		-		-	
	Top chords { Initial st.,	-		-		-		461		-	
	Bottom chords { Load .	4388.889		-		87778		-		-	
	Bottom chords { Wind .	-		-		-		-		109.7222	
	Verticals	-		40.74075		-		-		1067.7777	
	Girder diagonals . . .	1388.889		13.88889		36667		-		366.6667	
	K . .	-		-		-		51915		448.8212	
	$2000n\,W =$	13342.597		+54.62964		+275741		+52376		+1992.9878	
11	Top chords { Load . .	8226.202		-		149568		-		-	
	Top chords { Initial st.,	-		-		-		428		-	
	Bottom chords { Load .	4820.936		-		87654		-		-	
	Bottom chords { Wind .	-		-		-		-		109.5668	
	Verticals	-		48.88888		-		-		1071.0130	
	Girder diagonals . . .	1377.410		16.66667		33392		-		404.0404	
	K . .	-		-		-		51595		447.8397	
	$2000n\,W =$	14424.548		+65.55555		+270614		+52023		+2032.4599	
12	Top chords { Load . .	8903.037		-		148384		-		-	
	Top chords { Initial st.,	-		-		-		399		-	
	Bottom chords { Load .	5246.912		-		87448		-		-	
	Bottom chords { Wind .	-		-		-		-		109.3106	
	Verticals	-		58.66667		-		-		1232.6646	
	Girder diagonals . . .	1388.889		20.00000		30650		-		441.3580	
	K . .	-		-		-		51521		455.0907	
	$2000n\,W =$	15538.838		+78.66667		+266482		+51920		+2238.4239	

Multiplying each of these 8 equations by ($h \div 20000$), we find the uniform panel weight, W, of bridge, in terms of h, thus:

$$n = 5, \quad W = \frac{16.7259 + 3.2136h + 0.08232557h^2}{-0.395926 + 0.5h - 0.00065555h^2}.$$

$$n = 6, \quad W = \frac{15.7407 + 2.964h + 0.08443751h^2}{-0.4516461 + 0.6h - 0.00098333h^2}.$$

$$n = 7, \quad W = \frac{15.0199 + 2.8102h + 0.08213503h^2}{-0.5030235 + 0.7h - 0.001311111h^2}.$$

$$n = 8, \quad W = \frac{14.5038 + 2.7137h + 0.08955518h^2}{-0.5584491 + 0.8h - 0.001748148h^2}.$$

$$n = 9, \quad W = \frac{14.0985 + 2.6539h + 0.0902714h^2}{-0.6115685 + 0.9h - 0.002185185h^2}.$$

$$n = 10, \quad W = \frac{13.78705 + 2.6188h + 0.09964939h^2}{-0.66712985 + h - 0.002731482h^2}.$$

$$n = 11, \quad W = \frac{13.5307 + 2.60115h + 0.10162299h^2}{-0.7212274 + 1.1h - 0.00327777h^2}.$$

$$n = 12, \quad W = \frac{13.3241 + 2.596h + 0.11192120h^2}{-0.7769419 + 1.2h - 0.00393333h^2}.$$

Differentiating, and putting $\frac{dW}{dh} = 0$, according to equation (470), we find,—

HEIGHT, h, ANSWERING TO MINIMUM VALUE OF nW.

Span l = 100 Feet, Uniform Live Load nL = 200 Tons.

Number of Panels, n.	5	6	7	8
Height in feet, h	15.43076	14.61593	14.32074	13.43154
Weight of bridge, nW. . . .	59.97030	57.05609	54.55331	54.38682
Panel length, feet, $l \div n$ . . .	20.00000	$16\frac{2}{3}$	$14\frac{2}{7}$	$12\frac{1}{2}$
Ratio of length to height . . .	6.48060	6.84180	6.98290	7.44520
Slope of diagonals, ϕ	37° 39′ 6″	41° 14′ 58″	45° 4′ 13″	47° 3′ 27″
Ratio of dead to live load . . .	0.29985	0.28528	0.27277	0.27193
Ratio of dead to total load . .	0.23068	0.22196	0.21431	0.21379
Weight of bridge per lin. ft., lbs.,	1199.00000	1141.00000	1091.00000	1088.00000
Weight of wood, tons	25.55650	22.32550	20.99450	18.46850
Weight of iron, tons.	34.41380	34.73060	33.55880	35.91830
Cost of iron, at $150	$5162 07	$5209 59	$5033 82	$5387 75
Cost of wood, at $15	383 35	334 88	301 42	277 03
Cost of bridge	5545 42	5544 47	5335 24	5664 78
Excess over least.	210 18	209 23	0	329 54
Cost per linear foot	55 46	55 45	53 35	56 65

Number of Panels, n.	9	10	11	12
Height in feet, h	13.10601	12.33301	12.04572	11.40376
Weight of bridge, nW. . . .	53.61297	54.43510	54.39900	55.64659
Panel length, feet, $l \div n$ . . .	$11\frac{1}{9}$	10.00000	$9\frac{1}{11}$	$8\frac{1}{3}$
Ratio of length to height . . .	7.63010	8.10830	8.30170	8.76900
Slope of diagonals, ϕ	49° 42′ 33″	50° 57′ 49″	52° 57′ 30″	53° 50′ 33″
Ratio of dead to live load . . .	0.26806	0.27218	0.27199	0.27823
Ratio of dead to total load . .	0.21140	0.21394	0.21384	0.21767
Weight of bridge per lin. ft., lbs.,	1072.00000	1089.00000	1088.00000	1113.00000
Weight of wood, tons	17.23600	16.27100	15.49800	14.86500
Weight of iron, tons.	36.37700	38.16410	38.90100	40.78160
Cost of iron, at $150	$5456 55	$5724 62	$5835 15	$6117 24
Cost of wood, at $15	258 54	244 07	232 47	222 98
Cost of bridge	5715 09	5968 69	6067 62	6340 22
Excess over least.	379 85	633 45	732 38	1004 98
Cost per linear foot	57 15	59 69	60 68	63 40

Here, for weight, the *minimum minimorum* is 53.61297 tons, $n = 9$, $\phi = 49° 42' 33''$; while for cost, at the assumed prices, the least is $5,335.24, answering to $n = 7$, and $\phi = 45° 4' 13''$.

Comparing these results with the corresponding ones in article 141, we conclude: —

1st, For a given span and number of panels, if we increase the live load, we should increase the height.

2d, As the live load increases, the ratio of dead to both live and total loads diminishes.

143. As another example, let the span $l = 200$ feet; uniform live load $nL = 200$ tons, or 1 ton per linear foot; other data as in articles 141 and 142. Compute for $n = 8, 9, 10, 11, 12, 13, 14, 15$.

1st, The floor weighs

$$F = \frac{2.5}{12} \times 17.5 \times 200 \times 50 = 36458 \text{ pounds.}$$

2d, By (432),

$$\text{Thickness of a joist, } b = \left\{\frac{9 \times 10 \times 2 \times 200}{n^2 \times 17.5 \times 7000}(36458 + 400000)\right\}^{\frac{1}{3}}$$

$$= \frac{10.51042}{n^{0.4}} \text{ inches.}$$

And, from (433),

$$\text{Weight of joists, } J = \frac{200 \times 17.5 \times 50}{144 \times 2}\left(\frac{10.51042^3}{n^{1.2}}\right) = \frac{705523}{n^{1.2}} \text{ pounds.}$$

3d, Depth of I floor beams, from (412),

$$d = 3.80122\left\{\left(\frac{705523}{n^{1.2}} + 436458\right)\frac{18.5 \times 4}{50000n}\right\}^{\frac{1}{3}}$$

$$= 0.4331885\left(\frac{J + 436458}{n}\right)^{\frac{1}{3}} \text{ inches.}$$

By (434),

Weight of I-beams,

$$P = 15.46068(n - 1) \times \frac{5}{18} \times 18.5\left(\frac{J + 436458}{n} \times \frac{18.}{5}\right.$$

$$= 1.031824(n - 1)\left(\frac{J + 436458}{n}\right)^{\frac{3}{4}} \text{ pounds.}$$

4th, Top horizontal diagonals, as in article 142, weig

$$6.718\sqrt{324n^2 + 40000} \text{ pounds,}$$

l now being 200.

$$\left.\begin{array}{l}\text{Weight of bottom}\\ \text{horizontal diag-}\\ \text{onals}\end{array}\right\} = X = \frac{2 \times 5 \times 4 \times 18 \times 6}{18 \times 24 \sin^2 \phi_1} W_1 \begin{cases} n^2 \ (n \text{ eve} \\ (n^2 - 1) \end{cases}$$

$$W_1 = \frac{hwl}{2n}, \quad \frac{1}{\sin^2 \phi_1} = 1 + \frac{40000}{18^2 n^2}.$$

5th, Top horizontal struts, as before, with change of 100 to 200.

Cross-section of one, due initial strain, $= S_1 = 10.19495 \sqrt{\frac{}{1 +}}$

From (461),

Cross-section of one, due wind, $= 2ac = 0.613\frac{h^2}{n}$ square in

From (465),

Weight of n top horizontal struts

$$= 37.028557h^2 + 611.697n^2\sqrt{\frac{1}{n^2 + 123.4568}} \text{ pounds}$$

$$= 7405.714\frac{h}{n} + 611.697n^2\sqrt{\frac{1}{n^2 + 123.4568}} \text{ pounds,}$$

by reason of (471).

6th, Weight to be added to floor beams, due to wind, is equal to

$$4P' = 15.4166666h\left(1 + \frac{65.712}{d^2}\right)\left\{\begin{array}{l} n \ (n \text{ even}), \\ \dfrac{n^2 - 1}{n} \ (n \text{ odd}). \end{array}\right.$$

7th, Weight to be added to posts to resist wind is equal to

$$\frac{12^2 \times 5 \times 4 \times 0.0225 \times 200h^3}{18 \times 25 \times 8 \times 2} = 1.8h^3 = \frac{72000}{n^2}h \text{ pounds.}$$

by reason of (471).

8th, Braces. From (459), if $b = 5$ feet, $d_2 = 4$ inches.

Weight of braces

$$= \frac{4 \times 6 \times 5 \times 0.0225 \times 200h^2}{18 \times 18 \times 0.70711^2}\left(1 + \frac{60^2}{3000 \times 4^2}\right) = 3.58333h^2$$

$$= \frac{716.666}{n}h,$$

by (471).

Computing for 8 values of n, we find, —

Weights of Components of K, in Pounds. $l = 200$ Feet, $nL = 200$ Tons.

n	8	9	10	11
Floor	36458.0000	36458.0000	36458.0000	36458.0000
Joists	58184.0000	50515.0000	44516.0000	39705.0000
I floor beams . . .	11294.0000	11809.0000	12282.0000	12721.0000
Do. from wind . . .	$150.9538h$	$170.5809h$	$194.9868h$	215.9620
Horizontal top struts	2859.0000	3465.0000	4092.0000	4734.0000
	$925.7142h$	$822.8571h$	$740.5714h$	673.2467
Horizontal diagonals	1656.0000	1729.0000	1808.0000	1891.0000
	$527.2219h$	$504.8319h$	$502.7783h$	495.8917
Braces	$89.5833h$	$79.6296h$	$71.6666h$	65.1515
Residual	2000.0000	2000.0000	2000.0000	2000.0000
K	$1693.4732h$	$1577.8995h$	$1510.0031h$	1450.2519
	+112451	+105976	+101156	+97509

n	12	13	14	15
Floor	36458.0000	36458.0000	36458.0000	36458.0000
Joists	35768.0000	32492.0000	29727.0000	27365.0000
I floor beams . . .	13131.0000	13518.0000	13884.0000	14231.0000
Do. from wind . . .	$240.9957h$	$263.1340h$	$288.8560h$	312.0545
Horizontal top struts	5386.0000	6031.0000	6708.0000	7373.0000
	$617.1428h$	$569.6703h$	$528.9796h$	493.7143
Horizontal diagonals	1978.0000	2067.0000	2161.0000	2257.0000
	$501.4819h$	$503.6705h$	$512.2314h$	520.3630
Braces	$59.7222h$	$55.1282h$	$51.1905h$	47.7777
Residual	2000.0000	2000.0000	2000.0000	2000.0000
K	$1419.3426h$	$1391.6030h$	$1381.2575h$	1373.9095
	+94721	+92566	+90938	+89684

9th, Taking $Q = 16.7442$ tons, as before, and $L = \frac{l}{n} = 2$
tons, we find

$$\left.\begin{array}{l}\text{Weight of top chords due vertical}\\ \quad\text{pressures, in pounds}\end{array}\right\} = \frac{5 \times 4 \times 200^2}{2 \times 18 \times 16.7442h}\left(W + \frac{200}{n}\right)$$

$$\left\{\begin{array}{l}\times \dfrac{2n^2 + 3n - 2}{n} \ (n \text{ even}),\\[2ex] \times \dfrac{2n^3 + 3n^2 - 2n - 3}{n^2} \ (n \text{ odd}).\end{array}\right.$$

Strain throughout each top chord due to initial strain of $\frac{24}{4} \times 0.99402 = 5.96412$ tons, along each diagonal between top chords, is

$$5.96412 \cos\phi_1 \text{ tons.}$$

$$\text{Allowed pressure on top chords} = \frac{16.7442}{4}$$

$$= 4.18605 \text{ tons per square inch.}$$

$$\left.\begin{array}{l}\text{Additional cross-section of iron for}\\ \quad\text{both top chords due to initial}\\ \quad\text{strain on top diagonals}\end{array}\right\} = \frac{2 \times 5.96412}{4.18605}\cos\phi_1$$

$$= \frac{569.904}{\sqrt{324n^2 + 40000}} \text{ square inches.}$$

$$\left.\begin{array}{l}\text{Additional weight for}\\ \quad\text{top chords due in-}\\ \quad\text{itial strain on top}\\ \quad\text{diagonals, pounds}\end{array}\right\} = \frac{12 \times 200 \times 5 \times 569.904}{18\sqrt{324n^2 + 200^2}} = \frac{379936}{\sqrt{324n^2 + 40000}}.$$

10th, From (425),

$$\left.\begin{array}{l}\text{Weight of bottom chords due}\\ \quad\text{vertical forces, pounds}\end{array}\right\} = \frac{5 \times 4 \times 200^2}{2 \times 18 \times 24h}\left(W + \frac{200}{n}\right)$$

$$\left\{\begin{array}{l}\times \dfrac{2n^3 - 3n^2 + 22n - 24}{n^2} \ (n \text{ even}),\\[2ex] \times \dfrac{2n^3 - 3n^2 + 22n - 21}{n^2} \ (n \text{ odd}).\end{array}\right.$$

From (436), e being zero, multiplying by 2,

$$\left.\begin{array}{l}\text{Weight of bottom chords due}\\ \quad\text{wind, in pounds}\end{array}\right\} = \frac{5 \times 4 \times 200^3 \times 0.0225h}{18 \times 24 \times 18}$$

$$\left\{\begin{array}{l} \times \dfrac{2n^3 - 3n^2 + 22n - 24}{n^3} \text{ (n even)}, \\[2ex] \times \dfrac{2n^3 - 3n^2 + 22n - 21}{n^3} \text{ (n odd)}. \end{array}\right.$$

11th, From (426), Q_1 being 8.181818 tons,

Weight of verticals due load, pounds,

$$= \frac{3 \times 5 \times 4Whn^2}{18 \times 8.181818} + \frac{5 \times 4 \times 200h}{2 \times 18 \times 8.181818}\left(\frac{7n^2 + 3n - 10}{n}\right)$$
(n even),

$$= \frac{3 \times 5 \times 4Wh(n^2 - 1)}{18 \times 8.181818} + \frac{5 \times 4 \times 200h}{2 \times 18 \times 8.181818}\left(\frac{7n^3 - 3n^2 - 7n + 3}{n^2}\right)$$
(n odd).

Weight of verticals due wind, pounds, by (467),

$$= \frac{144 \times 5 \times 4 \times 0.0225 \times 200}{18 \times 2 \times 25 \times 8}h^3 = \frac{72000h}{n^2}$$

approximately, (471).

12th, From (428), where $\dfrac{1}{\sin^2 \phi} = 1 + \dfrac{l^2}{n^2h^2}$,

Weight of girder diagonals

$$= \frac{4 \times 5 \times 200 \times 4h}{18 \times 24 \sin^2 \phi}\left(\frac{n^2 - 1}{n}\right) + \frac{3 \times 5 \times 4h}{18 \times 24 \sin^2 \phi}Wn^2$$
(n even),

$$= \frac{4 \times 5 \times 200 \times 4h}{18 \times 24 \sin^2 \phi}\left(\frac{n^2 - 1}{n}\right) + \frac{3 \times 5 \times 4h}{18 \times 24 \sin^2 \phi}W(n^2 - 1)$$
(n odd).

Weights in Pounds, W in Tons, h in Feet, $nL = 200$ Tons.

n			$\frac{W}{h}$	Wh	$\frac{1}{h}$	h^2	h
8	Top chords { Load .	24884	-	622100	-	-	
	Top chords { Initial st.,	-	-	-	1542	-	
	Bottom chords { Load .	14236	-	355900	-	-	
	Bottom chords { Wind .	-	-	-	-	889.757	
	Verticals	-	26.0741	-	-	1909.258	
	Girder diagonals . . .	5556	8.8889	182292	-	291.667	
	K . .	-	-	-	112451	1693.473	
	$2000nW =$	44676	+34.9630	+1160292	+113993	+4784.155	
9	Top chords { Load . .	27526	-	611689	-	-	
	Top chords { Initial st.,	-	-	-	1476	-	
	Bottom chords { Load .	15912	-	353600	-	-	
	Bottom chords { Wind .	-	-	-	-	884.012	
	Verticals	-	32.5926	-	-	1693.644	
	Girder diagonals . . .	5487	11.1111	162577	-	329.218	
	K . .	-	-	-	105976	1577.900	
	$2000nW =$	48925	+43.7037	+1127866	+107452	+4484.774	
10	Top chords { Load . .	30259	-	605180	-	-	
	Top chords { Initial st.,	-	-	-	1412	-	
	Bottom chords { Load .	17556	-	351120	-	-	
	Bottom chords { Wind .	-	-	-	-	877.778	
	Verticals	-	40.7408	-	-	1697.778	
	Girder diagonals . . .	5556	13.8889	146667	-	366.667	
	K . .	-	-	-	101156	1510.003	
	$2000nW =$	53371	+54.6297	+1102967	+102568	+4452.226	
11	Top chords { Load . .	32905	-	598272	-	-	
	Top chords { Initial st.,	-	-	-	1350	-	
	Bottom chords { Load .	19284	-	350618	-	-	
	Bottom chords { Wind .	-	-	-	-	876.534	
	Verticals	-	48.8889	-	-	1591.675	
	Girder diagonals . . .	5510	16.6666	133567	-	404.040	
	K . .	-	-	-	97509	1450.252	
	$2000nW =$	57699	+65.5555	+1082457	+98859	+4322.501	
12	Top chords { Load . .	35612	-	593533	-	-	
	Top chords { Initial st.,	-	-	-	1291	-	
	Bottom chords { Load .	20988	-	349800	-	-	
	Bottom chords { Wind .	-	-	-	-	874.486	
	Verticals	-	58.6667	-	-	1670.165	
	Girder diagonals . . .	5556	20.0000	122599	-	441.357	
	K . .	-	-	-	94721	1419.343	
	$2000nW =$	62156	+78.6667	+1065932	+96012	+4405.351	

Weights in Pounds, W in tons, h in Feet, $nL = 200$ Tons.

n											
13	Top chords { Load . .	38260	$\frac{W}{h}$	-	Wh	588615	$\frac{1}{h}$	-	h^2	-	h
	Top chords { Initial st.,	-		-		-		1235		-	
	Bottom chords { Load .	22801		-		350785		-		-	
	Bottom chords { Wind .	-		-		-		-		874.930	
	Verticals	-		68.4444		-		-		1614.025	
	Girder diagonals . . .	5523		23.3333		113286		-		478.633	
	K . .	-		-		-		92566		1391.603	
	$2000n\,W =$	66584		+91.7777		+1052686		+93801		+4359.191	
14	Top chords { Load . .	40952		-		585028		-		-	
	Top chords { Initial st.,	-		-		-		1181		-	
	Bottom chords { Load .	24490		-		349857		-		-	
	Bottom chords { Wind .	-		-		-		-		874.636	
	Verticals	-		79.8519		-		-		1729.182	
	Girder diagonals . . .	5556		27.2222		105280		-		515.874	
	K . .	-		-		-		90938		1381.257	
	$2000n\,W =$	70998		+107.0741		+1040165		+92119		+4500.949	
15	Top chords { Load . .	43602		-		581360		-		-	
	Top chords { Initial st.,	-		-		-		1131		-	
	Bottom chords { Load .	26272		-		350293		-		-	
	Bottom chords { Wind .	-		-		-		-		875.720	
	Verticals	-		91.2593		-		-		1699.028	
	Girder diagonals . . .	5531		31.1111		98326		-		553.086	
	K . .	-		-		-		89684		1373.909	
	$2000n\,W =$	75405		+122.3704		+1029979		+90815		+4501.743	

Multiplying each of these 8 equations by ($h \div 20000$), we find the uniform panel weight, W, of bridge, in terms of h, thus:

$$n = 8, \quad W = \frac{58.0146 + 5.69965h + 0.2392078h^2}{-2.2338 + 0.8h - 0.00174815h^2}.$$

$$n = 9, \quad W = \frac{56.3933 + 5.3726h + 0.2242387h^2}{-2.44625 + 0.9h - 0.002185185h^2}.$$

$$n = 10, \quad W = \frac{55.14835 + 5.1284h + 0.2226113h^2}{-2.66855 + h - 0.002731485h^2}$$

$$n = 11,\quad W = \frac{54.12285 + 4.94295h + 0.216125h^2}{-2.88495 + 1.1h - 0.003277777h^2}.$$

$$n = 12,\quad W = \frac{53.2966 + 4.8006h + 0.2202675h^2}{-3.1078 + 1.2h - 0.003933333\tfrac{1}{3}h^2}.$$

$$n = 13,\quad W = \frac{52.6343 + 4.69005h + 0.2179595h^2}{-3.3292 + 1.3h - 0.004588888h^2}.$$

$$n = 14,\quad W = \frac{52.00825 + 4.60595h + 0.2250475h^2}{-3.5499 + 1.4h - 0.005353705h^2}.$$

$$n = 15,\quad W = \frac{51.49895 + 4.54075h + 0.2250872h^2}{-3.77025 + 1.5h - 0.006111852h^2}.$$

Differentiating these equations according to the form (470), and solving for h and W, we find as follows:—

HEIGHT, h, ANSWERING TO MINIMUM VALUE OF nW.

Span $l = 200$ Feet, Uniform Live Load $nL = 200$ Tons.

Number of Panels, n.	8	9	10	11
Height in feet, h	19.42424	19.40333	19.03250	18.89406
Weight of bridge, nW	163.83300	155.38700	151.80500	147.73400
Panel length, feet, $l \div n$	25.00000	$22\frac{2}{9}$	20.00000	$18\frac{2}{11}$
Ratio of length to height	10.29700	10.30800	10.50800	10.58500
Slope of diagonals, ϕ	37° 50′ 46″	41° 7′ 33″	43° 34′ 48″	46° 5′ 54″
Ratio of dead to live load	0.81900	0.77700	0.75900	0.73800
Ratio of dead to total load	0.45030	0.43720	0.43150	0.42490
Weight of bridge, lbs. to lin. ft.,	1638.00000	1554.00000	1518.00000	1477.00000
Weight of wood, tons	47.32100	43.48700	40.48700	38.08100
Weight of iron, tons	116.51200	111.90000	111.31800	109.65300
Cost of iron, at \$150	\$17476 80	\$16785 00	\$16697 70	\$16447 95
Cost of wood, at \$15	709 82	652 31	607 31	571 22
Cost of bridge	18186 62	17437 31	17305 01	17019 17
Excess over least	1167 45	418 14	285 84	0
Cost per linear foot	90 93	87 19	86 53	85 10

HEIGHT, h, ANSWERING TO MINIMUM VALUE OF nW.— *Concluded.*

Span $l = 200$ Feet, Uniform Live Load $nL = 200$ Tons.

Number of Panels, n.	12	13	14	15
Height in feet, h	18.44963	18.26650	17.80470	17.59645
Weight of bridge, nW	147.06400	145.26000	146.09100	145.51100
Panel length, feet, $l \div n$	$16\frac{2}{3}$	$15\frac{5}{13}$	$14\frac{2}{7}$	$13\frac{1}{3}$
Ratio of length to height	10.84100	10.94900	11.23300	11.36600
Slope of diagonals, ϕ	47° 54′ 24″	49° 53′ 42″	51° 15′ 29″	52° 50′ 52″
Ratio of dead to live load	0.73500	0.72600	0.73000	0.72800
Ratio of dead to total load	0.42370	0.42070	0.42210	0.42110
Weight of bridge, lbs. to lin. ft.,	1471.00000	1453.00000	1461.00000	1455.00000
Weight of wood, tons	36.11300	34.47500	33.09200	31.91200
Weight of iron, tons	110.95100	110.78500	112.99900	113.59900
Cost of iron, at $150	$16642 65	$16617 75	$16949 85	$17039 85
Cost of wood, at $15	541 71	517 13	496 38	478 68
Cost of bridge	17184 36	17134 88	17446 23	17518 53
Excess over least	165 19	115 71	427 06	499 36
Cost per linear foot	85 92	85 68	87 23	87 59

Of the bridge weights in this case, the ***minimum minimorum*** is 145.260 tons, $n = 13$, $\phi = 49° 53' 42''$; while of the costs at the assumed prices, the least is $17,019.17, corresponding to $n = 11$, $\phi = 46° 5' 54''$.

144. From articles 141 and 143, exemplifying 2 bridges of different spans but under the same live load per linear foot, we may deduce, —

1st, That, as the length increases, the bridge weight per linear foot increases; or, the ratio of dead to live load increases nearly as the length.

2d, That the dead load increases nearly as the square of the length.

3d, That an odd number of panels is more favorable to weight than an even number.

4th, That the height of each panel should be a little greater than its length.

5th, That the ratio of length to height of girder depends upon the span, as well as upon the live load, seen by comparing articles 141, 142, 143.

These principles are to be seen in this table.

COMPARATIVE VIEW OF RESULTS.

Span, Feet, l.	Uniform Live Load, Tons, nL.	Best Number of Panels, n.	Best Height of Girder, Feet, h.	Least Weight of Bridge, Tons, nW.	Slope of Diagonals, ϕ.	Ratio of Dead to Live Load, $W \div L$.	Bridge Weight per Lin. Ft., Pounds.	Ratio of Length to Height, $l \div h$.	Panel Length, Feet, $l \div n$.
100	100	9	11.592	37.835	46° 12′ 51″	0.378	757	8.626	$11\frac{1}{9}$
100	200	9	13.106	53.613	49° 42′ 33″	0.268	1072	7.630	$11\frac{1}{9}$
200	200	13	18.266	145.260	49° 53′ 42″	0.726	1453	10.949	$15\frac{5}{13}$

These examples may suffice to illustrate a mode of determining economical proportions for girders of all classes.

SECTION 2.

The Pratt Truss of Single System under Varying Live Load, without taking Account of Wind Pressure.

145. We shall here resume the example of article 36, the span being 100 feet of 10 panels, and the live load 2 locomotives of given weight and wheel base.

Take n = number of panels.

W = unknown panel weight of bridge.

h = 20 feet = height of girders, pin to pin.

q = 14 feet = width of bridge, in clear.

q_1 = 16 feet = width of bridge, extreme.

Single track, 2 rails, 56 pounds per yard each.
Ties, $6 \times 8 \times 84$ inches, spaced 8 inches in clear.
2 track stringers, $12 \times d$ inches each.
Ties and stringers, pine, 40 pounds per cubic foot.
Weight of 2 rails $= 2 \times 100 \times \frac{56}{9} = 3,733$ pounds.

Weight of 75 ties $= \dfrac{6 \times 8}{144} \times 75 \times 7 \times 40 = 7,000$ pounds.

Panel length of stringers = 120 inches.
Panel weight of rails = 373 pounds.
Panel weight of ties = 700 pounds.
$2 \times$ weight on 1 pair of drivers = 42,000 pounds distributed.
Maximum weight on 2 stringers = 43,073 pounds uniformly distributed.

Then, for both stringers,

$$b = \text{breadth} = 24 \text{ inches.}$$
$$d = \text{height} = 15 \text{ inches.}$$

Take $f = 10$ = factor of safety for pine.
$B = 8,000$ = breaking-weight for pine.

From equation (52),

$$M = \tfrac{1}{8}wl^2 = \tfrac{1}{8} \times 43073 \times 120,$$

where $wl = 43,073$ pounds; and, from (160),

$$R \div f = \tfrac{1}{60}Bbd^2 = \frac{24 \times 8000}{60}d^2,$$

$$\therefore \quad d^2 = \frac{43073 \times 120 \times 60}{8 \times 24 \times 8000},$$

$$d = 14.21 \text{ inches.}$$

Call $d = 15$ inches,

$$\text{Weight of 2 stringers} = \frac{2 \times 12 \times 15 \times 100 \times 40}{144} = 10000 \text{ pounds.}$$

Suppose 2 wrought-iron I-beams suspended at each panel joint, and assume the load on these beams to be concentrated at their centre.

Greatest load on 2 beams,

From rails and ties,	1073 pounds,
From stringers,	1000 pounds,
From locomotive,	28612 pounds (article 36),
Total,	30685 pounds,

at centre has the momental effect of 61,370 pounds uniformly distributed along the double beam.

Hence, for each single I-beam, D of (412) is equal to 30,685 pounds, and $q_1 = 16$ feet.

Take $f = 6$ = factor of safety.

$B = 50{,}000$ pounds.

From (412),

$$d_2 = 3.80122\left(\frac{30685 \times 16 \times 6}{50000}\right)^{\frac{1}{3}} = 14.79 \text{ inches} = \text{required depth.}$$

From (413),

$$\text{Area of cross-section of 1 beam} = S = 1.28839\left(\frac{30685 \times 16 \times 6}{50000}\right)^{\frac{2}{3}}$$
$$= 19.508 \text{ inches.}$$

Now the "heavy 15-inch I-beam" of the Union Iron Mills, Pittsburgh, Penn., weighs 67 pounds to the foot, and its section consequently $= 67 \times \frac{3}{10} = 20.1$ inches.

We will, therefore, use the heavy 15-inch beam of 67 pounds to the foot.

Weight of 9 pairs 15-inch I-beams, 67 pounds, 16 feet;

$$2 \times 9 \times 16 \times 67 = 19296 \text{ pounds.}$$

Weight of 11 head struts, 14 feet, 20 pounds;

$$11 \times 14 \times 20 = 3080 \text{ pounds.}$$

Weight of 40 horizontal diagonals, $1\frac{1}{8}$ diameter, 3.359 pounds, 18 feet;

$$40 \times 18 \times 3.359 = 2419 \text{ pounds.}$$

Weight of the residue,

$$10 \times 200 = 2000 \text{ pounds.}$$

RECAPITULATION.

Rails	=	3733 pounds,
Ties	=	7000 pounds,
Stringers	=	10000 pounds,
Beams	=	19296 pounds,
Head struts	=	3080 pounds,
Horizontal diagonals	=	2419 pounds,
Residue	=	2000 pounds.
K	=	47528 pounds.

For determining the girder strains in this example, we have already found, article 36, the greatest moments and greatest differences of moment due the given rolling-load.

From (65), we have the moments due W,

$$M = \frac{Wl}{2n}r(n - r) = 5Wr(10 - r),$$

$$\begin{aligned}
\therefore\ M_1 &= 45W + 478.32, & H_1 &= 2.25W + 23.916 \text{ tons;}\\
M_2 &= 80W + 863.71, & H_2 &= 4.00W + 43.186 \text{ tons;}\\
M_3 &= 105W + 1105.88, & H_3 &= 5.25W + 55.294 \text{ tons;}\\
M_4 &= 120W + 1236.95, & H_4 &= 6.00W + 61.848 \text{ tons;}\\
M_5 &= 125W + 1276.38, & H_5 &= 6.25W + 63.819 \text{ tons.}
\end{aligned}$$

$$\begin{aligned}
\Delta_1 H &= 2.25W + 23.916 = \text{difference of horizontal strains.}\\
\Delta_2 H &= 1.75W + 19.270\\
\Delta_3 H &= 1.25W + 14.050\\
\Delta_4 H &= 0.75W + 10.451\\
\Delta_5 H &= 0.25W + 7.395\\
\Delta_6 H &= 4.785\\
\Delta_7 H &= 2.579\\
\Delta_8 H &= 0.932\\
\Delta_9 H &= 0.058
\end{aligned}$$

Let ϕ = angle of elevation of any diagonal,

$$\therefore \quad \tan\phi = \tfrac{20}{10} = 2, \qquad \begin{aligned} \log\tan\phi &= 0.3010300, \\ \log\sin\phi &= 9.9515452, \\ \log\cos\phi &= 9.6505152. \end{aligned}$$

In top chord, take ratio of panel length to least diameter $= 12$; then, by (400),

$$Q = \frac{18}{1 + \frac{12^2}{3000}} = \frac{P}{S} \text{ of the Gordon formula } = 17.176 \text{ tons.}$$

Take ratio of length of vertical to its least diameter $= 40$; then, by (400),

$$Q_1 = \frac{18}{1 + \frac{40^2}{3000}} = 11.74 \text{ tons.}$$

Let T = 50,000 pounds = 25 tons = limit of tension.
$f = 5$ = factor of safety.
Summing the strains on the equal panel lengths, we have

$$\begin{aligned} \text{Weight of top chords} &= \frac{2f}{Q}(23.75W + 248.063) \times 10 \times \tfrac{10}{3} \\ &= 460.93W + 4815 \text{ pounds.} \end{aligned}$$

Calling the strain on end panels of the bottom chord the same as the strain on the adjacent panel, we have strains in bottom chord,

$$\begin{aligned} H_1 &= 2.25W + 23.916 \\ H_2 &= 2.25W + 23.916 \\ H_3 &= 4.00W + 43.186 \\ H_4 &= 5.25W + 55.294 \\ H_5 &= 6.00W + 61.848 \\ \hline \Sigma H &= 19.75W + 208.160 \end{aligned}$$

$\therefore$ Weight of bottom chords, $\frac{2f}{T}(19.75\,W + 208.16) \times 10 \times \frac{10}{8}$

$$= 263.33\,W + 2776 \text{ pounds.}$$

$$\text{Strain on a vertical} = Z = \Delta H \tan\phi,$$

by the formulæ for Class IX.;

$$\begin{aligned}
\therefore\quad Z_1 &= 4.50\,W + 47.832 \\
Z_2 &= 3.50\,W + 38.540 \\
Z_3 &= 2.50\,W + 28.100 \\
Z_4 &= 1.50\,W + 20.902 \\
Z_5 &= 0.50\,W + 14.790 \\
Z_6 &= 9.570 \\
\Sigma Z &= 12.50\,W + 159.734
\end{aligned}$$

Weight of verticals, $\frac{2f}{Q_1}(12.5\,W + 159.734) \times 20 \times \frac{10}{8}$

$$= \frac{2 \times 5}{11.74}(12.5\,W + 159.734) \times \frac{200}{8} = 709.86\,W + 9071 \text{ pounds,}$$

where Z_6 is used twice to provide resistance to lateral shocks.

$$\text{Strain on a diagonal} = Y = \Delta H \div \cos\phi.$$

If we call the strain on each of the first 5 counters equal to that on the fifth, from live load alone, we have

$$\begin{aligned}
Y_1 = Y_2 = Y_3 = Y_4 = Y_5 &= 4.785 && \div \cos\phi, \\
Y_6 &= (7.395 + 0.25\,W) && \div \cos\phi, \\
Y_7 &= (10.541 + 0.75\,W) && \div \cos\phi, \\
Y_8 &= (14.050 + 1.25\,W) && \div \cos\phi, \\
Y_9 &= (19.270 + 1.75\,W) && \div \cos\phi, \\
Y_{10} &= (23.916 + 2.25\,W) && \div \cos\phi. \\
\Sigma Y &= (99.007 + 6.25\,W) && \div \cos\phi.
\end{aligned}$$

$$\text{Weight of diagonals} = \frac{2f}{T}(99.007 + 6.25\,W) \times \frac{10 \times 10}{3\cos^2\phi}$$

$$= 416.17\,W + 6601 \text{ pounds.}$$

RECAPITULATION.

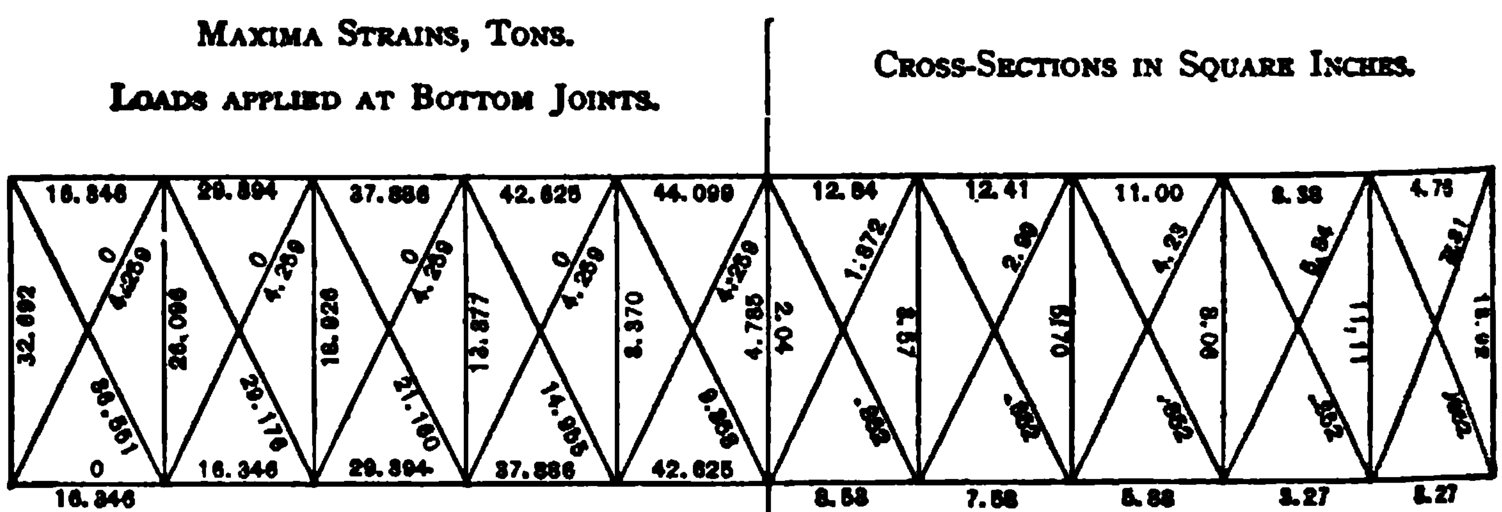
Weight of top chords	=	$460.93\,W$	+	4815 pounds,
Weight of bottom chords	=	$263.33\,W$	+	2776 pounds,
Weight of verticals	=	$709.86\,W$	+	9071 pounds,
Weight of diagonals	=	$416.17\,W$	+	6601 pounds,
Weight of girders $= G$	=	$1850.29\,W$	+	23263 pounds,
K	=			47528 pounds,

$$\text{Weight of bridge} = 2000nW = 1850.29\,W + 70791 = G + K.$$

$$\therefore \quad W = 3.9004 \text{ tons} = \text{panel weight.}$$

$$\text{Weight of bridge} = nW = 39.004 \text{ tons.}$$

Substituting 3.9004 for W in the expressions for H, Y, and Z, we have this strain sheet: —



EACH OF TWO GIRDERS.

FIG. 115.

146. If the dead and live loads are applied at the upper joints, instead of the lower joints, the structure becomes a deck bridge; and the compressions here found for the verticals must be increased by the panel weight of dead load plus the greatest apex load from the locomotives; viz., for each girder we must augment Z by

$$\frac{3.9004 + 14.3063}{2} = 9.1034 \text{ tons.}$$

147. As another example of varying load applied at the lower joints of the Pratt Truss, we will, in accordance with the practice of some engineers, assume a certain panel weight of engine, of tender, and of train, and determine the strains thence resulting, and also the weight of the bridge.

Let us take the example given by Col. Merrill for this truss (see "Iron Truss Bridges for Railroads," by Col. William E. Merrill, U.S.A.); viz., —

Span $= 200$ feet $= nc = l$.
Length of panel $= 12.5$ feet $= c$.
Height of truss $= 18.75$ feet $= h$.
Number of panels $= 16 = n$.
On each of 2 trusses,
Panel weight of engine $=$ 17,600 pounds.
Panel weight of tender $=$ 16,160 pounds.
Panel weight of cars $=$ 13,152 pounds.

The engine is supposed to cover 2 panels, the tender 2 panels, and the cars follow. We therefore have,

1st panel weight of moving-load $= W_5 = 8.800$ tons, engine;
2d panel weight of moving-load $= W_4 = 8.800$ tons, engine;
3d panel weight of moving-load $= W_3 = 8.080$ tons, tender;
4th panel weight of moving-load $= W_2 = 8.080$ tons, tender;
5th panel weight of moving-load $= W_1 = 6.576$ tons, cars;
6th, etc., the same.

To find the strains due to this rolling-load, we employ equations (91); and for convenience, after dividing by the height, $h = 18.75$ feet, we may let r_1 denote the number of panel weights of cars, and $u =$ the number of panel weights of engine and tender on the girder at any time. We shall then have for the different positions of the load, by summing equations (91), and putting

$$X = (n - r_1 - 1) W_2 + (n - r_1 - 2) W_3 + (n - r_1 - 3) W_4 + \dots (n - r_1 - u) W_{u+1},$$

$$Y = \frac{r_1(r_1 + 1)}{2} W_1 + (r_1 + 1) W_2 + (r_1 + 2) W_3$$
$$+ (r_1 + 3) W_4 + \ldots (r_1 + u) W_{u+1}.$$

HORIZONTAL STRAINS AT JOINTS FOR ROLLING-LOAD.

$$H_1 = \frac{c}{nh}\left\{\left[r_1 n - \frac{r_1(r_1 + 1)}{2}\right] W_1 + X\right\},$$

$$H_2 = \frac{2c}{nh}\left\{\left[\left(r_1 - \frac{1}{2}\right)n - \frac{r_1(r_1 + 1)}{2}\right] W_1 + X\right\} = 2H_1 - \frac{cW_1}{h},$$

$$H_3 = \frac{3c}{nh}\left\{\left[\left(r_1 - \frac{2}{2}\right)n - \frac{r_1(r_1 + 1)}{2}\right] W_1 + X\right\} = 3H_1 - \frac{3cW_1}{h},$$

$$H_4 = \frac{4c}{nh}\left\{\left[\left(r_1 - \frac{3}{2}\right)n - \frac{r_1(r_1 + 1)}{2}\right] W_1 + X\right\} = 4H_1 - \frac{6cW_1}{h},$$

$$H_5 = \frac{5c}{nh}\left\{\left[\left(r_1 - \frac{4}{2}\right)n - \frac{r_1(r_1 + 1)}{2}\right] W_1 + X\right\} = 5H_1 - \frac{10cW_1}{h},$$

. . . .

$$H_{r_1} = \frac{r_1 c}{nh}\left\{\left[\left(r_1 - \frac{r_1 - 1}{2}\right)n - \frac{r_1(r_1 + 1)}{2}\right] W_1 + X\right\}$$
$$= r_1 H_1 - \frac{(r_1 - 1)r_1}{2} \times \frac{cW_1}{h},$$

$$H_{r_1 + 1} = \frac{(r_1 + 1)c}{nh}\left\{\left[\left(r_1 - \frac{r_1}{2}\right)n - \frac{r_1(r_1 + 1)}{2}\right] W_1 + X\right\}$$
$$= (r_1 + 1) H_1 - \frac{r_1(r_1 + 1)}{2} \times \frac{cW_1}{h},$$

$$H_{r_1 + 2} = \frac{c}{nh}\left\{\frac{r_1(r_1 + 1)}{2}(n - r_1 - 2) W_1\right.$$
$$+ (r_1 + 1)(n - r_1 - 2) W_2 + (r_1 + 2)(n - r_1 - 2) W_3$$
$$\left. + (r_1 + 2)(n - r_1 - 3) W_4 + \ldots (r_1 + 2)(n - r_1 - u) W_{u+1}\right\},$$

$$H_{r_1+3} = \frac{c}{nh}\left\{\frac{r_1(r_1+1)}{2}(n - r_1 - 3)W_1\right.$$

$$+ (r_1+1)(n - r_1 - 3)W_2 + (r_1+2)(n - r_1$$

$$+ (r_1+3)(n - r_1 - 3)W_4 + \ldots (r_1+3)(n -$$

$$H_{r_1+4} = \frac{c}{nh}\left\{\frac{r_1(r_1+1)}{2}(n - r_1 - 4)W_1\right.$$

$$+ (r_1+1)(n - r_1 - 4)W_2 + (r_1+2)(n - r_1$$

$$+ (r_1+3)(n - r_1 - 4)W_4 + \ldots (r_1+4)(n -$$

. . . .

$$H_{r_1+u} = \frac{c}{nh}\left\{\frac{r_1(r_1+1)}{2}(n - r_1 - u)W_1\right.$$

$$+ (r_1+1)(n - r_1 - u)W_2$$

$$+ (r_1+2)(n - r_1 - u)W_3 + (r_1+3)(n -$$

$$\left. + \ldots (r_1+u)(n - r_1 - u)W_{u+1}\right\}$$

$$= \frac{c(n - r_1 - u)}{nh}Y.$$

$$H_{r_1+u+1} = \frac{c(n - r_1 - u - 1)}{nh}Y,$$

$$H_{r_1+u+2} = \frac{c(n - r_1 - u - 2)}{nh}Y,$$

. . . .

$$H_{n-1} = \frac{c}{nh}Y.$$

Computing these values of H from the abov
position of the train as it advances, a panel at a
to right, preceded by the engine and tender, we
of horizontal strains at the joints, in tons; vi
directly by the set of equations (91), —

STRAINS IN HORIZONTAL LINES ARE SIMULTANEOUS.

Joints loaded.	H_1	H_2	H_3	H_4	H_5	H_6	H_7	H_8	H_9	H_{10}	H_{11}	H_{12}	H_{13}	H_{14}	H_{15}
1	5.5														0.3$\frac{3}{8}$
1-2	10.6$\frac{1}{8}$	15.4													1.1
1-3	14.95	24.51$\frac{1}{8}$	28.21												2.17
1-4	18.93	32.47$\frac{1}{8}$	40.63	42.92											3.57$\frac{3}{8}$
1-5	21.63$\frac{1}{8}$	38.882$\frac{3}{8}$	50.742$\frac{1}{8}$	57.221$\frac{1}{8}$	57.830$\frac{3}{8}$										5.257$\frac{1}{8}$
1-6	24.062$\frac{3}{8}$	43.741$\frac{1}{8}$	59.036	68.944	73.465$\frac{1}{8}$	72.12									7.212
1-7	26.218	48.052	65.502	78.568	86.247$\frac{1}{8}$	88.54	84.916								9.440$\frac{3}{8}$
1-8	28.099$\frac{1}{8}$	51.814$\frac{3}{8}$	71.146	86.09$\frac{1}{8}$	96.65$\frac{3}{8}$	101.8$\frac{1}{8}$	101.62$\frac{1}{8}$	95.54$\frac{3}{8}$							11.94$\frac{1}{8}$
1-9	29.70$\frac{3}{8}$	55.029$\frac{1}{8}$	75.968	92.522$\frac{3}{8}$	104.69$\frac{1}{8}$	112.48	114.88	111.89$\frac{1}{8}$	103.04						14.72
1-10	31.04	57.696	79.968	97.856	111.36	120.48	125.216	124.565$\frac{1}{8}$	118.528	106.624					17.770$\frac{3}{8}$
1-11	32.099$\frac{1}{8}$	59.814$\frac{3}{8}$	83.146	102.09$\frac{1}{8}$	116.65$\frac{3}{8}$	126.836	132.631$\frac{1}{8}$	134.042$\frac{3}{8}$	130.067$\frac{1}{8}$	120.705$\frac{1}{8}$	105.47$\frac{3}{8}$				21.095$\frac{1}{8}$
1-12	32.884$\frac{3}{8}$	61.385$\frac{1}{8}$	85.502	105.234$\frac{3}{8}$	120.58$\frac{1}{8}$	131.548	138.128$\frac{3}{8}$	140.325$\frac{1}{8}$	138.138	130.564	117.60$\frac{1}{8}$	98.776			24.694
1-13	33.396	62.408	87.036	107.28	123.14	134.616	141.708	144.416	142.74	136.68	125.2$\frac{1}{8}$	108.4	85.7		28.5$\frac{3}{8}$
1-14	33.6$\frac{1}{8}$	62.882$\frac{3}{8}$	87.784	108.229$\frac{1}{8}$	124.32$\frac{3}{8}$	136.04	143.369$\frac{1}{8}$	146.314$\frac{3}{8}$	144.876	139.05$\frac{1}{8}$	128.84$\frac{3}{8}$	113.25$\frac{1}{8}$	92.27$\frac{1}{8}$	65.42$\frac{3}{8}$	32.71$\frac{1}{8}$
1-15	33.59$\frac{3}{8}$	62.809$\frac{1}{8}$	87.638	108.082$\frac{3}{8}$	124.14$\frac{1}{8}$	135.80	143.112$\frac{3}{8}$	146.021$\frac{1}{8}$	144.546	138.68$\frac{3}{8}$	128.44$\frac{1}{8}$	113.816	93.802	68.401$\frac{1}{8}$	37.234

The blanks in this table may be filled by continually adding to itself each number in the right-hand column. It follows, therefore, that this right-hand column expresses the negative differences of simultaneous horizontal strains at adjacent joints due to rolling-load. It is evident, that, in this case, these negative differences are numerically the greatest differences of horizontal strains at adjacent joints, and may therefore be employed to find the maxima vertical and diagonal strains due live load.

The table shows (as was to be expected from this load, but contrary to the assumption made by Col. Merrill) that the horizontal strains are not maxima throughout when the foremost end of the engine is at the last joint, but that the greater part of these strains reach their greatest values when the foremost end (that is, forward panel weight) of the engine is at the fourteenth joint.

Since we require only the greatest horizontal strains, we need not compute the whole table, but only enough of the higher values of H to be certain that we find the highest at each joint. In the present example, it will suffice to compute values of H for the positions of load when $r_1 = 11$, $r_1 = 10$, $r_1 = 9$, and these for only the last 8 joints, since the first 7 horizontal joint strains are smaller than the last 7 by reason of the unequal loading.

We have, then, the following brief solution:—

1. For maxima differences of horizontal strain due live load.

$$H_{n-1} = H_{15} = \frac{c}{nh}Y = \frac{c}{nh}\left\{\frac{r_1(r_1+1)}{2}W_1 + (r_1+1)W_2 \right.$$

$$\left. + (r_1+2)W_3 + (r_1+3)W_4 + (r_1+4)W_5\right\}.$$

$c = 12.5,\ k = 18.75,\ n = 16,\ W_1 = 6.576,\ W_2 = W_3 = 8.08,\ W_4 = W_5 = 8.8.$

r_1	W_1	W_2	W_3	W_4	W_5	H_{n-1} =	H_n
-3	0	0	0	0	8.8	$\frac{1}{24}\times 8.8$	$0.3\frac{2}{3}$
-2	0	0	0	8.8	8.8	$(\frac{1}{24}+\frac{2}{24})8.8$	1.1
-1	0	0	8.08	8.8	8.8	$\frac{5}{24}\times 8.8+\frac{1}{24}\times 8.08$	2.17
0	0	8.08	8.08	8.8	8.8	$\frac{7}{24}\times 8.8+\frac{3}{24}\times 8.08$	$3.57\frac{2}{3}$
1	6.576	8.08	8.08	8.8	8.8	$\frac{9}{24}\times 8.8+\frac{5}{24}\times 8.08+\frac{1}{24}\times 6.576$	$5.257\frac{1}{3}$
2	6.576	8.08	8.08	8.8	8.8	$5.257\frac{1}{3}+\frac{1}{24}(2\times 8.08+2\times 8.8+2\times 6.576)$	7.212
3	6.576	8.08	8.08	8.8	8.8	$7.212\ +\frac{1}{24}(33.76+3\times 6.576)$	$9.440\frac{2}{3}$
4	6.576	8.08	8.08	8.8	8.8	$9.440\frac{2}{3}+1.40\frac{2}{3}+\frac{4}{24}\times 6.576$	$11.94\frac{1}{3}$
5	6.576	8.08	8.08	8.8	8.8	$11.94\frac{1}{3}+1.40\frac{2}{3}+\frac{5}{24}\times 6.576$	14.72
6	6.576	8.08	8.08	8.8	8.8	$14.72\ +1.40\frac{2}{3}+\ 6\times 0.274$	$17.770\frac{2}{3}$
7	6.576	8.08	8.08	8.8	8.8	$17.770\frac{2}{3}+1.40\frac{2}{3}+\ 7\times 0.274$	$21.095\frac{1}{3}$
8	6.576	8.08	8.08	8.8	8.8	$21.095\frac{1}{3}+1.40\frac{2}{3}+\ 8\times 0.274$	24.694
9	6.576	8.08	8.08	8.8	8.8	$24.694\ +1.40\frac{2}{3}+\ 9\times 0.274$	$28.5\frac{2}{3}$
10	6.576	8.08	8.08	8.8	8.8	$28.5\frac{2}{3}+1.40\frac{2}{3}+10\times 0.274$	$32.71\frac{1}{3}$
11	6.576	8.08	8.08	8.8	8.8	$32.71\frac{1}{3}+1.40\frac{2}{3}+11\times 0.274$	37.134

2. For the maxima horizontal strains due live load, as already computed and tabulated above.

$$r_1 = 9, \quad H_1 = \frac{c}{nh}\left\{\left[r_1 n - \frac{r_1(r_1+1)}{2}\right]W_1 + (n-r_1-1)W_2 + (n-r_1-2)W_3 + (n-r_1-3)W_4 + (n-r_1-4)W_5\right\} = 33.396$$

$$H_8 = 8H_1 - 28\frac{cW_1}{h} = 144.416$$

$$H_9 = 9H_1 - 36 \times 4.384 = 142.74$$

$$H_{10} = 10H_1 - 45 \times 4.384 = 136.68$$

$$H_{11} = \tfrac{1}{24}(45 \times 5 \times 6.576 + 10 \times 5 \times 8.08 + 11 \times 5 \times 8.08 + 11 \times 4 \times 8.8 + 11 \times 3 \times 8.8) = 125.233\tfrac{1}{3}$$

$$H_{12} = \tfrac{1}{24}(45 \times 4 \times 6.576 + 10 \times 4 \times 8.08 + 11 \times 4 \times 8.08 + 12 \times 4 \times 8.8 + 12 \times 3 \times 8.8) = 108.4$$

$$H_{13} = \tfrac{1}{24}(45 \times 3 \times 6.576 + 10 \times 3 \times 8.08 + 11 \times 3 \times 8.08 + 12 \times 3 \times 8.8 + 13 \times 3 \times 8.8) = 85.7$$

$$\tfrac{2}{3}H_{13} = H_{14} = \tfrac{1}{24}(45 \times 2 \times 6.576 + 10 \times 2 \times 8.08 + 11 \times 2 \times 8.08 + 12 \times 2 \times 8.8 + 13 \times 2 \times 8.8) = 57.1\tfrac{1}{3}$$

$$\tfrac{1}{3}H_{13} = H_{15} = \frac{c}{nh}Y = 28.5\tfrac{2}{3}$$

$$r_1 = 10, \quad H_1 = 33.6\tfrac{1}{3}$$

$$\text{(max)} \quad H_8 = 8H_1 - 28\frac{cW_1}{h} = 146.314\tfrac{2}{3}$$

$$\text{(max)} \quad H_9 = 9H_1 - 36 \times 4.384 = 144.876$$

$$\text{(max)} \quad H_{10} = 10H_1 - 45 \times 4.384 = 139.05\tfrac{1}{3}$$

$$\text{(max)} \quad H_{11} = 11H_1 - 55 \times 4.384 = 128.846\tfrac{2}{3}$$

$$H_{12} = \tfrac{1}{24}(55 \times 4 \times 6.576 + 11 \times 4 \times 8.08 + 12 \times 4 \times 8.08 + 12 \times 3 \times 8.8 + 12 \times 2 \times 8.8) = 113.25\tfrac{1}{3}$$

$$H_{13} = \tfrac{1}{24}(55 \times 3 \times 6.576 + 11 \times 3 \times 8.08 + 12 \times 3 \times 8.08 + 13 \times 3 \times 8.8 + 13 \times 2 \times 8.8) = 92.27\tfrac{1}{3}$$

$$H_{14} = \tfrac{1}{24}(55 \times 2 \times 6.576 + 11 \times 2 \times 8.08 + 12 \times 2 \times 8.08 + 13 \times 2 \times 8.8 + 14 \times 2 \times 8.8) = 65.42\tfrac{2}{3}$$

$$H_{15} = \tfrac{1}{2}H_{14} = 32.71\tfrac{1}{3}$$

$$r_1 = 11, \quad H_1 = 33.59\tfrac{2}{3}$$

$$H_8 = 8H_1 - 28\frac{cW_1}{h} = 146.021\tfrac{1}{3}$$

$$H_9 = 9H_1 - 36 \times 4.384 = 144.546$$

$$H_{10} = 10H_1 - 45 \times 4.384 = 138.686\tfrac{2}{3}$$

$$H_{11} = 11H_1 - 55 \times 4.384 = 128.44\tfrac{1}{3}$$

$$\text{(max)} \quad H_{12} = 12H_1 - 66 \times 4.384 = 113.816$$

$$\text{(max)} \quad H_{13} = \tfrac{1}{24}(66 \times 3 \times 6.576 + 12 \times 3 \times 8.08 + 13 \times 3 \times 8.08 + 13 \times 2 \times 8.8 + 13 \times 1 \times 8.8) = 93.802$$

$$\text{(max)} \quad H_{14} = \tfrac{1}{24}(66 \times 2 \times 6.576 + 12 \times 2 \times 8.08 + 13 \times 2 \times 8.08 + 14 \times 2 \times 8.8 + 14 \times 1 \times 8.8) = 68.401\tfrac{1}{3}$$

$$\text{(max)} \quad H_{15} = \tfrac{1}{24}(66 \times 1 \times 6.576 + 12 \times 1 \times 8.08 + 13 \times 1 \times 8.08 + 14 \times 1 \times 8.8 + 15 \times 1 \times 8.8) = 37.134$$

It is manifest that the labor of computing the maxima values of H would be much lessened if we could legitimately assume that the horizontal strains are greatest throughout when the head of the engine is at the last joint or at any particular joint.

To find the strains due the unknown bridge weight, $2nW$, we have the panel weight of bridge on each girder $= W$, and find from equation (65), after dividing by h,

$$H = \frac{Wl}{2nh} r(n - r).$$

		1st difference.	H maximum, tons.
$\therefore\ r = 0,$	$H_0 = 0,$		0
		$5\ W.$	
$r = 1,$	$H_1 = 5\ W,$		$5\ W + 37.134$
		$4\frac{1}{3} W.$	
$r = 2,$	$H_2 = 9\frac{1}{3} W,$		$9\frac{1}{3} W + 68.401\frac{1}{3}$
		$3\frac{2}{3} W.$	
$r = 3,$	$H_3 = 13\ W,$		$13\ W + 93.802$
		$3\ W.$	
$r = 4,$	$H_4 = 16\ W,$		$16\ W + 113.816$
		$2\frac{1}{3} W.$	
$r = 5,$	$H_5 = 18\frac{1}{3} W,$		$18\frac{1}{3} W + 128.84\frac{2}{3}$
		$1\frac{2}{3} W.$	
$r = 6,$	$H_6 = 20\ W,$		$20\ W + 139.05\frac{1}{3}$
		$1\ W.$	
$r = 7,$	$H_7 = 21\ W,$		$21\ W + 144.876$
		$\frac{1}{3} W.$	
$r = 8,$	$H_8 = 21\frac{1}{3} W.$		$21\frac{1}{3} W + 146.314\frac{2}{3}$

Bottom chord.

Top chord.

Panel.	Maximum Difference of Horizontal Strain $= \Delta H$.	No.	Maximum Vertical Strain $= \Delta H \tan\phi = \frac{3}{2}\Delta H$.	Maximum Diagonal Strain.
1	$0 - 5W$			0 or 5
2	$0.3\frac{2}{3} - 4\frac{1}{3}W$			0 or 5
3	$1.1 - 3\frac{2}{3}W$			0 or 5
4	$2.17 - 3W$			0 or 5
5	$3.57\frac{2}{3} - 2\frac{1}{3}W$			0 or 5
6	$5.257\frac{1}{3} - 1\frac{2}{3}W$			0 or 5
7	$7.212 - W$			$(7.212 - W)\sec\phi$
8	$9.440\frac{2}{3} - \frac{1}{3}W$	9	$\frac{3}{2}(9.440\frac{2}{3} - \frac{1}{3}W)$	$(9.440\frac{2}{3} - \frac{1}{3}W)\sec\phi$
9	$11.94\frac{1}{3} + \frac{2}{3}W$	10	$\frac{3}{2}(11.94\frac{1}{3} + \frac{1}{3}W)$	$(11.94\frac{1}{3} + \frac{1}{3}W)\sec\phi$
10	$14.72 + W$	11	$\frac{3}{2}(14.72 + W)$	$(14.72 + W)\sec\phi$
11	$17.770\frac{2}{3} + 1\frac{2}{3}W$	12	$\frac{3}{2}(17.770\frac{2}{3} + 1\frac{2}{3}W)$	$(17.770\frac{2}{3} + 1\frac{2}{3}W)\sec\phi$
12	$21.095\frac{1}{3} + 2\frac{1}{3}W$	13	$\frac{3}{2}(21.095\frac{1}{3} + 2\frac{1}{3}W)$	$(21.095\frac{1}{3} + 2\frac{1}{3}W)\sec\phi$
13	$24.694 + 3W$	14	$\frac{3}{2}(24.694 + 3W)$	$(24.694 + 3W)\sec\phi$
14	$28.5\frac{2}{3} + 3\frac{2}{3}W$	15	$\frac{3}{2}(28.5\frac{2}{3} + 3\frac{2}{3}W)$	$(28.5\frac{2}{3} + 3\frac{2}{3}W)\sec\phi$
15	$32.71\frac{1}{3} + 4\frac{1}{3}W$	16	$\frac{3}{2}(32.71\frac{1}{3} + 4\frac{1}{3}W)$	$(32.71\frac{1}{3} + 4\frac{1}{3}W)\sec\phi$
16	$37.134 + 5W$	17	$\frac{3}{2}(37.134 + 5W)$	$(37.134 + 5W)\sec\phi$

In the first 6 panels we shall introduce counters of 1 square inch cross-section, and therefore capable of resisting safely 5 tons where theoretically no strain appears. Also, we shall call the strain on the bottom chords in the first panel equal to that of the second panel; viz.,

$$5W + 37.134 \text{ tons.}$$

For each panel length of top chord, take ratio of length to least diameter = 15. Then the Gordon formula becomes, equation (400),

$$\frac{P}{S} = Q \doteq \frac{18}{1 + \frac{15^2}{3000}} = 16.744 \text{ tons per square inch.}$$

$$\frac{16.744}{5} = 3.3488 \text{ tons} = \text{allowed inch strain on top chords.}$$

For each vertical strut, take ratio of length to least diameter $= 37.5$. Then

$$\frac{P}{S} = Q'' = \frac{18}{1 + \frac{37.5^2}{3000}} = 12.255 \text{ tons per square inch.}$$

$$\frac{12.255}{5} = 2.451 \text{ tons} = \text{allowed inch strain on verticals.}$$

In tension, 5 tons allowed. Factor of safety $f = 5$ for all parts of girder; panel length of chords $= 12.5$ feet; length of verticals $= 18.75$ feet; length of diagonals $= 23$ feet; wrought-iron, $\frac{5}{18}$ pound per cubic inch.

From these data we find,

Weight of top chords

$$= 4(124W + 872.244) \times \frac{12.5 \times 10}{3.3488 \times 3} = 6171.36W + 43411 \text{ pounds.}$$

Weight of bottom chords

$$= 4(107\tfrac{5}{8}W + 763.06\tfrac{1}{8}) \times \frac{12.5 \times 10}{5 \times 3} = 3588.89W + 25435 \text{ pounds.}$$

Weight of verticals

$$= 4(21\tfrac{1}{8}W + 193.357\tfrac{7}{8}) \times \tfrac{3}{4} \times \frac{18.75 \times 10}{2.451 \times 3} = 3238.47W + 29585 \text{ pounds.}$$

Weight of diagonals

$$= 4\left(\frac{6 \times 5}{\sec\theta} + 20W + 205.290\right) \times \frac{23 \times 10 \sec\theta}{5 \times 3} = 2211.38W + 24539 \text{ pounds.}$$

Weight of 2 girders $= G$ $= 15210.10W + 122970$ pounds.

W is in tons.

$$\tan\phi = \frac{18.75}{12.5} = \frac{3}{2}, \quad \sec\phi = \sqrt{1^2 + \left(\frac{3}{2}\right)^2} = 1.80278.$$

To find the constant part of the bridge weight $= K$.
2 rails, 200 feet, 56 pounds per yard, = 7,467 pounds.
Rails 5 feet between centres.
Ties 6×8 inches, 16 inches between centres.

$\frac{12 \times 200}{16} = 150$ ties, 7 feet long, 40 pounds per cubic foot.

Weight of ties $= 150 \times \frac{6 \times 8}{144} \times 7 \times 40 = 14{,}000$ pounds.

2 track stringers, each $15 \times 18\frac{1}{2}$ inches, 40 pounds per cubic foot, $\frac{2 \times 15 \times 18\frac{1}{2}}{144} \times 200 \times 40 = 30{,}833$ pounds.

Depth of stringer is thus found:—
Length between bearings = 12.5 feet.
Uniform load $= \frac{1}{16}$ of rails and ties $= \frac{21467}{16} = 1{,}342$ pounds.
Concentrated load = panel weight of engine $= 2 \times 17{,}600 = 35{,}200$ pounds, which is equivalent to uniform load of 70,400 pounds.

$\therefore$ Uniform load on 2 stringers 30 inches wide = 71742 pounds.

For pine, take factor of safety = 10, and the ultimate inch resistance to cross-breaking $B = 8{,}000$ pounds;

$\therefore$ Moment due external forces $= \frac{1}{8} \times 71742 \times 12 \times 12.5$,

from equation (52).

Moment of resistance due internal forces, equation (160),

$$= \tfrac{1}{6}Bbd^2 = \frac{8000 \times 30}{6}d^2,$$

which becomes, after introducing the factor of safety,

$$\frac{8000 \times 30d^2}{60} = 4000d^2.$$

Equating moments of external and allowable internal forces, we find

$$\tfrac{1}{8} \times 71742 \times 12 \times 12.5 = 4000d^2,$$

$$\therefore \quad \text{Depth} = d = 18.338 \text{ inches, called } 18\tfrac{1}{4}.$$

15 pairs I-beams, heavy 15-inch, 67 pounds, 16 feet, = 32,160 pounds.

Depth of beam is thus determined:

Panel weight of rails	=	467 pounds,
Panel weight of ties	=	875 pounds,
Panel weight of stringers	=	1927 pounds,
Panel weight of engine	=	35200 pounds.
Weight on each pair of I-beams	=	38469 pounds.

Now, this weight is actually concentrated at two points 5 feet apart, under the rails, each point being $5\tfrac{1}{2}$ feet from the nearer end of the I-beams.

The moment at the centre of the double beam is therefore, according to equation (43),

$$M_c = 2 \times 19234.5 \times \tfrac{8}{16} \times 5\tfrac{1}{2} \times 12 = \tfrac{1}{4}W' \times 16 \times 12,$$

by equation (46), if W' is the weight at centre producing an equivalent moment;

$$\therefore \quad W' = 26447.4,$$

and 52,895 pounds is the equivalent load uniformly distributed for 2 beams. Therefore uniform load on 1 beam is 26,447 pounds. And, if 6 is the factor of safety for beams, equation (412) gives, B being $= 50{,}000$,

$$\text{Depth of beam} = d_2 = 3.80122\left(\frac{26447 \times 16 \times 6}{50000}\right)^{\frac{1}{3}} = 14.076 \text{ inches.}$$

From (413),

$$\text{Area} = 1.28839\left(\frac{26447 \times 16 \times 6}{50000}\right)^{\frac{2}{3}} = 17.667 \text{ square inches.}$$

$$\text{Area of similar beam 15 inches deep} = 17.667 \times \frac{15^2}{14.076^2}$$

$$= 20.062 \text{ square inches.}$$

Area of heavy 15-inch 67-pound beam of the Union Iron Mills, Pittsburgh, Penn., $\Big\}$ = 20.1 square inches.

Hence we may with safety use this beam.

17 head struts, 14 feet, 25 pounds, = 5,950 pounds.

64 horizontal diagonals, $1\frac{1}{8}$ diameter, $19\frac{1}{2}$ feet, 3.359 pounds, = 4,192 pounds.

Residue, 200 pounds per panel, = 3,200 pounds.

RECAPITULATION.

Weight of rails	=	7467 pounds,
Weight of ties	=	14000 pounds,
Weight of stringers	=	30833 pounds,
Weight of I-beams	=	32160 pounds,
Weight of head struts	=	5950 pounds,
Weight of horizontal diagonals	=	4192 pounds,
Weight of residue	=	3200 pounds.
K	=	97802 pounds.

Weight of girders $= 122970 + 15210.10W.$

Weight of bridge $= 220772 + 15210.1W$ pounds

$= 2nW \text{ tons} = 2 \times 16 \times 2000W \text{ pounds}$

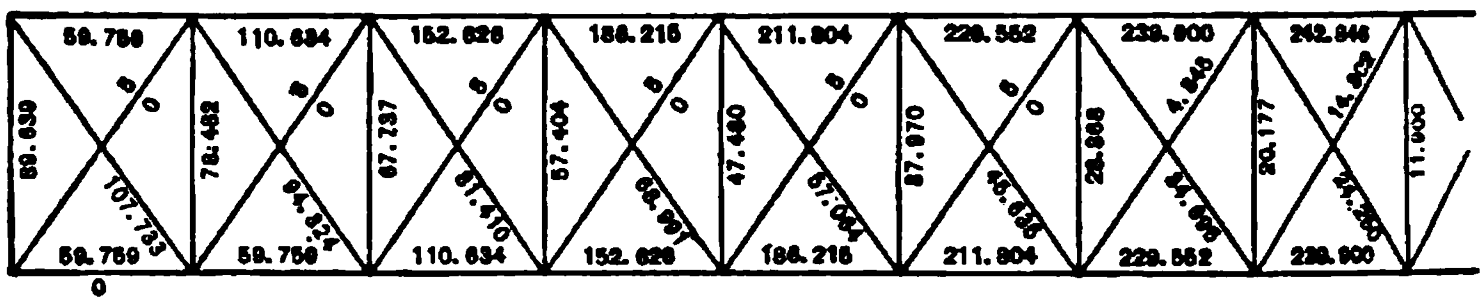
$= 64000W.$

$$\therefore\ 48789.9W = 220772, \quad W = 4.52493 \text{ tons.}$$

$$\text{Weight of bridge} = 32W = 144.798 \text{ tons.}$$

Substituting this value of W in the expressions already found for greatest strains, we have, —

MAXIMA STRAINS, IN TONS, FOR EACH OF TWO GIRDERS.



LOADS APPLIED AT BOTTOM JOINTS.

FIG. 116.

To find the allowed cross-sections in square inches, we divide strains in top chord by 3.3488, strains in verticals by 2.451, and in bottom chord and diagonals by 5.

CHAPTER X.

CALCULATION OF THE WEIGHT OF BRIDGES HAVING GIRDERS OF CLASS I., AND DETERMINATION OF THE NUMBER OF PANELS AND THE HEIGHT OF GIRDER, WHICH RENDER THE BRIDGE WEIGHT LEAST FOR A GIVEN SPAN AND UNIFORM LIVE LOAD. — LIMITING SPAN FOUND.

SECTION I.

General Specifications for Iron Bridges, issued in 1879 by the New York, Lake Erie, and Western Railroad Company. O. Chanute, Chief Engineer.

148. General Specifications for Iron Bridges.

NEW YORK, LAKE ERIE, AND WESTERN RAILROAD COMPANY.

1879.

GENERAL DESCRIPTION.

1. All parts of the superstructure shall be of wrought-iron, except bed plates and washers, which may be of cast-iron.

2. The following modes of construction shall preferably be employed:— Kinds of bridges.

Spans up to 17 feet	Rolled beams.
Spans 17 to 40 feet	Riveted plate girders.
Spans 40 to 75 feet	Riveted lattice girders.
Spans over 75 feet	Pin-connected trusses.

In calculating strains, the length of span shall be understood to be the distance between centres of end pins for trusses, and between centres of bearing-plates for all beams and girders.

Spacing of girders. 3. The girders shall be spaced (with reference to the axis of the bridge) as required by local circumstances, and directed by the chief engineer of the railroad company.*

Head room. 4. In all through bridges, there shall be a clear head room of 20 feet above the base of the rails.

Floor. 5. The wooden floor will consist of transverse floor timbers extending the full width of the bridge, supporting the rails and guard beams. Their scantling will vary with circumstances. They will be furnished and put on by the railroad company.

Loads. 6. Bridges shall be proportioned to carry the following loads:—

1st, The weight of iron in the structure.

2d, A floor weighing 400 pounds per lineal foot of *track*, to consist of the rails, ties, and guard timbers only.

These two items taken together shall constitute the "dead load."

3d, A moving-load for each *track*, supposed to be moving in either direction, and consisting of two "consolidation" engines coupled, followed by a train weighing 2,240 pounds per running foot; this "live load" being concentrated upon points distributed as in the diagram on p. 415.

Stresses. The maximum strains due to all positions of the above "live load," and of the "dead load," shall be taken to proportion all the parts of the structure.

Lateral stresses. 7. To provide for wind strains and vibrations, the top lateral bracing in deck bridges, and the bottom lateral bracing in through bridges, shall be proportioned to resist a lateral force of 450 pounds for each foot of the span; 300 pounds of this to be treated as a moving-load.

The bottom lateral bracing in deck bridges, and the top lateral bracing in through bridges, shall be proportioned to resist a lateral force of 150 pounds for each foot of the span.

Temperature. 8. Variations in temperature to the extent of 150 degrees shall be provided for.

* Generally, in through bridges, the clear width between trusses shall be 15 feet for single track, and 28 feet for double track. In deck bridges, and for the floor system of all bridges, the spacing between the centres of trusses and girders shall generally be as follows:—

Description.	Single Track.	Double Track.	
		2 Trusses.	3 Trusses.
Deck truss bridges . . .	12 feet or over.	16 feet or over.	10 feet or over.
Deck plate girders. . . .	8 feet or over.	16 feet or over.	10 feet or over.
Floor stringers	8 feet or over.	10 feet or over.	8 feet or over.

The centres of beams and plate girders shall be not less than 4 feet (on either side) from the centre of the broad gauge track.

The standard distance between centres of tracks is 13 feet.

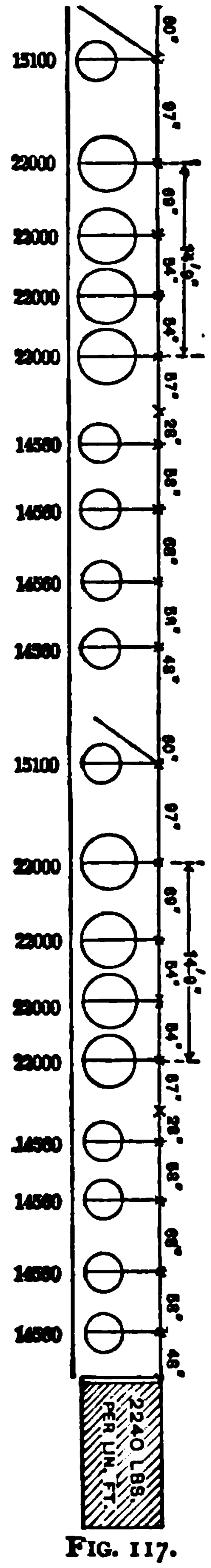


FIG. 117.

9. All parts shall be so designed that the strains coming upon them can be accurately calculated.

Plans and strain sheets. 10. Strain sheets and a general plan showing the dimensions of the parts and general details must accompany each proposal.

11. Upon the acceptance of a proposal, a full set of working drawings must be submitted for approval by the chief engineer of the railroad company before the work is commenced.

Form of truss. 12. Unless otherwise specified, the form of truss may be selected by the builder; but, to secure uniformity in appearance, it is desired that all "through" trusses shall be built with inclined end posts.

13. In comparing competitive plans, the relative cost of the wooden floors required will be taken into consideration.

14. The following clauses are all intended to apply to iron construction. Parties proposing to substitute steel for particular parts will be required to furnish evidence of its strength, elasticity, uniformity in production, and adaptability to the intended purpose.

PROPORTION OF PARTS.

Tensile strains. 1. All parts of the structures shall be so proportioned that the maximum strains produced shall in no case cause a greater tension than the following: —

	Pounds per Sq. Inch.
On lateral bracing	15000
On solid rolled beams, used as cross floor beams and stringers	10000
On bottom chords and main diagonals . .	10000
On counter rods and long verticals . . .	8000
On bottom flange of riveted cross girders, net section	8000
On bottom flange of riveted longitudinal plate girders *over* 20 ft. long, net section,	8000
On bottom flange of riveted longitudinal plate girders *under* 20 ft. long, net section,	7000
On floor beam hangers, and other similar members liable to sudden loading . . .	6000

Compressive strains.

2. Compression members shall be so proportioned that the maximum load shall in no case cause a greater strain than that determined by the following formulæ: —

$$P = \frac{8000}{1 + \frac{L^2}{40000R^2}} \text{ for square end compression members.}$$

$$P = \frac{8000}{1 + \frac{L^2}{30000R^2}} \text{ for compression members with one pin and one square end.}$$

$$P = \frac{8000}{1 + \frac{L^2}{20000R^2}} \text{ for compression members with pin bearings.}$$

$P =$ the allowed compression per square inch of cross-section.

$L =$ the length of compression member, in inches.

$R =$ the least radius of gyration of the section, in inches.

3. The lateral struts shall be proportioned by the above formulæ to resist the resultant due to an assumed initial strain of 10,000 pounds per square inch upon all the rods attaching to them, produced by adjusting the bridge.

4. In beams and girders, compression shall be limited, as follows: —

	Pounds per Square Inch.
In rolled beams used as cross floor beams and stringers . .	10000
In riveted plate girders used as cross floor beams, gross section,	6000
In riveted longitudinal plate girders *over* 20 feet long, gross section	6000
In riveted longitudinal plate girders *under* 20 feet long, gross section	5000

5. Members subjected to alternate strains of tension and compression shall be proportioned to resist each of them. The strains, however, shall be assumed to be increased by an amount equal to eight-tenths of the least strain.

Shearing-strains.

6. The rivets and bolts connecting all parts of the girders must be so spaced that the shearing-strain per square inch shall not exceed 6,000 pounds, nor the pressure upon the bearing-surface exceed 12,000 pounds

per square inch of the projected semi-intrados (diameter × thickness of piece) of the rivet or bolt hole.

Bending-strains.

7. Pins shall be so proportioned that the shearing-strain shall not exceed 7,500 pounds per square inch, nor the crushing-strain upon the projected area of the semi-intrados (diameter × thickness of piece) of any member connected to the pin be greater than 12,000 pounds per square inch, nor the bending-strain exceed 15,000 pounds per square inch, when the centres of bearings of the strained members are taken as the points of application of the strains.

8. In case any member is subjected to a bending-strain from local loadings (such as distributed floors on deck bridges), in addition to the strain produced by its position as a member of the structure, it must be proportioned to resist the combined strains.

Plate girders.

9. Plate girders shall be proportioned upon the supposition that the bending or chord strains are resisted entirely by the upper and lower flanges, and that the shearing or web strains are resisted entirely by the web plate.

10. The compression flanges of beams and girders shall be stayed against transverse crippling when their length is more than thirty times their width.

11. The unsupported width of any plate subjected to compression shall never exceed thirty times its thickness.

12. In members subject to tensile strains, full allowance shall be made for reduction of section by rivet holes, screw threads, etc.

13. The iron in the web plates shall not have a shearing-strain greater than 4,000 pounds per square inch, and no web plate shall be less than $\frac{1}{4}$ inch in thickness.

14. No wrought-iron shall be used less than $\frac{5}{16}$ inch thick, except in places where both sides are always accessible for cleaning and painting.

DETAILS OF CONSTRUCTION.

1. All the connections and details of the several parts of the structure shall be of such strength, that, upon testing, rupture shall occur in the body of the members rather than in any of their details or connections.

2. Preference will be had for such details as will be most accessible for inspection, cleaning, and painting.

3. The web of plate girders must be spliced at all joints by a plate on each side of the web. T-iron must not be used for splices.

4. When the least thickness of the web is less than one-eightieth of the depth of a girder, the web shall be stiffened at intervals not over twice the depth of the girder.

5. The pitch of rivets in all classes of work shall never exceed 6 inches, nor sixteen times the thinnest outside plate, nor be less than three diameters of the rivet.

6. The rivets used will generally be $\frac{3}{4}$ and $\frac{7}{8}$ inch diameter.

7. The distance between the edge of any piece and the centre of a rivet hole must never be less than $1\frac{1}{4}$ inches, except for bars less than $2\frac{1}{2}$ inches wide; when practicable, it shall be at least two diameters of rivets.

8. When plates more than 12 inches wide are used in the flanges of plate or lattice girders, an extra line of rivets, with a pitch of not over 9 inches, shall be driven along each edge, to draw the plates together, and prevent the entrance of water

9. In punching plate or other iron, the diameter of the dye shall in no case exceed the diameter of the punch by more than $\frac{1}{16}$ of an inch.

10. All rivet holes must be so accurately punched, that, when the several parts forming one member are assembled together, a rivet $\frac{1}{16}$ inch less in diameter than the hole can be entered, hot, into any hole, without reaming or straining the iron by "drifts."

11. The rivets, when driven, must completely fill the holes.

12. The rivet heads must be hemispherical, and of a uniform size for the same-sized rivets throughout the work. They must be full and neatly made, and be concentric to the rivet hole.

13. Whenever possible, all rivets must be machine-driven.

14. The several pieces forming one built member must fit closely together, and, when riveted, shall be free from twists, bends, or open joints.

15. All joints in riveted work, whether in tension or compression members, must be fully spliced, as no reliance will be placed upon abutting joints. The ends, however, must be dressed straight and true, so that there shall be no open joints.

Lower chords and suspension bars.

16. The heads of eye-bars shall be so proportioned that the bar will break in the body instead of in the eye. The form of the head and the mode of manufacture shall be subject to the approval of the chief engineer of the railroad company.

17. The bars must be free from flaws, and of full thickness in the necks. They shall be perfectly straight before boring. The holes shall be in the centre of the head, and on the centre line of the bar.

18. The bars must be bored of exact lengths, and the pin hole $\frac{1}{50}$ inch larger than the diameter of the pin.

19. The lower chord shall be packed as narrow as possible.

Pins.

20. The pins shall be turned straight and smooth, and shall fit the pin holes within $\frac{1}{50}$ of an inch.

21. The diameter of the pin shall not be less than two-thirds the largest dimension of any tension member attached to it. Its effective length shall not be greater than the breadth of the foot of the post plus four times the diameter of the pin. The several members attaching to the pin shall be packed close together, and all vacant spaces between the chords and posts must be filled with wrought-iron filling-rings.

Upset screw ends.

22. All rods and hangers with screw ends shall be upset at the ends, so that the diameter at the bottom of the threads shall be $\frac{1}{16}$ inch larger than any part of the body of the bar.

23. All threads must be of the United States standard, except at the ends of the pins.

Floor beam hangers.

24. Floor beam hangers shall be so placed that they can be readily examined at all times. When fitted with screw ends, they shall be provided with check nuts.

25. When bent loops are used, they must fit perfectly around the pin throughout its semi-circumference.

Compression members.

26. Compression members shall be of wrought-iron of approved forms.

27. The pitch of rivets, for a length of two diameters at the ends, shall not be over four times the diameter of the rivets.

28. The open sides of all trough-shaped sections shall be stayed by diagonal lattice work at distances not exceeding the width of the member. The size of bars shall be duly proportioned to the width.

29. All pin holes shall be re-enforced by additional material, so as not to exceed the allowed pressure on the pins. These re-enforcing plates must contain enough rivets to transfer the proportion of pressure which comes upon them.

30. Pin holes shall be bored exactly perpendicular to a vertical plane passing through the centre line of each member, when placed in a position similar to that it is to occupy in the finished structure.

Abutting joints.

31. The ends of all square-ended members shall be planed smooth, and exactly square to the centre line of strain.

32. All members must be free from twists or bends. Portions exposed to view shall be neatly finished.

Splicing of top chord.

33. The sections of the top chord shall be connected at the abutting ends by splices sufficient to hold them truly in position.

Lateral bracing.

34. In no case shall any lateral or diagonal rod have a less area than $\frac{3}{4}$ of a square inch.

35. The attachment of the lateral system to the chords shall be thoroughly efficient. If connected to suspended floor beams, the latter shall be stayed against all motion.

Transverse diagonal bracing.

36. All through bridges with top lateral bracing shall have wrought-iron portals of approved design at each end of the span, connected rigidly to the end posts.

37. When the height of the trusses exceeds 25 feet, overhead diagonal bracing shall be attached to each post and to the top lateral struts.

38. Pony trusses and through-plate or lattice girders shall be stayed by knee braces or gusset plates attached to the top chords, at the ends, and at intermediate points not more than 10 feet apart, and attached below to the cross floor beams or to the transverse struts.

In all deck bridges, diagonal bracing shall be provided at each panel. In double-track bridges, this bracing shall be proportioned to resist the unequal loading of the trusses. The diagonal bracing at the ends shall be of the same equivalent strength as the end top lateral bracing.

Bed plates. 39. All bed plates must be of such dimensions that the greatest pressure upon the masonry shall not exceed 250 pounds per square inch.

Friction rollers. 40. All bridges over 50 feet span shall have at one end nests of turned friction rollers formed of wrought-iron, running between planed surfaces. The rollers shall not be less than 2 inches diameter, and shall be so proportioned that the pressure per lineal inch of rollers shall not exceed the product of the square root of the diameter of the roller, in inches, multiplied by 500 pounds ($500\sqrt{d}$).

41. Bridges less than 50 feet span will be secured at one end to the masonry, and the other end shall be free to move by sliding upon planed surfaces.

Camber. 42. All bridges will be given a camber by making the panel lengths of the top chord longer than those of the bottom chord in the proportion of $\frac{1}{8}$ of an inch to every 10 feet.

QUALITY OF IRON.

1. All wrought-iron must be tough, fibrous, and uniform in character. It shall have a limit of elasticity of not less than 26,000 pounds per square inch.

Finished bars must be thoroughly welded during the rolling, and free from injurious seams, blisters, buckles, cinder spots, or imperfect edges.

2. For all tension members, the muck bars shall be rolled into flats, and again cut, piled, and rolled into finished sizes. They shall stand the following tests: —

Tension tests. 3. Full-sized pieces of flat, round, or square iron, not over $4\frac{1}{2}$ inches in sectional area, shall have an ultimate strength of 50,000 pounds per square inch, and stretch $12\frac{1}{2}$ per cent in their whole length.

Bars of a larger sectional area than $4\frac{1}{2}$ square inches, when tested in the usual way, will be allowed a reduction of 1,000 pounds per square inch for each additional square inch of section, down to a minimum of 46,000 pounds per square inch.

4. When tested in specimens of uniform sectional area of at least $\frac{1}{2}$ square inch for a distance of 10 inches taken from tension members which have been rolled to a section not more than $4\frac{1}{2}$ square inches, the iron shall show an ultimate strength of 52,000 pounds per square inch, and stretch 18 per cent in a distance of 8 inches.

Specimens taken from bars of a larger cross-section than $4\frac{1}{2}$ inches will be allowed a reduction of 500 pounds for each additional square inch of section, down to a minimum of 50,000 pounds.

5. The same-sized specimen taken from *angle* and other shaped iron shall have an ultimate strength of 50,000 pounds per square inch, and elongate 15 per cent in 8 inches.

6. The same-sized specimen taken from *plate* iron shall have an ultimate strength of 48,000 pounds, and elongate 15 per cent in 8 inches.

7. All iron for tension members must bend cold, for about 90 degrees, to a curve whose diameter is not over twice the thickness of the piece, without cracking. At least one sample in three must bend 180 degrees to this curve without cracking. When nicked on one side, and bent by a blow from a sledge, the fracture must be nearly all fibrous, showing but few crystalline specks. **Bending-tests.**

8. Specimens from *angle*, *plate*, and *shaped* iron must stand bending cold through 90 degrees, and to a curve whose diameter is not over three times its thickness, without cracking.

When nicked or bent, its fracture must be mostly fibrous.

9. Rivets and pins shall be made from the best double-refined iron.

10. The cast-iron must be of the best quality of soft gray iron. **Cast-iron.**

11. All facilities for inspection of iron and workmanship shall be furnished by the contractor. He shall furnish, without charge, such specimens (prepared) of the several kinds of iron to be used as may be required to determine their character. **Tests.**

12. Full-sized parts of the structure may be tested at the option of the chief engineer of the railroad company; but, if tested to destruction, such material shall be paid for at cost, less its scrap value, to the contractor, if it proves satisfactory. If it does not stand the specified tests, it will be considered rejected material, and be solely at the cost of the contractor.

WORKMANSHIP.

1. All workmanship shall be first-class in every particular.

2. Abutting joints in truss bridges shall be in exact contact throughout.

3. Bars which are to be placed side by side in the structure shall be bored at the same temperature, and of such equal length, that, upon being piled on each other, the pins shall pass through the holes at both ends without driving.

4. Whenever necessary for the protection of the thread, provision shall be made for the use of pilot nuts, in erection.

PAINTING.

1. All work shall be painted at the shop with one good coat of selected iron-ore paint and pure linseed-oil.

2. In riveted work, all surfaces coming in contact shall be painted before being riveted together.

Bed plates, the inside of closed sections, and all parts of the work which will not be accessible for painting after erection, shall have two coats of paint.

3. Pins, bored pin holes, and turned friction rollers shall be coated with white lead and tallow before being shipped from the shop.

4. After the structure is erected, the ironwork shall be thoroughly and evenly painted with two additional coats of paint mixed with pure linseed-oil, of such color as may be directed; the tension members being, however, generally of lighter color than the compression members.

ERECTION.

1. The railroad company will take down the old bridge if any exists. It will furnish the lower falseworks, or supporting-trestles, only. The use of these falseworks by the contractor shall be construed as his approval of them.

2. The contractor shall furnish all other staging (the plan and construction of which must be approved by the chief engineer), and shall erect and adjust all the ironwork complete.

3. The contractor shall so conduct all his operations as not to impede the running of the trains or the operations of the road.

4. The contractor shall assume all risks of accidents to men or material during the manufacture and erection of the bridge.

ADDITIONAL STIPULATIONS.

The above specifications are approved.

Chief Engineer N. Y., L. E., & W. R.R.

The above specifications are accepted.

Contractor.

These specifications will be modified to suit the character of the bridges here considered.

Section 2.

The Brunel Girder of Single System.

149. Take the form of girder shown in Fig. 16, Class I., article 49, where the end lengths of top chord are shorter than other segments. Bridge to have 2 equal parabolic double-bow girders, top and bottom chords of the same curvature. Floor carrying load attached by vertical struts and suspenders to bottom and top panel points, and in the plane of the axes of the two girders.

To compute the dimensions, let l = span in feet, h = height of the two equal parabolas composing each girder at the centre of span. Take the axis of x horizontal, that of y vertical; then the equation to the upper parabola, origin at top, is

$$x^2 = ay.$$

But, with origin at left end, the equation is

$$y = \frac{2h}{l}\left(x - \frac{x^2}{l}\right). \tag{472}$$

If there are n equal panels, then the value of y at the r^{th} panel point, where $x = \frac{rl}{n}$, becomes

$$y = \frac{2h}{n^2}r(n-r); \tag{473}$$

and the whole height is

$$2y = \frac{4h}{n^2}r(n-r). \tag{474}$$

By (473),

For $r = 0$, $y = 0$;

$r = 1$, $y = 2h\frac{n-1}{n^2}$;

$r = 2$, $y = 2h\frac{2n-4}{n^2}$;

For $r = 3$, $y = 2h\frac{3n-9}{n^2}$;

$r = 4$, $y = 2h\frac{4n-16}{n^2}$;

$r = 5$, $y = 2h\frac{5n-25}{n^2}$; etc.

Twice these values of y will be the heights between curves of parabolas; but, as we shall here assume each chord to be polygonal, that is, to be straight from apex to apex, the actual heights of girder at these points are manifestly

$$h_1 = y_1 + \tfrac{1}{2}(y_0 + y_2) = \frac{2h}{n^2}(2n - 3),$$

$$h_2 = y_2 + \tfrac{1}{2}(y_1 + y_3) = \frac{2h}{n^2}(4n - 9),$$

$$h_3 = y_3 + \tfrac{1}{2}(y_2 + y_4) = \frac{2h}{n^2}(6n - 19),$$

. . . .

$$h_r = y_r + \tfrac{1}{2}(y_{r-1} + y_{r+1}) = \frac{2h}{n^2}e. \tag{475}$$

VALUES OF e IN (475).

r	$n=4$	6	8	10	12	14	16	18	20	22	24
1	5	9	13	17	21	25	29	33	37	41	45
2	7	15	23	31	39	47	55	63	71	79	87
3		17	29	41	53	65	77	89	101	113	125
4			31	47	63	79	95	111	127	143	159
5				49	69	89	109	129	149	169	189
6					71	95	119	143	167	191	215
7						97	125	153	181	209	237
8							127	159	191	223	255
9								161	197	233	269
10									199	239	279
11										241	285
12											287

Also we have

$$\frac{1}{\cos^2 \alpha_1} = 1 + \frac{4h^2}{n^2 l^2}(n - 1)^2, \quad \frac{1}{\cos^2 \alpha_3} = 1 + \frac{4h^2}{n^2 l^2}(n - 8)^2;$$

$$\frac{1}{\cos^2 \alpha_2} = 1 + \frac{4h^2}{n^2 l^2}(n - 4)^2, \quad \frac{1}{\cos^2 \alpha_4} = 1 + \frac{4h^2}{n^2 l^2}(n - 12)^2;$$

and generally

$$\frac{1}{\cos^2 \alpha_r} = 1 + \frac{4h^2}{n^2 l^2} \varepsilon_1^2. \qquad (476)$$

VALUES OF ε_1 IN (476).

r	$n = 4$	6	8	10	12	14	16	18	20	22	24
1	3	5	7	9	11	13	15	17	19	21	23
2	0	2	4	6	8	10	12	14	16	18	20
3			0	2	4	6	8	10	12	14	16
4					0	2	4	6	8	10	12
5							0	2	4	6	8
6									0	2	4

In the same manner, we derive

$$\frac{1}{\cos^2 \beta_1} = 1 + \frac{4h^2}{n^2 l^2}(n - 2)^2, \quad \frac{1}{\cos^2 \beta_3} = 1 + \frac{4h^2}{n^2 l^2}(n - 10)^2;$$

$$\frac{1}{\cos^2 \beta_2} = 1 + \frac{4h^2}{n^2 l^2}(n - 6)^2, \quad \frac{1}{\cos^2 \beta_4} = 1 + \frac{4h^2}{n^2 l^2}(n - 14)^2;$$

$$\frac{1}{\cos^2 \beta_r} = 1 + \frac{4h^2}{n l^2} \varepsilon_2^2. \qquad (477)$$

VALUES OF ε_2 IN (477).

r	$n = 4$	6	8	10	12	14	16	18	20	22	24
1	2	4	6	8	10	12	14	16	18	20	22
2		0	2	4	6	8	10	12	14	16	18
3				0	2	4	6	8	10	12	14
4						0	2	4	6	8	10
5								0	2	4	6
6										0	2

$$\frac{1}{\cos^2\theta_1} = 1 + \frac{4h^2}{n^2l^2}(3n - 5)^2, \quad \frac{1}{\cos^2\theta_3} = 1 + \frac{4h^2}{n^2l^2}(11n - 61)^2;$$

$$\frac{1}{\cos^2\theta_2} = 1 + \frac{4h^2}{n^2l^2}(7n - 25)^2, \quad \frac{1}{\cos^2\theta_4} = 1 + \frac{4h^2}{n^2l^2}(15n - 113)^2;$$

$$\frac{1}{\cos^2\theta_r} = 1 + \frac{4h^2}{n^2l^2}\varepsilon_3^2. \tag{478}$$

VALUES OF ε_3 IN (478).

r	$n=4$	6	8	10	12	14	16	18	20	22	24
1	7	13	19	25	31	37	43	49	55	61	67
2	3	17	31	45	59	73	87	101	115	129	143
3		5	27	49	71	93	115	137	159	181	203
4			7	37	67	97	127	157	187	217	247
5				9	47	85	123	161	199	237	275
6					11	57	103	149	195	241	287
7						13	67	121	175	229	283
8							15	77	139	201	263
9								17	87	157	227
10									19	97	175
11										21	107
12											23

α denoting slope of any segment of top chord,
β denoting slope of any segment of bottom chord,
θ denoting slope of any Z web member, as shown in Fig. 16.

$\dfrac{l}{n\cos\alpha}$ = length of end segment of top chord.

$\dfrac{2l}{n\cos\alpha}$ = length of any other segment of top chord.

$\dfrac{2l}{n\cos\beta}$ = length of any segment of bottom chord.

$\dfrac{l}{n\cos\theta}$ = length of any Z member.

Take $q_1 = 18$ feet = extreme width of bridge.

$q = 16$ feet = width of floor.

ϕ_1 = inclination to plane of girder, of any horizontal diagonal.

Then

$$\frac{1}{\sin^2\phi_1} = 1 + \frac{l^2}{18^2 n^2}. \qquad (479)$$

$\dfrac{18}{\sin\phi_1}$ = length of horizontal diagonal supposed to reach from end to end of the transverse I floor beams.

150. To compute the Moments and Strains in Chords due to the Total Panel Weight, $W + L$, applied at Each Apex, Top and Bottom. — We have, from equation (65), moments at the vertical sections through these apices,

$$M_r = \frac{W + L}{2n} l(n - r)r = \frac{W + L}{2n} l e_4. \qquad (480)$$

VALUES OF e_4 IN (480).

r	$n = 4$	6	8	10	12	14	16	18	20	22	24
1	3	5	7	9	11	13	15	17	19	21	23
2	4	8	12	16	20	24	28	32	36	40	44
3		9	15	21	27	33	39	45	51	57	63
4			16	24	32	40	48	56	64	72	80
5				25	35	45	55	65	75	85	95
6					36	48	60	72	84	96	108
7						49	63	77	91	105	119
8							64	80	96	112	128
9								81	99	117	135
10									100	120	140
11										121	143
12											144

Therefore, since $H^r = \dfrac{M_r}{h_r}$, equations (475) and (480) give

$$H = \frac{W + L}{h} \times \frac{ln}{4} \times \frac{\varepsilon_4}{\varepsilon}, \tag{481}$$

which is the horizontal component of the greatest strain in each chord, at a point directly above or below an opposite apex. Hence, in the present case:—

Strains in Top Chord.

$$P_1 = \frac{H_1}{\cos \alpha_1}, \quad P_2 = \frac{H_2}{\cos \alpha_2}, \quad P_3 = \frac{H_4}{\cos \alpha_3}, \quad P_4 = \frac{H_6}{\cos \alpha_4}, \text{ etc.} \tag{482}$$

Strains in Bottom Chord.

$$U_1 = \frac{H_1}{\cos \beta_1}, \quad U_2 = \frac{H_3}{\cos \beta_2}, \quad U_3 = \frac{H_5}{\cos \beta_3}, \quad U_4 = \frac{H_7}{\cos \beta_4}, \text{ etc.} \tag{483}$$

Values of $\dfrac{\varepsilon_4}{\varepsilon} = 0.5 + \dfrac{1}{2\varepsilon}$, in (481).

r	$n = 4$	6	8	10	12	14	16	18	20	22	24
1	0.600000	0.555555	0.538462	0.529412	0.523810	0.520000	0.517241	0.515152	0.513514	0.512195	0.511111
2	0.571429	0.533333	0.521739	0.516129	0.512821	0.510638	0.509091	0.507937	0.507042	0.506329	0.505747
3		0.529412	0.517241	0.512195	0.509434	0.507692	0.506494	0.505618	0.504950	0.504425	0.504000
4			0.516129	0.510638	0.507937	0.506329	0.505263	0.504505	0.503937	0.503497	0.503145
5				0.510204	0.507246	0.505618	0.504587	0.503876	0.503356	0.502959	0.502646
6					0.507042	0.505263	0.504202	0.503497	0.502994	0.502618	0.502326
7						0.505155	0.504000	0.503268	0.502762	0.502392	0.502110
8							0.503937	0.503145	0.502618	0.502242	0.501961
9								0.503115	0.502538	0.502146	0.501859
10									0.502513	0.502092	0.501792
11										0.502075	0.501754
12											0.501742

Owing to the peculiar form $\left(0.5 + \frac{1}{2\varepsilon}\right)$, these values are easily written from a table of reciprocals.

151. Weights of these Wrought-Iron Chords.

W = unknown panel weight of bridge.
L = given panel weight of live load.
Q = allowed inch strain in top chord.
T = allowed inch strain in bottom chord, all in tons, say.
$\frac{P}{Q}$ = cross-section of top chord, square inches.
$\frac{U}{T}$ = cross-section of bottom chord, square inches.

VOLUME OF SEGMENTS OF TOP CHORD, CUBIC INCHES.

$$\frac{12lH_1}{Qn\cos^2\alpha_1}, \quad \frac{24lH_2}{Qn\cos^2\alpha_2}, \quad \frac{24lH_4}{Qn\cos^2\alpha_3}, \quad \frac{24lH_6}{Qn\cos^2\alpha_4}, \text{ etc.}$$

VOLUME OF SEGMENTS OF BOTTOM CHORD, CUBIC INCHES.

$$\frac{24lH_1}{Tn\cos^2\beta_1}, \quad \frac{24lH_3}{Tn\cos^2\beta_2}, \quad \frac{24lH_5}{Tn\cos^2\beta_3}, \quad \frac{24lH_7}{Tn\cos^2\beta_4}, \text{ etc.}$$

$$\left.\begin{array}{l}\text{Weight of top chords,}\\ \text{in pounds,}\end{array}\right\} = \frac{5}{18} \times \frac{24l}{nQ}\Sigma\frac{H}{\cos^2\alpha}$$

$$= \frac{5}{18} \times \frac{24l}{nQ} \times \frac{W+L}{h} \times \frac{ln}{4}\Sigma_{\varepsilon}^{\varepsilon_4}\left(1 + \frac{4h^2}{n^2l^2}\varepsilon_1^2\right)$$

$$= \frac{5}{3} \times \frac{W+L}{Qh}\left(l^2\Sigma_{\varepsilon}^{\varepsilon_4} + \frac{4h^2}{n^2}\Sigma_{\varepsilon}^{\varepsilon_4}\varepsilon_1^2\right), \quad (484)$$

by reason of (476) and (481).

In summing (484), it will be seen that only one of the two extreme panels is to be counted.

VALUES OF $\frac{\varepsilon_4}{\varepsilon}$ TO BE USED IN (484); THE BRACE INCLUDING NUMBERS TO BE TAKEN TWICE IN SUMMING.

r	$n = 4$	6	8	10	12	14	16	18	20	22	24
1	0.600000	0.555555	0.538462	0.529412	0.523810	0.520000	0.517241	0.515152	0.513514	0.512195	0.511111
2	0.571429	0.533333	0.521739	0.516129	0.512821	0.510638	0.509091	0.507937	0.507042	0.506329	0.505747
4			0.516129	0.510638	0.507937	0.506329	0.505263	0.504505	0.503937	0 503497	0.503145
6					0.507042	0.505263	0.504202	0.503497	0.502994	0.502618	0.502326
8							0.503937	0.503145	0.502618	0.502242	0.501961
10									0.502513	0.502092	0.501792
12											0.501742
Σ	1.171429	1.622222	2.098069	2.582946	3.072368	3.564460	4.058290	4.553320	5.049209	5.545751	6.042795

VALUES OF $\frac{\varepsilon_4}{\varepsilon}\varepsilon_1{}^2$ TO BE USED IN SUMMING (484).

r	$n = 4$	6	8	10	12	14	16	18	20	22	24
1	5.4000	13.8889	26.3846	42.8824	63.3810	87.8800	116.3792	148.8788	185.3786	225 8780	270.3780
2	0	2.1333	8.3478	18.5806	32.8205	51.0638	73.3091	99.5556	129.8028	164.0506	202 2990
4			0	2.0426	8.1270	18.2278	32.3368	50 4505	72.5669	98.6854	128 8050
6					0	2.0211	8.0672	18.1259	32.1916	50.2618	72.3350
8							0	2.0126	8.0419	18.0807	32.1255
10									0	2.0084	8.0277
12											0
Σ	5.4000	18.1555	43.0803	84.1288	145.2760	230.5054	343.8054	489.1680	670.5850	892.0518	1157 5624

We shall here assume that the ratio of the panel length of top chord to its least radius of gyration is 100, and that sizes of iron can be exactly fitted to meet the required strains. We have, then, for the segments of the top chord, by our specifications for columns with flat ends,

$$Q = \frac{4}{1 + \frac{100^2}{40000}} = 3.2 \text{ tons per square inch of section.}$$

These values of $\sum\frac{\varepsilon_4}{\varepsilon}$, $\sum\frac{\varepsilon_4}{\varepsilon}\varepsilon_1^2$, and Q, put in (484), give,

		n
Weight of top chords, in pounds, $\Bigr\} = \dfrac{W + L}{h}$	$(0.610120l^2 + 0.70313h^2)$	4
	$(0.844908l^2 + 1.05067h^2)$	6
	$(1.092744l^2 + 1.40235h^2)$	8
	$(1.345284l^2 + 1.75268h^2)$	10
	$(1.600192l^2 + 2.10179h^2)$	12
	$(1.856490l^2 + 2.45010h^2)$	14
	$(2.113692l^2 + 2.79790h^2)$	16
	$(2.371521l^2 + 3.14537h^2)$	18
	$(2.629797l^2 + 3.49263h^2)$	20
	$(2.888413l^2 + 3.83976h^2)$	22
	$(3.147290l^2 + 4.18679h^2)$	24

Similarly we find, from (477) and (481),

$$\left.\begin{array}{l}\text{Weight of bottom chords,}\\ \quad\text{in pounds,}\end{array}\right\} = \frac{5}{18} \times \frac{24l}{nT}\sum\frac{H}{\cos^2\beta}$$

$$= \frac{5}{3}\,\frac{W + L}{Th}\left(l^2\sum\frac{\varepsilon_4}{\varepsilon} + \frac{4h^2}{n^2}\sum\frac{\varepsilon_4}{\varepsilon}\varepsilon_2^2\right), \qquad (485)$$

to be summed as follows :—

VALUES OF $\frac{\varepsilon_4}{\varepsilon}$ IN (485). BRACE INCLUDES NUMBERS TO BE TAKEN TWICE.

r	$n = 4$	6	8	10	12	14	16	18	20	22	24
1	0.600000	0.555555	0.538462	0.529412	0.523810	0.520000	0.517241	0.515152	0.513514	0.512195	0.511111
3	.	0.529412	0.517241	0.512195	0.509434	0.507692	0.506494	0.505618	0.504950	0.504425	0.504000
5				0.510204	0.507246	0.505618	0.504587	0.503876	0.503356	0.502959	0.502646
7						0.505155	0.504000	0.503268	0.502762	0.502392	0.502110
9								0.503115	0.502538	0.502146	0.501859
11										0.502075	0.501754
Σ	1.200000	1.640523	2.111406	2.593418	3.080980	3.571775	4.064644	4.558943	5.054240	5.550309	6.046960

VALUES OF $\frac{\varepsilon_4}{\varepsilon}\varepsilon_2{}^2$ IN (485) TO BE USED TWICE.

r	$n = 4$	6	8	10	12	14	16	18	20	22	24
1	2.4000	8.8888	19.3846	33.8824	52.3810	74.8800	101.3792	131.8788	166.3785	204.8780	247.3778
3		0	2.0690	8.1951	18.3396	32.4923	50.6494	72.8090	98.9702	129.1328	163.2690
5				0	2.0290	8.0899	18.1651	32.2481	50.3356	72.4261	98.5186
7						0	2.0160	8.0523	18.0994	32.1531	50.2110
9								0	2.0101	8.0343	18.0669
11										0	2.0070
Σ	4.8000	17.7778	42.9072	84.1550	145.4992	230.9844	344.4194	489.9764	671.5876	893.2486	1158.9006

Since $T = 5$ tons, we have, from (485),

		n
Weight of bottom chords, in pounds, $= \dfrac{W+L}{h}$	$(0.400000l^2 + 0.40000h^2)$	4
	$(0.546841l^2 + 0.65844h^2)$	6
	$(0.703802l^2 + 0.89390h^2)$	8
	$(0.864473l^2 + 1.12207h^2)$	10
	$(1.026993l^2 + 1.34722h^2)$	12
	$(1.190592l^2 + 1.57091h^2)$	14
	$(1.354881l^2 + 1.79385h^2)$	16
	$(1.519648l^2 + 2.01636h^2)$	18
	$(1.684746l^2 + 2.23863h^2)$	20
	$(1.850103l^2 + 2.46074h^2)$	22
	$(2.015653l^2 + 2.68264h^2)$	24

152. **To find the Greatest Strains and the Weights in the Web System.** — Calling $L = 0$ in (481), and taking first differences, we find horizontal component of strain in any girder diagonal due to dead load, nW, thus:

$$\Delta H_W = \frac{W}{h}\frac{ln}{4}\Delta\frac{e_4}{e}. \tag{486}$$

VALUES OF $\Delta\frac{\varepsilon_4}{\varepsilon}$ TO BE USED IN (486).

$r_{x+1}-r_x$	$n=4$	6	8	10	12	14	16	18	20	22	24
2– 1	−0.028571	−0.022222	−0.016723	−0.013283	−0.010989	−0.009362	−0.008150	−0.007215	−0.006472	−0.005866	−0.005364
4– 3		+0.003921	−0.001112	−0.001557	−0.001497	−0.001363	−0.001231	−0.001113	−0.001013	−0.000928	−0.000855
6– 5			+0.004498	+0.000434	−0.000204	−0.000355	−0.000385	−0.000379	−0.000362	−0.000341	−0.000320
8– 7				+0.003934	+0.000691	+0.000108	−0.000063	−0.000123	−0.000144	−0.000150	−0.000149
10– 9					+0.003387	+0.000711	+0.000202	+0.000030	−0.000025	−0.000054	−0.000067
12–11						+0.002946	+0.000676	+0.000229	+0.000080	+0.000017	−0.000012
14–13							+0.002597	+0.000629	+0.000232	+0.000096	+0.000038
16–15								+0.002319	+0.000581	+0.000226	+0.000102
18–17									+0.002092	+0.000538	+0.000216
20–19										+0.001904	+0.000499
22–21											+0.001747
Σ	−0.028571	−0.018301	−0.013337	−0.010472	−0.008612	−0.007315	−0.006354	−0.005623	−0.005031	−0.004558	−0.004165

These values of $\Delta\frac{\varepsilon_4}{\varepsilon}$ are the alternate first differences; the only ones required in this calculation, since the whole set of Z diagonals, Fig. 16, is strained equally with the whole set of Y diagonals under the same uniform dead load and same live load supposed to pass either way. And, moreover, we shall compute only weights of diagonals due to the greatest compressive strains developed in them; since, although they are alternately in compression and tension under live load, the allowed unit strain in tension, 5 tons per square inch, is so much greater than that allowed in compression, that, if a diagonal can resist the compression, it certainly can resist the tension coming upon it.

We shall, however, augment the cross-section of all girder diagonals by eight-tenths of the section computed to resist the compressive strain, though not quite in accord with our general specifications. Also, each girder diagonal is to be rigidly connected, at or near its centre, with the floor system, so that virtually the unsupported length of these struts is but half of what it otherwise would be. In these ways we guard against lateral shocks received by the girder diagonals.

It may be noted here that the differences, $\Delta\frac{\varepsilon_4}{\varepsilon}$, in (486), would all vanish if both chords coincided with the parabolic curve throughout. Also, these differences, being negative, increase the counter strains, and diminish the main strains, due live load, contrary to what results when the chords are horizontal.

For the live load, nL, using the values of h_r in (475), and taking M_r from equation (64), we have

$$(H_L)_r = \frac{M_r}{h_r} = \frac{\frac{Ll}{2n^2}r(r+1)(n-r)}{\frac{2h}{n^2}\varepsilon_r} = \frac{L}{h} \times \frac{l}{4}\varepsilon_5 \qquad (487)$$

if $\varepsilon_5 = \dfrac{r(r+1)(n-r)}{\varepsilon_r}$.

$(H_l)_r$ = the horizontal component of chord strain at the foremost end of live load, due the same.

Also, from (68) and (475), we have

$$(H_L)_{r+1} = \frac{M_{r+1}}{h_{r+1}} = \frac{\frac{Ll}{2n^2}r(r+1)(n-r-1)}{\frac{2h}{n^2}e_{r+1}} = \frac{L}{h} \times \frac{l}{4}e_6 \qquad (488)$$

if $e_6 = \dfrac{r(r+1)(n-r-1)}{e_{r+1}}$.

$(H_L)_{r+1}$ = horizontal component of chord strain due live load at one interval before its foremost end, and is simultaneous with $(H_L)_r$ in (487);

$$\therefore \quad \Delta H_L = \frac{L}{h} \times \frac{l}{4}(e_6 - e_5), \qquad (489)$$

which is the horizontal component of greatest strain on diagonals due live load alone, and made positive, and added to ΔH, also made positive, gives the total horizontal component of maximum diagonal strain, as thus expressed:

$$\Delta H = \frac{l}{4h}\left\{L(e_5 - e_6) - nW\Delta\frac{e_4}{e}\right\}. \qquad (490)$$

Here also we require only the alternate values of e_5 and e_6, that is, those values which correspond to the instants when the foremost panel weight of live load is directly under the upper end of the Z member.

VALUES OF ε_5 AND ε_6 IN (487), (488), AND (490).

n	4		6		8		10		12		14		16		18		20		22		24	
r	ε_5	ε_6	ε_5	ε_6	ε_5	ε_6	ε_5	ε_6	ε_5	ε_6	ε_5	ε_6	ε_5	ε_6	ε_5	ε_6	ε_5	ε_6	ε_5	ε_6	ε_5	ε_6
1	$\frac{6}{5}$	$\frac{4}{7}$	$\frac{10}{9}$	$\frac{8}{15}$	$\frac{14}{13}$	$\frac{12}{23}$	$\frac{18}{17}$	$\frac{16}{31}$	$\frac{22}{21}$	$\frac{20}{39}$	$\frac{26}{25}$	$\frac{24}{47}$	$\frac{30}{29}$	$\frac{28}{55}$	$\frac{34}{33}$	$\frac{32}{63}$	$\frac{38}{37}$	$\frac{36}{71}$	$\frac{42}{41}$	$\frac{40}{79}$	$\frac{46}{45}$	$\frac{44}{87}$
3			$\frac{36}{17}$	$\frac{24}{15}$	$\frac{60}{29}$	$\frac{48}{31}$	$\frac{84}{41}$	$\frac{72}{47}$	$\frac{108}{53}$	$\frac{96}{63}$	$\frac{132}{65}$	$\frac{120}{79}$	$\frac{156}{77}$	$\frac{144}{95}$	$\frac{180}{89}$	$\frac{168}{111}$	$\frac{204}{101}$	$\frac{192}{127}$	$\frac{228}{113}$	$\frac{216}{143}$	$\frac{252}{125}$	$\frac{240}{159}$
5					$\frac{90}{29}$	$\frac{60}{23}$	$\frac{150}{49}$	$\frac{120}{47}$	$\frac{210}{69}$	$\frac{180}{71}$	$\frac{270}{89}$	$\frac{240}{95}$	$\frac{330}{109}$	$\frac{300}{119}$	$\frac{390}{129}$	$\frac{360}{143}$	$\frac{450}{149}$	$\frac{420}{167}$	$\frac{510}{169}$	$\frac{480}{191}$	$\frac{570}{189}$	$\frac{540}{215}$
7							$\frac{168}{41}$	$\frac{112}{31}$	$\frac{280}{69}$	$\frac{224}{63}$	$\frac{392}{97}$	$\frac{336}{95}$	$\frac{504}{125}$	$\frac{448}{127}$	$\frac{616}{153}$	$\frac{560}{159}$	$\frac{728}{181}$	$\frac{672}{191}$	$\frac{840}{209}$	$\frac{784}{223}$	$\frac{952}{237}$	$\frac{896}{255}$
9									$\frac{270}{53}$	$\frac{180}{39}$	$\frac{450}{89}$	$\frac{360}{79}$	$\frac{630}{125}$	$\frac{540}{119}$	$\frac{810}{161}$	$\frac{720}{159}$	$\frac{990}{197}$	$\frac{900}{199}$	$\frac{1170}{233}$	$\frac{1080}{239}$	$\frac{1350}{269}$	$\frac{1260}{279}$
11											$\frac{396}{65}$	$\frac{264}{47}$	$\frac{660}{109}$	$\frac{528}{95}$	$\frac{924}{153}$	$\frac{792}{143}$	$\frac{1188}{197}$	$\frac{1056}{191}$	$\frac{1452}{241}$	$\frac{1320}{239}$	$\frac{1716}{285}$	$\frac{1584}{287}$
13													$\frac{546}{77}$	$\frac{364}{55}$	$\frac{910}{129}$	$\frac{728}{111}$	$\frac{1274}{181}$	$\frac{1092}{167}$	$\frac{1638}{233}$	$\frac{1456}{223}$	$\frac{2002}{285}$	$\frac{1820}{279}$
15															$\frac{720}{89}$	$\frac{480}{63}$	$\frac{1200}{149}$	$\frac{960}{127}$	$\frac{1680}{209}$	$\frac{1440}{191}$	$\frac{2160}{269}$	$\frac{1920}{255}$
17																	$\frac{918}{101}$	$\frac{612}{71}$	$\frac{1530}{169}$	$\frac{1224}{143}$	$\frac{2142}{237}$	$\frac{1836}{215}$
19																			$\frac{1440}{113}$	$\frac{760}{79}$	$\frac{1900}{189}$	$\frac{1520}{159}$
21																					$\frac{1386}{125}$	$\frac{924}{87}$

VALUES OF $e_5 - e_6$ IN (490).

r	$n=4$	6	8	10	12	14	16	18	20	22	24
1	0.62857	0.57778	0.55518	0.54269	0.53480	0.52936	0.52539	0.52236	0.51999	0.51806	0.51647
3		0.51765	0.52058	0.51686	0.51393	0.51179	0.51018	0.50896	0.50799	0.50721	0.50657
5			0.49475	0.50803	0.50827	0.50739	0.50651	0.50578	0.50516	0.50466	0.50424
7				0.48466	0.50242	0.50440	0.50444	0.50413	0.50378	0.50344	0.50315
9					0.47895	0.49923	0.50218	0.50276	0.50277	0.50263	0.50246
11						0.47529	0.49716	0.50076	0.50167	0.50189	0.50189
13							0.47273	0.49570	0.49975	0.50089	0.50127
15								0.47084	0.49463	0.49901	0.50033
17									0.46939	0.49381	0.49844
19										0.46825	0.49317
21											0.46731
Σ	0.62857	1.09543	1.57051	2.05224	2.53837	3.02746	3.51859	4.01129	4.50511	4.99985	5.49530

Since the girder diagonals are to have their end bearings on the chord pins, and are to have their centres attached to the floor system, we shall take the ratio of whole length of diagonal to least radius of gyration = 100, and the allowed inch strain

$$Q_1 = \frac{4}{1 + \frac{100^2}{20000}} = \frac{8}{3} \text{ tons.}$$

We have, then, after multiplying by 1.8, as above explained,

Cross-section of 1 diagonal $= \dfrac{1.8\Delta H}{Q_1 \cos\theta}$ square inches.

Volume of 1 diagonal $= \dfrac{12l}{n} \times \dfrac{1.8\Delta H}{Q_1 \cos^2\theta}$ cubic inches.

Weight of 1 diagonal $= \dfrac{5}{18} \times \dfrac{12l}{n} \times \dfrac{1.8\Delta H}{Q_1 \cos^2\theta}$ pounds.

$$\left.\begin{array}{l}\text{Weight of all girder diag-}\\ \quad\text{onals, in pounds,}\end{array}\right\} = 2 \times \frac{5}{18} \times \frac{12l}{n} \times \frac{3 \times 1.8}{8}\Sigma\frac{\Delta H}{\cos^2\theta}$$

$$= \frac{L}{h}\left\{\frac{9l^2}{8n}\Sigma(e_5 - e_6) + \frac{9h^2}{2n^3}\Sigma e_3^2(e_5 - e_6)\right\}$$

$$- \frac{W}{h}\left\{\frac{9l^2}{8}\Sigma\Delta\frac{e_4}{e} + \frac{9h^2}{2n^2}\Sigma e_3^2\Delta\frac{e_4}{e}\right\}, \quad (491)$$

from (478) and (490).

VALUES OF $\varepsilon_3^2(\varepsilon_5 - \varepsilon_6)$ IN (491).

r	$n = 4$	6	8	10	12	14	16	18	20	22	24
1	30.800	97.645	200.420	339.181	513.943	724.694	971.446	1254.186	1572.969	1927.701	2318 434
3		149.601	500.277	1046.641	1788.990	2727.330	3861.552	5191.901	6718.168	8440.482	10358.850
5			360.673	1219.780	2562.189	4388.416	6698.595	9492.985	12770.950	16533.166	20779.226
7				663.500	2255.363	4745.900	8136.113	12426.300	17616.683	23706.486	30696.678
9					1058.001	3606.937	7597.481	13032.042	19910.195	28232.224	37998.538
11						1544.217	5274.370	11117.373	19076.002	29150.273	41340.178
13							2122.085	7257.544	15304.844	26267.172	40146.213
15								2791.610	9556.746	20160 503	34607.326
17									3552.813	12171.923	25684.115
19										4405.764	15103.331
21											5350 232
Σ	30.800	247.246	1061.370	3269.102	8178.486	17737.494	34661.642	62563.941	106079.370	170995.694	264383.121

VALUES OF $\varepsilon_3^2\Delta\frac{\varepsilon_4}{\varepsilon}$ IN (491).

r	$n = 4$	6	8	10	12	14	16	18	20	22	24
1	−1.4000	−3.7555	−6.0370	−8.3019	−10.5604	−12.8166	−15 0694	−17.3232	−19.5778	−21.8274	−24.0790
3		+1.1332	−1.0686	−3.1529	− 5.2111	− 7.2634	− 9.3174	·11.3537	−13.3969	−15.4428	−17.4839
5			+3.2790	+1.0420	− 1.0284	− 3.0704	− 5.0916	− 7.1135	− 9.1517	−11.1715	−13.1869
7				+5.3856	+ 3.1019	+ 1.0162	− 1.0161	− 3.0318	− 5.0355	− 7 0633	− 9.0903
9					+ 7.4819	+ 5.1370	+ 3.0561	+ 0.7776	− 0.9900	− 3 0331	− 5.0669
11						+ 9.5716	+ 7.1717	+ 5.0840	+ 3.0420	+ 0.9874	− 0.9884
13							+11.6579	+ 9.2092	+ 7.1050	+ 5.0343	+ 3.0434
15								+13.7493	+11.2255	+ 9.1306	+ 7.0552
17									+15.8343	+13.2612	+11.1303
19										+17.9147	+15.2819
21											+20.0014
Σ	−1.4000	−2.6223	−3.8266	−5.0272	6.2161	− 7.4256	− 8.6088	−10.0021	−10.9451	12.2099	−13.3832

Placing the values of $\Sigma(\varepsilon_5 - \varepsilon_6)$, $\Sigma\varepsilon_3{}^2(\varepsilon_5 - \varepsilon_6)$, $\Sigma\Delta\frac{\varepsilon_4}{\varepsilon}$, and $\Sigma\varepsilon_3{}^2\Delta\frac{\varepsilon_4}{\varepsilon}$, in (491), we find

Weight of girder diagonals, in pounds,

				n
$= \frac{W}{h}$	$(0.032142l^2 + 0.39375h^2)$	$+ \frac{L}{h}$	$(0.176785l^2 + 2.16563h^2)$	4
	$(0.020589l^2 + 0.32779h^2)$		$(0.205393l^2 + 5.15096h^2)$	6
	$(0.015004l^2 + 0.26906h^2)$		$(0.220852l^2 + 9.32845h^2)$	8
	$(0.011781l^2 + 0.22622h^2)$		$(0.230877l^2 + 14.71096h^2)$	10
	$(0.009688l^2 + 0.19425h^2)$		$(0.237972l^2 + 21.29814h^2)$	12
	$(0.008229l^2 + 0.17048h^2)$		$(0.243278l^2 + 29.08846h^2)$	14
	$(0.007148l^2 + 0.15133h^2)$		$(0.247401l^2 + 38.08042h^2)$	16
	$(0.006326l^2 + 0.13892h^2)$		$(0.250705l^2 + 48.27465h^2)$	18
	$(0.005660l^2 + 0.12313h^2)$		$(0.253412l^2 + 59.66964h^2)$	20
	$(0.005128l^2 + 0.11352h^2)$		$(0.255654l^2 + 72.26527h^2)$	22
	$(0.004686l^2 + 0.10456h^2)$		$(0.257592l^2 + 86.06221h^2)$	24

Weight of girders without head system, in pounds,

				n
$= \frac{W}{h}$	$(1.042262l^2 + 1.49688h^2)$	$+ \frac{L}{h}$	$(1.186905l^2 + 3.26876h^2)$	4
	$(1.412338l^2 + 2.03690h^2)$		$(1.597142l^2 + 6.86007h^2)$	6
	$(1.811550l^2 + 2.56531h^2)$		$(2.017398l^2 + 11.62470h^2)$	8
	$(2.221543l^2 + 3.10097h^2)$		$(2.440634l^2 + 17.58571h^2)$	10
	$(2.636873l^2 + 3.64326h^2)$		$(2.865157l^2 + 24.74715h^2)$	12
	$(3.055311l^2 + 4.19149h^2)$		$(3.290360l^2 + 33.10947h^2)$	14
	$(3.475721l^2 + 4.74308h^2)$		$(3.715974l^2 + 42.67217h^2)$	16
	$(3.897495l^2 + 5.30065h^2)$		$(4.141874l^2 + 53.43638h^2)$	18
	$(4.320203l^2 + 5.85439h^2)$		$(4.567955l^2 + 65.40090h^2)$	20
	$(4.743644l^2 + 6.41402h^2)$		$(4.994170l^2 + 78.56577h^2)$	22
	$(5.167629l^2 + 6.97399h^2)$		$(5.420535l^2 + 92.93164h^2)$	24

153. The floor; to be made of $2\frac{1}{2}$-inch oak planks weighing 52 pounds per cubic foot. Width $= q = 16$ feet in (432).

$$\therefore \quad \text{Weight of floor} = F = \frac{2.5}{12} \times 16 \times 52l = \tfrac{520}{3}l \text{ pounds.} \qquad (492)$$

154. The joists of oak; longitudinal, spaced 2 feet between centres, and consequently 9 in number, all of equal size.

By (431) and (432), we have

$$\text{Depth of joists} = b^2 = \left\{\frac{9 \times 10 \times 2l}{16 \times 10000n^2}(F + 2000nL)\right\}^{\frac{2}{5}} \text{ inches,} \qquad (493)$$

calling $B = 10{,}000$ pounds = breaking inch strain for oak, $f = 10$ = factor of safety, $g = 2$ feet = distance between centres of joists.

And, from (433),

$$\text{Weight of 9 joists} = J = \frac{9 \times 52l}{144}\left\{0.001125\frac{l}{n^2}(F + 2000nL)\right\}^{\frac{3}{5}}$$

$$= \tfrac{13}{4}l\left\{0.001125\frac{l}{n^2}(F + 2000nL)\right\}^{\frac{3}{5}} \text{ pounds.} \qquad (494)$$

155. The wrought-iron I floor beams, $n - 1$ in number (single or in pairs, according to live load, as shown below), each bearing 1 panel weight of live load, of floor, and of joists, besides its own weight, which is provided for as explained in article 124.

Also, the I-beams must bear the longitudinal strain due to wind pressure, and to initial strain on the horizontal diagonals inserted between them in each panel.

Taking the proportions of the beam's section as in article 124, and calling $\dfrac{B}{f} = 10{,}000$ pounds = allowed inch strain for wrought-iron beams, (412) gives

$$\text{Depth of I-beam} = d_2 = 3.80122\left\{\frac{(F + J + 2000nL)18}{10000n}\right\}^{\frac{1}{3}}, \qquad (495)$$

in inches.

Cross-section of I-beam, from (413), is

$$S = \frac{107}{1200} d_2{}^2 \text{ square inches.}$$

$$\left.\begin{array}{l}\text{Weight of } (n-1) \\ \text{I-beams, due to} \\ \text{load, pounds,}\end{array}\right\} = P = 15.46068 \times \tfrac{5}{18}$$

$$\times 18(n-1)\left\{\frac{18(F+J+2000nL)}{10000n}\right\}^{\frac{2}{3}}$$

$$= 77.3034(n-1)\left\{\frac{18(F+J+2000nL)}{10000n}\right\}^{\frac{2}{3}}, \quad (496)$$

by reason of (414).

156. Or, in order to avoid the complicated expressions for weight of joists in terms of L and l, we may proceed as follows: —

By equation (408),

$$\text{Weight of floor} = F = ulqt \text{ pounds,}$$

$$\therefore \quad uqt\frac{l}{n} + 2000L \quad = \text{panel weight of floor and live load,}$$

$$\frac{g}{q}\left(uqt\frac{l}{n} + 2000L\right) = \text{uniform load on each panel length of joist.}$$

Putting this weight for lw in equation (52), we have, for each panel length of joist,

$$\text{Moment due floor and live load, } M = \frac{1}{8}\frac{g}{q}\left(uqt\frac{l}{n} + 2000L\right)\frac{12l}{n}$$
$$= \tfrac{1}{6}B_1bd^2, \quad (497)$$

by (161); B_1 being the allowed inch strain.

Now we will take $bd^2 = \frac{l}{n}S$;

$$\therefore \quad d = \frac{l}{n}$$

(d in inches, l in feet),

$$\therefore \quad S = \frac{n}{l} bd^2 \text{ square inches.} \qquad (498)$$

That is, the cross-section, S, of a joist is taken equal to $\frac{n}{l}$ times its breadth, b, multiplied by the square of its depth, d. Then, calling $g = 2$ feet, $q = 16$ feet, $u = 52$ pounds, $t = \frac{2.5}{12}$ feet, $B_1 = 1{,}000$ pounds per square inch for oak, equation (497) becomes, after reducing,

$$S = 0.001125\left(\frac{520}{3}\frac{l}{n} + 2000L\right) \text{ square inches,} \qquad (499)$$

$$\therefore \quad J = \frac{9 \times 52 l}{144} S = 0.00365625\left(\frac{520}{3}\frac{l}{n} + 2000L\right)l \text{ pounds.} \qquad (500)$$

$J = 7.3125Ll + l^2$	$\frac{0.63375}{n}$ pound,	n
	0.1584375 pound,	4
	0.1056250 pound,	6
	0.0792187 pound,	8
	0.0633750 pound,	10
	0.0528125 pound,	12
	0.0452679 pound,	14
	0.0396094 pound,	16
	0.0352083 pound,	18
	0.0316875 pound,	20
	0.0288068 pound,	22
	0.0264063 pound,	24

which are the weights of the 9 joists for oak.

157. If, instead of wood, we use wrought-iron I-beams to support the floor and load, we may assume the beam's cross-section to have such proportions (see manufacturers' tables) that

$$I = \frac{l}{10n} Sd, \qquad (501)$$

where I = moment of inertia of section; S = area of section, in square inches; d = depth of beam, in inches; l = length of bridge, in feet.

Then, from equations (52) and (187), we have moments of external and internal forces,

$$M = \frac{1}{8} \times \frac{g}{q}\left(\frac{F + 2000nL}{n}\right) \times \frac{12l}{n} = \frac{2B_1 I}{d} = \tfrac{1}{6}B_1\frac{l}{n}S;$$

$$\left.\begin{aligned} \therefore\ \frac{I}{d} &= \frac{M}{2B_1}, \text{ to be used with makers' tables,} \\ S &= 0.00015\left(\frac{520}{3}\frac{l}{n} + 2000L\right) \text{ square inches,} \end{aligned}\right\} \quad (502)$$

if $g = 3.2$ feet, $q = 16$ feet, $B_1 = 10{,}000$ pounds per square inch.

Weight of 6 wrought-iron longitudinal I-beams, in pounds, $\Big\} = J_1 = 6 \times \frac{5}{18} \times 12lS.$ (503)

$J_1 = 6Ll + l^2$		n
	$\frac{0.52}{n}$ pound,	n
	0.1300000 pound,	4
	0.0866666 pound,	6
	0.0650000 pound,	8
	0.0520000 pound,	10
	0.0433333 pound,	12
	0.0371429 pound,	14
	0.0325000 pound,	16
	0.0288889 pound,	18
	0.0260000 pound,	20
	0.0236364 pound,	22
	0.0216667 pound.	24

158. For the transverse I-beams supporting the longitudinal I-beams, the floor, and load, we have on each, exclusive of its own weight,

$$\frac{F + J_1 + 2000nL}{n} \text{ pounds.}$$

From (52) and (187),

$$M = \frac{1}{8}\left(\frac{F + J_1 + 2000nL}{n}\right)12q_1 = \frac{2B_1 I}{d}. \qquad (504)$$

Take $B_1 = 10{,}000$ pounds per square inch.
$q_1 = 18$ feet $=$ entire length of beam.

We shall assume the cross-section of each transverse I-beam to be such that

$$I = 2Sd \qquad (505)$$

whether the beam be rolled or made up of plate and angle iron.

Therefore, from (504) and (505),

$$S = 0.000675\left(\frac{520}{3}\frac{l}{n} + \frac{6Ll}{n} + \frac{0.52l^2}{n^2} + 2000L\right). \qquad (506)$$

Weight of $(n - 1)$ transverse I-beams due load, in pounds,

$$= (n - 1) \times \tfrac{5}{18} \times 12 \times 18S$$

$$= 0.0405(n - 1)\left(\frac{520l}{3n} + \frac{0.52l^2}{n^2} + \frac{6Ll}{n} + 2000L\right) \qquad (507)$$

					n
$= l$	$5.2650 + l^2$	$0.003949 + Ll$	$0.18225 + L$	243	4
	5.8500	0.002925	0.20250	405	6
	6.1425	0.002303	0.21262	567	8
	6.3180	0.001895	0.21870	729	10
	6.4350	0.001609	0.22275	891	12
	6.5186	0.001397	0.22564	1053	14
	6.5812	0.001234	0.22781	1215	16
	6.6300	0.001105	0.22950	1377	18
	6.6690	0.001000	0.23085	1539	20
	6.7009	0.000914	0.23195	1701	22
	6.7275	0.000841	0.23287	1863	24

159. In order that the wind pressure may be a function of the height, h, of the girders, as it manifestly ought to be, we will assume, for wind pressure against these highway bridges, 50 pounds per square foot of actual vertical surface presented by both girders, estimated at $\frac{100}{l} \times \frac{h}{2}$ square feet, to the running-foot of bridge. Therefore

$$\text{Wind pressure per linear foot} \quad = 2500\frac{h}{l} \text{ pounds.}$$

$$\text{Wind pressure per panel length} = W_1 = 2500\frac{h}{n} \text{ pounds.}$$

No account is here taken of vertical wind force.

Let the strains due this horizontal force of wind be provided for along the floor system which is placed midway between the top and bottom chords (that is, insert horizontal diagonals in each panel between the transverse I-beams), and increase the cross-section already found for these transverse beams by an amount required by the wind force; and let the longitudinal strain due wind be taken up by 2 longitudinal wrought-iron bars, or channels, extending the entire length of bridge, securely attached to the outside of the outer longitudinal floor beams, to the top of every transverse beam, and to each girder diagonal, as already explained.

These two longitudinal bars or channels, and the two outside floor beams, are to be made continuous throughout, and capable of resisting either tension or compression. It will be noticed, that the floor and load use only one-half the capacity of these two outside longitudinal floor beams; also, that all floor beams, longitudinal and transverse, are, from the manner of their loading, unable to deflect horizontally.

160. The Horizontal Diagonals, $2n$ in Number.—To provide for travelling gusts of wind, we shall here assume that

this panel pressure, $W_1 = 2{,}500\frac{h}{n}$ pounds, is a uniform live load. Therefore the strains upon the diagonals are given in Fig. 112 if we put W_1 in the place of L, and make $W = 0$.

But, since we must provide for this wind pressure coming upon either side of the bridge, it is plain that all horizontal diagonals must be "mains," and the two in any panel equal in size. We must, therefore, take $\frac{n}{2}$ terms of the following series four times:

Strain on horizontal diagonals due wind

$$= \frac{W_1}{2n\sin\phi_1}\left\{n(n-1), (n-1)(n-2), (n-2)(n-3), \ldots \frac{n}{2}\text{terms}\right\}, \quad (508)$$

in same denomination as W_1.

Take the inch strain $T_1 = 15{,}000$ pounds.

Weight of horizontal diagonals, pounds,

$$= 4 \times \frac{12q_1}{\sin\phi_1} \times m \times \frac{W_1}{2nT_1\sin\phi_1}\left\{\begin{array}{r} n(n-1) + (n-1)(n-2) \\ + (n-2)(n-3) \\ + (n-3)(n-4) \\ \ldots \frac{n}{2}\text{ terms} \end{array}\right\}$$

$$= \frac{24mq_1W_1}{nT_1\sin^2\phi_1}\left(\tfrac{7}{24}n^3 - \tfrac{1}{6}n\right)$$

$$= \tfrac{5}{8}h\left(7n - \frac{4}{n}\right)\left(1 + \frac{l^2}{324n^2}\right), \quad (509)$$

since $m = \frac{5}{18}$ pounds per cubic inch for wrought-iron, $q_1 = 18$ feet = length of transverse I-beams.

$$\frac{1}{\sin^2\phi_1} = 1 + \left(\frac{l}{nq_1}\right)^2,$$

by (479).

From (509), we find

Weight of horizontal diagonals, pounds,

$= h$	22.500	$+ hl^2$ 0.0043403	n = 4
	34.444	0.0029531	6
	46.250	0.0022304	8
	58.000	0.0017901	10
	69.722	0.0014944	12
	81.429	0.0012823	14
	93.125	0.0011225	16
	104.815	0.0009985	18
	116.500	0.0008989	20
	128.182	0.0008174	22
	139.861	0.0007494	24

161. The Horizontal Struts; that is, in this Case, the Quantity of Iron to be added to the Transverse I-Beams by Reason of Wind Pressure. — If we divide the terms of equation (508) by 15,000, we shall have the cross-sections of the horizontal diagonals in square inches. And, if each of these sections be multiplied by $10{,}000 \sin \phi_{,,}$ the product will be the longitudinal pressure brought upon the end of any transverse I-beam by one horizontal diagonal. Now, by our specifications, the pressure so brought upon these horizontal struts by the horizontal diagonals attached to the end of each is the end pressure to be provided for in these struts or I-beams. We therefore have

Longitudinal pressure upon end of transverse I-beams

$$= \frac{W_1}{3n}\left\{\left[\begin{matrix} n(n-1) \\ +(n-1)(n-2)\end{matrix}\right], \left[\begin{matrix}(n-1)(n-2) \\ (n-2)(n-3)\end{matrix}\right], \text{etc.}\right\}$$

$$= \frac{2W_1}{3n}[(n-1)^2, (n-2)^2, (n-3)^2, \text{etc.}]. \quad (510)$$

These I-beams being unable to deflect horizontally, and having considerable depth, we may take for them, under this wind pressure, the unit strain,

$$Q_2 = \frac{8000}{1 + \dfrac{(12q_1)^2}{20000 \times \dfrac{2.5l}{n}}} = \frac{8000}{1 + 0.93312\dfrac{n}{l}} \text{ pounds per square inch,}$$

where $\frac{2.5l}{n}$ is put for the square of the radius of gyration about an axis normal to web of beam.

Hence the areas of sections to be added to the I-beams, by reason of wind, are

$$S = \frac{2W_1}{3nQ_2}[(n-1)^2, (n-2)^2, (n-3)^2, \text{etc.}]. \quad (511)$$

Taking $m = \frac{5}{18}$, $q_1 = 18$, $W_1 = 2{,}500\frac{h}{n}$, we find

Weight of iron to be added to transverse I-beams, on account of wind, in pounds,

$$= 4 \times 12q_1 \times \frac{mW_1}{3nQ_2}\left\{\begin{matrix}(n-1)^2 + (n-2)^2 + (n-3)^2 \\ \ldots\left(\frac{n}{2} - 1\right) \text{terms} + \frac{n^2}{8}\end{matrix}\right\}$$

$$= \frac{2mq_1W_1}{3Q_2}(7n^2 - 12n + 2) \ (n \text{ even}),$$

$$= h\left(\frac{25}{24} + 0.9722\frac{n}{l}\right)\left(7n - 12 + \frac{2}{n}\right) \quad (512)$$

			n
$= h$	18.188 $+ \frac{h}{l}$	64.15	4
	31.597	176.91	6
	46.111	344.09	8
	60.625	565.71	10
	75.173	841.75	12
	89.733	1172.18	14
	104.297	1557.15	16
	118.866	1996.49	18
	133.438	2490.27	20
	148.008	3038.47	22
	162.587	3641.11	24

162. The Chords required in the Horizontal System to resist Wind Force. — The strains generated in these chords by the panel pressure of wind, W_{11}, are given in Fig. 112 if, for $N = \frac{W + L}{2nh}l$, we put $N = \frac{W_1 l}{2nq_1}$, thus:

Maxima chord strains in horizontal system

$$= \frac{W_1 l}{2nq_1}[(n-1), 2(n-2), 3(n-3), \text{etc.}], \quad (513)$$

for each chord, since the wind may act on either side of the bridge.

Now, since these strains will be sometimes in tension and sometimes in compression, these wind chords must be constructed so as to resist either kind of strain. Then, of course, a cross-section sufficient for the greatest compressive strain will be ample for the maximum tensile strain.

And, because these chords are to be securely attached to the girder diagonals, to the outside longitudinal I-beams whose strength is only one-half taxed in supporting the floor and live load, and to the transverse I-beams, we shall take

$$\frac{\text{unsupported length}}{\text{radius of gyration}} = 100,$$

as in case of the top chords, article 151; also, call the ends fixed. The axis of gyration is normal to the plane of girder, since the floor prevents these struts from deflecting horizontally. Then

$$Q = 3.2 \text{ tons} = 6400 \text{ pounds} = \text{allowed inch strain.}$$

Cross-section of wind chords for each panel

$$= \frac{W_1 l}{2nq_1 Q}\left\{n - 1, 2(n-2), 3(n-3), \ldots \frac{n}{2}\left(n - \frac{n}{2}\right)\right\}. \quad (514)$$

Weight of wind chords, in pounds,

$$= 4 \times \frac{5}{18} \times \frac{12l}{n}$$

$$\times \frac{W_1 l}{2nq_1 Q}\left\{(n-1) + 2(n-2) + 3(n-3), \ldots \frac{n}{2} \text{ terms}\right\}$$

$$= 0.006028164\left(2 + \frac{3}{n} - \frac{2}{n^2}\right)hl^2 \qquad (515)$$

$= hl^2$			
	0.0158239	$n =$	4
	0.0147353		6
	0.0141285		8
	0.0137442		10
	0.0134800		12
	0.0132866		14
	0.0131395		16
	0.0130238		18
	0.0129304		20
	0.0128534		22
	0.0127889		24

163. The vertical supports for the transverse floor beams and their load must also resist the moment due that part of the wind pressure which acts upon the chords and girder diagonals.

1st, The total weight to be upheld by each of these vertical struts and hangers is the $(2n)^{\text{th}}$ part of the sum of the weights of the live load, the floor, the longitudinal I-beams, the transverse I-beams taken $\frac{n}{n-1}$ times, the horizontal diagonals in the floor system, and the wind chords; all of which have now been found in terms of l and L. Call this weight e_n pounds on each strut and on each hanger, n being the number of panels. Each strut, of course, transmits the load, e_n, to the panel point below; while each hanger or suspender transmits the load, e_n, to the alternate panel points above.

For the struts, we may take

$$Q_3 = \frac{8000}{1 + \frac{100^2}{20000}} \doteq 5333 \text{ pounds}$$

as the allowed inch strain in compression.

$\therefore$ $\left.\begin{array}{l}\text{Cross-section of a strut due}\\ \quad\text{vertical forces}\end{array}\right\} = S = \frac{e_n}{Q_3}$ square inches. (516)

$\left.\begin{array}{l}\text{Weight of all struts due}\\ \quad\text{vertical forces}\end{array}\right\} = 2 \times \frac{5}{18} \times \frac{e_n}{Q_3} \times 12\Sigma y$

$$= 0.0002083\tfrac{1}{3}\left(n - \frac{4}{n}\right)he_n \text{ pounds,} \qquad (517)$$

since, from (473), for lower parabola,

$$\Sigma y = \frac{2h}{n^2}\left\{(2n - 4) + (4n - 16) + (6n - 36)\ldots\left(\frac{n}{2} - 1\right)\text{terms}\right\}$$

$$= \tfrac{1}{6}h\left(n - \frac{4}{n}\right).$$

Similarly, for the suspenders, take $T_1 = 6{,}000$ pounds = the allowed inch strain in tension.

$\left.\begin{array}{l}\text{Cross-section of suspender}\\ \quad\text{due vertical forces}\end{array}\right\} = S_1 = \frac{e_n}{T_1}$ square inches. (518)

$\left.\begin{array}{l}\text{Weight of all suspenders due}\\ \quad\text{vertical forces}\end{array}\right\} = 2 \times \frac{5}{18} \times \frac{e_n}{T_1} \times 12\Sigma y$

$$= 0.000\dot{1}8\dot{5}\left(n + \frac{2}{n}\right)he_n \text{ pounds,} \qquad (519)$$

since, for the upper parabola,

$$\Sigma y = \frac{2h}{n^2}\left\{(n - 1) + (3n - 9) + (5n - 25)\ldots\frac{n}{2}\text{ terms}\right\}$$

$$= \tfrac{1}{6}h\left(n + \frac{2}{n}\right).$$

2d, We shall assume, that, of the wind pressure, 125 pounds act upon each running-foot of the two top chords and of the two bottom chords, tending to rotate each girder about its longitudinal axis. We may note, that, in general, these forces acting upon one chord will be nearly balanced by the forces acting upon the other; but in certain cases a gale may strike one chord and not the other.

Acting, then, in lines normal to the plane of each girder, at each panel point or apex, is the pressure of $62.5 \times \frac{2l}{n} = 125\frac{l}{n}$ pounds, with the lever arm, y, causing a moment, at the wide end of the strut or suspender, of $125\frac{l}{n}y$ foot-pounds. We take no account here of the fact that each end segment of the top chord is only about one-half the length of any other.

Let these struts and suspenders, acting also as lateral braces to the chords where there is no lateral head system, have a breadth of effective base equal to $\frac{1}{10}y$. The broad end of the suspender is to be attached to the top chord and head lateral strut whenever it would obstruct unduly the roadway below. Otherwise, and in all cases where head laterals are wanting, the suspender has its broad end securely bolted to the transverse I-beam in the floor system.

Then, if S_2 = cross-section of the two members or flanges of each strut or suspender, we have, at the broad end of each, this equality of moments,

$$125\frac{l}{n}y = \tfrac{1}{2}S_2 \times \tfrac{1}{10}yB_1,$$

$$S_2 = \frac{2500l}{nB_1} = \frac{75}{170}\,\frac{l}{n} \text{ square inches,} \qquad (520)$$

if $B_1 = \frac{1}{2}(5{,}333 + 6{,}000) = 5{,}667$ pounds = allowed inch strain in bending.

It will be noticed that the cross-section, S_2, is uniform throughout the member if the two flanges meet at one end, as we shall assume they do, and shall illustrate in specifications.

We have, then,

Weight of verticals required to resist bending-moment due wind, in pounds,

$$= 2 \times \frac{5}{18} \times \frac{75}{170}\frac{l}{n} \times 12\Sigma y = 0.9803922\left(1 - \frac{1}{n^2}\right)hl, \quad (521)$$

since now

$$\Sigma y = \frac{2h}{n^2}[(n-1) + 2(n-2) + 3(n-3) \ldots (n-1) \text{ terms}]$$

$$= \tfrac{1}{3}h\left(n - \frac{1}{n}\right).$$

From (517), (519), and (521), we find

Weight of all vertical supports, in pounds,

$$= h\left\{e_n\left(0.000393518n - \frac{0.000462963}{n}\right) + 0.9803922\left(1 - \frac{1}{n^2}\right)l\right\} \quad (522)$$

$$= h \left| \begin{array}{l|r} & n \\ (0.001138Ll + 1.5174L + 0.000000735l^3 + 0.00002466l^2 + 0.95447l + 0.0031) & 4 \\ (0.001188Ll + 2.3765L + 0.000000673l^3 + 0.00001716l^2 + 0.99024l + 0.0081) & 6 \\ (0.001206Ll + 3.2154L + 0.000000632l^3 + 0.00001306l^2 + 1.00373l + 0.0152) & 8 \\ (0.001214Ll + 4.0464L + 0.000000604l^3 + 0.00001052l^2 + 1.01035l + 0.0244) & 10 \\ (0.001218Ll + 4.8733L + 0.000000584l^3 + 0.00000880l^2 + 1.01470l + 0.0358) & 12 \\ (0.001221Ll + 5.6980L + 0.000000570l^3 + 0.00000756l^2 + 1.01758l + 0.0494) & 14 \\ (0.001223Ll + 6.5211L + 0.000000559l^3 + 0.00000662l^2 + 1.01988l + 0.0650) & 16 \\ (0.001224Ll + 7.3434L + 0.000000550l^3 + 0.00000589l^2 + 1.02172l + 0.0828) & 18 \\ (0.001225Ll + 8.1650L + 0.000000540l^3 + 0.00000531l^2 + 1.02342l + 0.1028) & 20 \\ (0.001225Ll + 8.9861L + 0.000000537l^3 + 0.00000483l^2 + 1.02487l + 0.1249) & 22 \\ (0.001226Ll + 9.8069L + 0.000000531l^3 + 0.00000443l^2 + 1.02626l + 0.1491) & 24 \end{array} \right.$$

since, for ε_n in (522), we have

$$\begin{aligned}
\varepsilon_4 &= 0.78037Ll + 1040.5L + 0.016908l^2 + 22.5441l + 0.002521hl^2 + 5.843h + 10.69\frac{h}{l},\\
\varepsilon_6 &= 0.52025Ll + 1040.5L + 0.007515l^2 + 15.0290l + 0.001474hl^2 + 6.030h + 17.69\frac{h}{l},\\
\varepsilon_8 &= 0.39019Ll + 1040.5L + 0.004227l^2 + 11.2720l + 0.001022hl^2 + 6.184h + 24.58\frac{h}{l},\\
\varepsilon_{10} &= 0.31215Ll + 1040.5L + 0.002705l^2 + 9.0176l + 0.000777hl^2 + 6.268h + 31.43\frac{h}{l},\\
\varepsilon_{12} &= 0.26013Ll + 1040.5L + 0.001879l^2 + 7.5147l + 0.000624hl^2 + 6.322h + 38.26\frac{h}{l},\\
\varepsilon_{14} &= 0.22297Ll + 1040.5L + 0.001380l^2 + 6.4412l + 0.000520hl^2 + 6.359h + 45.08\frac{h}{l},\\
\varepsilon_{16} &= 0.19509Ll + 1040.5L + 0.001057l^2 + 5.6360l + 0.000446hl^2 + 6.387h + 51.90\frac{h}{l},\\
\varepsilon_{18} &= 0.17342Ll + 1040.5L + 0.000835l^2 + 5.0098l + 0.000389hl^2 + 6.407h + 58.71\frac{h}{l},\\
\varepsilon_{20} &= 0.15608Ll + 1040.5L + 0.000676l^2 + 4.5088l + 0.000346hl^2 + 6.424h + 65.53\frac{h}{l},\\
\varepsilon_{22} &= 0.14189Ll + 1040.5L + 0.000559l^2 + 4.0989l + 0.000311hl^2 + 6.437h + 72.34\frac{h}{l},\\
\varepsilon_{24} &= 0.13006Ll + 1040.5L + 0.000470l^2 + 3.7574l + 0.000282hl^2 + 6.448h + 79.15\frac{h}{l},
\end{aligned}$$

which, as above defined, is the weight upheld vertically by each support.

It will be observed, that, in all terms involving h^2 in the expression for weight of vertical supports, equation (522), h^2 has been replaced by $\frac{1}{8}hl$. This substitution is simply for convenience, and, being made in these small terms only, does not practically affect the accuracy of our resulting equations, while we are hereby relieved of higher powers of h than the second, in the value of W.

164. As additional security against deflection, out of the plane of the girder, by the top chord, we shall insert a system of head lateral bracing between the two top chords where the

height is sufficient. For this purpose, the two top chords are the flanges of a great longitudinal strut or column, whose tendency to deflect laterally must be overcome by this head web system of diagonals and struts.

Suppose the moment at the centre of this system be that due to $\frac{1}{4}W_1$ acting upon each panel length, or to $\frac{1}{2}W_1$ acting upon the windward side at each joint of the windward top chord; that is, by (480), where now we must put $\frac{1}{2}n$ for n, and $\frac{1}{4}n$ for r, and $2,500\frac{h}{n}$ pounds for W_{1}, we have

$$\text{Moment at centre} = M = \tfrac{1}{16}W_1 ln = 78.125hl;$$

and the longitudinal horizontal strain,

$$H = \frac{M}{q_1} = 78.125\frac{hl}{q_1} \text{ pounds at centre,}$$

which in each flange may be considered to decrease uniformly to the ends, as is practically the case with that part of the strain due to bending-moment in a pillar.

Therefore, for each double panel length,

$$\Delta H = H \times \frac{1}{\frac{1}{4}n} = 312.5\frac{hl}{nq_1};$$

requiring each diagonal tie to resist

$$\frac{312.5}{\cos\alpha\cos\phi_2} \times \frac{hl}{nq_1} \text{ pounds,}$$

and to have a cross-section

$$S = \frac{312.5hl}{15000\cos\alpha\cos\phi_2 nq_1} = \frac{0.0208\frac{1}{3}hl}{nq_1\cos\phi_2} \text{ square inches,} \qquad (523)$$

since $\cos\alpha$ may, for these central panels receiving the head system, be put $= 1$ without practical error.

$$\text{Length of each head diagonal} = \frac{2l}{n\cos\phi_2}\ \text{practically.}$$

Weight of $2\left(\frac{n}{2} - 3\right)$ wrought-iron head diagonals, in pounds,

$$= 2\left(\frac{n}{2} - 3\right) \times \frac{5}{18} \times \frac{12 \times 2l}{n\cos\phi_2} \times \frac{0.0208\frac{1}{3}hl}{18n\cos\phi_2}$$

$$= 0.007716\frac{n-6}{n^2} \times \frac{hl^2}{\cos^2\phi_2}$$

$$= 0.007716(n-6)\left(\frac{hl^2}{n^2} + 81h\right) \quad (524)$$

$= h$	$+ hl^2$	n
-	-	4
0	0	6
1.25	0.0002411	8
2.50	0.0003086	10
3.75	0.0003215	12
5.00	0.0003149	14
6.25	0.0003014	16
7.50	0.0002857	18
8.75	0.0002701	20
10.00	0.0002551	22
11.25	0.0002411	24

since

$$q_1 = 18 \text{ feet,} \quad \text{and} \quad \frac{1}{\cos^2\phi_2} = 1 + \left(\frac{18n}{2l}\right)^2 = 1 + \tan^2\phi_2.$$

If we multiply S in (523) by $2 \times 10{,}000\cos\phi_2\tan\phi_2$, we have, by our specifications, the end pressure brought by each pair of diagonals upon the end of each head strut; and, calling the inch strain in compression on these struts 2,500 pounds, we have the cross-section of each head strut, in square inches,

$$S = \frac{312.5 \times 2 \times 10000hl \tan\phi_2}{15000 \times 2500nq_1} = \frac{1}{12}h; \qquad (525)$$

h being in feet, and $\tan\phi_2 = \dfrac{nq_1}{2l}$.

Weight of $\left(\frac{n}{2} - 2\right)$ head struts, in pounds,

$$= \left(\frac{n}{2} - 2\right) \times \frac{5}{18} \times 12 \times 18 \times \frac{h}{12} = 5h\left(\frac{n}{2} - 2\right) \qquad (526)$$

		n
$= h$	0	4
	5	6
	10	8
	15	10
	20	12
	25	14
	30	16
	35	18
	40	20
	45	22
	50	24

h in feet.

Since we have already provided, in the floor system, for the whole bending-force of the wind, and are now simply stiffening the head system laterally as a column, or to meet adjustment strains, and strains due to imperfections in workmanship, it will manifestly suffice if we call the additional chord strain in each segment of top chord within the head system equal to

$$\Delta H = 312.5\frac{hl}{nq^1} = 17.36\tfrac{1}{9}\frac{hl}{n} \text{ pounds.}$$

And, as 6,400 pounds is the allowed inch strain in top chords, we have

Cross-section to be added to each segment of top chord due to strain on head diagonals, square inches,

$$= S = 0.002712267\frac{hl}{n}. \qquad (527)$$

Weight of added iron in $\left(\frac{n}{2} - 3\right)$ of the central double panels, for top chords, in pounds,

$$= 2\left(\frac{n}{2} - 3\right) \times \frac{5}{18} \times \frac{12 \times 2l}{n}S = 0.0361689\left(\frac{\frac{1}{2}n - 3}{n^2}\right)hl^2 \quad (528)$$

$= hl^2$

	n
-	4
0	6
0.0005651	8
0.0007234	10
0.0007535	12
0.0007381	14
0.0007064	16
0.0006698	18
0.0006329	20
0.0005978	22
0.0005651	24

165. To find the Necessary Amount of Material for the Triangular Web System of Latticed Struts or Columns.

Let l' = length of strut.

d = effective width of strut.

A = area of both flanges in section normal to axis of strut, in square inches.

A_1 = area of diagonals in the same section normal to axis of strut, and not to its own axis, in square inches.

$\theta = 45°$ = inclination of diagonal to axis of strut.

B_1 = allowed inch strain, both in flanges and diagonals, in the present case.

Then, moment at centre,

$$M = \tfrac{1}{8}AdB_1.$$

Longitudinal flange strain at centre $= H = \dfrac{M}{d} = \tfrac{1}{8}AB_1.$

Now, since H decreases uniformly from the centre to the ends, at least practically,

$$\Delta H = \frac{d}{\frac{1}{2}l'}H = \frac{AdB_1}{l'},$$

which is the longitudinal component of diagonal strain.

And

$$\Delta H + \cos\theta = \frac{AdB_1}{l'\cos\theta} = \text{strain on diagonal};$$

$$\therefore \quad A_1 = \frac{Ad}{l'\cos^2\theta} = \frac{2Ad}{l'}, \tag{529}$$

$$\frac{A_1}{A} = \frac{2d}{l'}, \tag{530}$$

$$\begin{aligned} &= \tfrac{1}{10} \text{ if } l' + d = 20, \\ &= \tfrac{1}{15} \text{ if } l' + d = 30, \\ &= \tfrac{1}{20} \text{ if } l' + d = 40, \\ &= \tfrac{1}{25} \text{ if } l' + d = 50. \end{aligned}$$

Since about one-half of each diagonal bar is cut away to receive its end pin, we have for use,

$$\frac{A_1}{A} = \frac{4d}{l'}, \tag{531}$$

$$= \frac{1}{5} \quad \text{if } l' \div d = 20,$$

$$= \frac{1}{7.5} \quad \text{if } l' \div d = 30,$$

$$= \frac{1}{10} \quad \text{if } l' \div d = 40,$$

$$= \frac{1}{12.5} \quad \text{if } l' \div d = 50,$$

$$= \frac{1}{15} \quad \text{if } l' \div d = 60,$$

which is the ratio of the section of the diagonal bar to that of the two flanges, the section being normal to axis of the strut in both cases.

This ratio must be doubled for square struts latticed against deflection both ways, and it becomes

$$\frac{A_1}{A} = \frac{8d}{l'}, \tag{532}$$

$$= \frac{1}{2.5} \quad \text{if } l' \div d = 20,$$

$$= \frac{1}{3.75} \quad \text{if } l' \div d = 30,$$

$$= \frac{1}{5} \quad \text{if } l' \div d = 40,$$

$$= \frac{1}{6.25} \quad \text{if } l' \div d = 50,$$

$$= \frac{1}{7.5} \quad \text{if } l' \div d = 60.$$

By reviewing our compression members, which are to be latticed in at least one direction, we find the girder diagonals

having the ratio of length to radius of gyration = 100, giving ratio of length to diameter = about 40: so that, by (531), the weight found in (491) should be augmented by one-tenth of itself. Also, the vertical supports have a mean ratio of length to width $= 2 \times 10 = 20$: so that, by (531), that part of their weight due to bending-moment, (521), should be augmented by one-fifth; or, which is approximately the same thing, the weight given in (522) is to be increased by one-tenth of itself. Similarly, we shall augment the weight of the lateral head struts, (526), by one-tenth of itself, on account of bracing.

In general, the longitudinal wind chords, being attached to the floor joists, to the transverse I-beams, and to the girder diagonals, will need diagonal bracing only when very long.

The top and bottom chords, however, though not having diagonal bracing in themselves, yet will need to have their weight, (484), (485), augmented by about one-tenth of itself, on account of the enlarged ends of I-bars, the re-enforcement of plates and rivets at joints, and the nuts and pins.

166. Weight of the Bridge. — Increasing, therefore, by one-tenth of itself, the weight of girders, of vertical supports, and of lateral head struts, and collecting all the weights which will then have been found in pounds, and expressed in terms of W, L, l, and h, for each value of n, the number of panels, and putting each sum $= 2000nW =$ total weight of bridge, in pounds also, since W, the panel weight of bridge, and L, the panel weight of uniform discontinuous live load, are in tons, we find the following values of W for the different values of n, remembering that h and l are in feet:

PARABOLIC DOUBLE BOW, OR LENTICULAR GIRDER. (SEE FIG. 16.)

$$n = 4.$$

$$W = \frac{\begin{array}{l} +h[L(6.18225l + 243) + 0.133949l^2 + 178.5983l] + 1.305595Ll^2 \\ \qquad + h^2[L(0.001252l + 5.2647) + 0.000000808l^3 + 0.02019132l^2 + 1.0499l + 40.6914 + 64.15l^{-1}] \end{array}}{-1.146488l^2 + 8000h - 1.64657h^2}, \quad (533)$$

$$= \frac{4.07998 + 1.616619h + 0.02116399h^2}{-0.286622 + 0.8h - 0.000164657h^2} \text{ if } l = nL = 50, \quad = 2.83403 \text{ tons, a minimum for } h = 15.03893.$$

$$n = 6.$$

$$W = \frac{\begin{array}{l} +h[L(6.2025l + 405) + 0.089592l^2 + 179.1833l] + 1.756856Ll^2 \\ \qquad + h^2[L(0.001307l + 10.16018) + 0.000000740l^3 + 0.0177073l^2 + 1.08926l + 66.05 + 176.91l^{-1}] \end{array}}{-1.553572l^2 + 12000h - 2.24059h^2}, \quad (534)$$

$$= \frac{3.6601166 + 1.514252h + 0.02536247h^2}{-0.388393 + 1.2h - 0.000224059h^2} \text{ if } l = nL = 50, \quad = 1.82101 \text{ tons, a minimum for } h = 13.0178.$$

$$n = 8.$$

$$W = \frac{\begin{array}{l} +h\,[L(6.21262l + 567) + 0.067303l^2 + 179.4758l] + 2.219138Ll^2 \\ \qquad + h^2[L(0.001327l + 16.32407) + 0.000000695l^3 + 0.0171795l^2 + 1.10410l + 104.628 + 344.09l^{-1}] \end{array}}{-1.992705l^2 + 16000h - 2.82184h^2}, \quad (535)$$

$$= \frac{27.73922 + 3.347388h + 0.0596678h^2}{-1.992705 + 1.6h - 0.000282184h^2} \text{ if } l = nL = 100, \quad = 3.93003 \text{ tons, a minimum for } h = 24.1921.$$

$$n = 10.$$

$$W = \frac{\begin{array}{l} +h[L(6.2187l + 729) + 0.053895l^2 + 179.651l] + 2.684697Ll^2 \\ \qquad + h^2[L(0.001335l + 23.7953) + 0.000000664l^3 + 0.016578l^2 + 1.11138l + 137.65 + 565.71l^{-1}] \end{array}}{-2.443697l^2 + 20000h - 3.41107h^2}, \quad (536)$$

$$= \frac{26.84697 + 3.201275h + 0.0660177h^2}{-2.443697 + 2h - 0.000341107h^2} \text{ if } l = nL = 100, \quad = 3.12130 \text{ tons, a minimum for } h = 22.6697.$$

$$n = 12.$$

$$W = \frac{+h[L(6.22275l + 891) + 0.044942l^2 + 179.7683l] + 3.151673Ll^2 + h^2[L(0.001339l + 32.58246) + 0.000000642l^3 + 0.0160591l^2 + 1.11619l + 170.66 + 841.75l^{-1}]}{-2.90056l^2 + 24000h - 4.00759h^2}, \quad (537)$$

$$= \frac{44.3204 + 2.53908h + 0.05584925h^2}{-3.26313 + 1.2h - 0.0002003795h^2} \text{ if } l = nL = 150, \quad = 5.23077 \text{ tons, a minimum for } h = 32.8472.$$

$$n = 14.$$

$$W = \frac{+h[L(6.22564l + 1053) + 0.0385399l^2 + 179.8519l] + 3.619396Ll^2 + h^2[L(0.0013431l + 42.68822) + 0.000000627l^3 + 0.0156302l^2 + 1.11934l + 203.716 + 1172.18l^{-1}]}{-3.360842l^2 + 28000h - 4.61064h^2}, \quad (538)$$

$$= \frac{103.41131 + 3.5171188h + 0.083867h^2}{-6.721684 + 1.4h - 0.000230532h^2} \text{ if } l = nL = 200, \quad = 7.7226 \text{ tons, a minimum for } h = 42.5850.$$

$$n = 16.$$

$$W = \frac{+h[L(6.22781l + 1215) + 0.033734l^2 + 179.9145l] + 4.087571Ll^2 + h^2[L(0.001345l + 54.1126) + 0.000000615l^3 + 0.0152773l^2 + 1.12187l + 236.74 + 1557.15l^{-1}]}{-3.823293l^2 + 32000h - 5.21739h^2}, \quad (539)$$

$$= \frac{344.8888 + 5.741154h + 0.1496117h^2}{-17.204818 + 1.6h - 0.000260869h^2} \text{ if } l = nL = 300, \quad = 15.7507 \text{ tons, a minimum for } h = 63.2967.$$

$$n = 18.$$

$$W = \frac{+h[L(6.2295l + 1377) + 0.029994l^2 + 179.9633l] + 4.5560614Ll^2 + h^2[L(0.001346l + 66.857718) + 0.000000605l^3 + 0.0149843l^2 + 1.12389l + 269.77 + 1996.49l^{-1}]}{-4.2872445l^2 + 36000h - 5.830715h^2}, \quad (540)$$

$$= \frac{1581.9658 + 11.11255h + 0.326663h^2}{-53.59056 + 1.8h - 0.00029153h^2} \text{ if } l = nL = 500, \quad = 47.96318 \text{ tons, a minimum for } h = 110.4054.$$

$$n = 20.$$

$$W = \frac{\begin{array}{l}+h[L(6.23085l + 1539) + 0.027l^2 + 180.002l] + 5.0247505Ll^2 \\ \quad +h^2[L(0.0013475l + 80.92249) + 0.000000594l^3 + 0.0147381l^2 + 1.12576l + 302.80 + 2490.27l^{-1}]\end{array}}{-4.7522233l^2 + 40000h - 6.439829h^2}, \quad (541)$$

$$= \frac{4308.7235 + 17.28761h + 0.569255h^2}{-116.42947 + 2h - 0.000321991 5h^2} \text{ if } l = nL = 700, \qquad = 110.6006 \text{ tons, a minimum for } h = 168.5605.$$

$$n = 22.$$

$$W = \frac{\begin{array}{l}+h[L(6.23195l + 1701) + 0.0245504l^2 + 180.0342l] + 5.493587Ll^2 \\ \quad +h^2[L(0.0013475l + 96.30706) + 0.0000005907l^3 + 0.014529l^2 + 1.12735l + 335.83 + 3038.47l^{-1}]\end{array}}{-5.2180084l^2 + 44000h - 7.055422h^2}, \quad (542)$$

$$= \frac{6392.5376 + 20.14436h + 0.719189h^2}{-166.97627 + 2.2h - 0.000352771h^2} \text{ if } l = nL = 800, \qquad = 151.3419 \text{ tons, a minimum for } h = 202.445.$$

$$n = 24.$$

$$W = \frac{\begin{array}{l}+h[L(6.23287l + 1863) + 0.022508l^2 + 180.0608l] + 5.962589Ll^2 \\ \quad +h^2[L(0.001349l + 113.0124) + 0.000000584l^3 + 0.0143494l^2 + 1.12888l + 368.86 + 3641.11l^{-1}]\end{array}}{-5.684392l^2 + 48000h - 7.67139h^2}, \quad (543)$$

$$= \frac{12422.0604 + 26.994836h + 1.0599919h^2}{-284.2196 + 2.4h - 0.0003835695h^2} \text{ if } l = nL = 1000, \qquad = 288.1332 \text{ tons, a minimum for } h = 283.8575,$$

$=$ infinity when $l = 2h = 3157$ feet,	$=$ infinity when $l = 10h = 853$ feet,
$=$ infinity when $l = 4h = 1947$ feet,	$=$ infinity when $l = 12h = 697$ feet,
$=$ infinity when $l = 6h = 1356$ feet,	$=$ infinity when $l = 14h = 599$ feet,
$=$ infinity when $l = 8h = 1034$ feet,	$=$ infinity when $l = 16h = 525$ feet,

which are limiting spans for $n = 24$.

In all these cases, the value of h which renders W a minimum has been found by the simple method of article 140, equations (469) and (470). The limiting spans just given have been determined by putting the denominator of (543) equal to zero, and substituting the assigned values of h. It will be seen that these limiting spans are independent of the live load, nL, and therefore represent the limit to the length of each girder imposed by its own weight. The effect of live load on the limiting span will be considered below.

167. Having found W and h, it is easy to compute the weights of all parts of the bridge from the expressions for weights in terms of W, h, L, and l. The computation affords a perfect verification of the accuracy of the work. We give below a table showing the number of panels and the height, which simultaneously render the total bridge weight, nW, a minimum for various spans ranging from 50 to 1,000 feet, and have thus probably extended the table far beyond any *economical* use of this girder.

Of course, we find great heights; but it should be remembered that one-half of this central height, h, is below the plane of the floor system, where the points of support are situated. Also the width, 18 feet, becomes too small for the highest girders; but it has been retained in this set of examples, to preserve uniformity in data.

To illustrate the change in central height and bridge weight, as the number of panels varies for the same span, we have given the solutions corresponding to three values of n, including that one which renders nW least. Also bridge weights and central heights are given for 2,000, 3,000, and 4,000 pounds of live load to the running-foot; the weights being minima values. The 30th line of this table exhibits the effect of a small live load upon the length of the limiting span, as resulting from the substitution of $\frac{1}{5}W$ for L in the equations for weight. Of course, we do not mean that the live load is small near the limit when W is infinite.

The reader cannot fail to notice how prolific in useful and interesting results these general equations for bridge weight are.

TWO DOUBLE PARABOLIC-BOW OR BRUNEL GIRDERS. HIGHWAY BRIDGE, FIG. 16.

Uniform Live and Dead Loads applied at all Apices. Height and Number of Panels yielding Minimum Bridge Weight. Single System.

1	Span	l	feet		50			100			150	
2	Uniform live load	nL	tons		50			100			150	
3	Number of panels	n		4	6	8	8	10	12	10	12	14
4	Best central height	h	feet	15.039	13.018	11.471	24.192	22.670	21.453	34.300	32.847	31.585
5	Least bridge weight	nW	tons	11.336	10.926	10.932	31.440	31.213	31.285	63.068	62.769	62.958
6	Bridge weight if $l = 10h$	nW	tons		12.69			37.37			77.57	
7	Panel length	$l \div n$	feet	$12\frac{1}{2}$	$8\frac{1}{3}$	$6\frac{1}{4}$	$12\frac{1}{2}$	10	$8\frac{1}{3}$	15	$12\frac{1}{2}$	$10\frac{5}{7}$
8	Ratio of length to height	$l \div h$			3.84			4.41			4.56	
9	Ratio of minimum dead to live load	$\frac{W}{L}$			0.219			0.312			0.418	
10	Ratio of minimum dead to total load	$\frac{W}{W+L}$			0.177			0.238			0.295	
	Weight of Parts, using Best Height and Two Thousand Pounds per Running-Foot of Span, as above, for Uniform Live Load.											
11	Top chords, Equation (484)		lbs.	1,892	1,969	2,136	8,783	9,141	9,524	22,098	22,734	23,432
12	Do. from head system, (528)		lbs.	0	0	17	137	164	162	558	550	524
13	Bottom chords, (485)		lbs.	1,225	1,271	1,374	5,653	5,872	6,112	14,797	14,588	15,028
14	Girder diagonals, (491)		lbs.	887	992	1,076	4,414	4,821	5,229	10,930	11,[illegible]22	12,426
15	Floor, (492)		lbs.	8,667	8,667	8,667	17,333	17,333	17,333	26,000	26,000	26,000
16	I-Beams, longitudinal, (503)		lbs.	4,075	2,717	2,038	8,150	6,520	5,433	14,670	12,225	10,478
17	I-Beams, transverse, (507), (512)		lbs.	3,715	4,215	4,526	9,181	9,657	10,057	14,622	15,203	15,724
18	Horizontal diagonals, (509)		lbs.	501	543	594	1,658	1,720	1,816	3,370	3,[illegible]	3,474
19	Wind chords, (515)		lbs.	594	479	405	3,416	3,112	2,891	10,604	9,[illegible]	9,512
20	Vertical supports, (522)		lbs.	1,116	999	892	3,800	3,585	3,393	8,197	7,867	7,506
21	(526)		lbs.	0	0	22	206	374	472	566	724	868
22	(524)		lbs.	0	0	115	89	127	148	324	302	382
23			lbs.	22,675	21,852	21,862	62,880	62,426	62,570	126,136	125,538	125,014
24	…unning-foot		lbs.	453	437	437	629	624	626	841	857	839
25	If live load = 3,000 lbs. per running-foot,	h	feet	16.84	14.51	12.79	27.18	25.07	23.65	37.94	36.14	34.04
26	then	nW	tons	14.04	13.44	13.40	39.37	39.28	39.35	80.23	79.77	80.17
27	If live load = 4,000 lbs. per running-foot,	h	feet	18.10	15.54	13.72	28.97	26.69	25.13	40.41	38.39	36.68
28	then	nW	tons	16.70	15.92	15.81	47.39	47.20	47.28	97.06	96.67	97.08
29	Limiting span when $L = 0$, and $l = 10h$,	l	feet	688	761	702		807	816			822
30	Limiting span when $W = 5L$, and $l = 5h$,	l	feet	901	1,024	1,083		1,115	1,132			1,142
31	Limiting span when $L = 0$, and $l = 5h$,	l	feet	1,320	1,461	1,520		1,550	1,568			1,580

Two Double Parabolic-Bow or Brunel Girders. Highway Bridge, Fig. 16.

Continued.

1	Span	l	feet	200			300			400		
2	Uniform live load	nL	tons	200			300			400		
3	Number of panels	n		**12**	**14**	**16**	**14**	**16**	**18**	**14**	**16**	**18**
4	Best central height	h	feet	43.960	42.585	41.417	64.575	63.297	62.063	87.800	86.466	85.156
5	Least bridge weight	nW	tons	108.279	108.117	108.503	252.620	252.011	252.837	494.398	491.375	491.618
6	Bridge weight if $l = 10h$	nW	tons		137.04			341.24			737.40	
7	Panel length	$l \div n$	feet	$16\frac{2}{3}$	$14\frac{2}{7}$	$12\frac{1}{2}$	$21\frac{3}{7}$	$18\frac{3}{4}$	$16\frac{2}{3}$	$28\frac{4}{7}$	25	$22\frac{2}{9}$
8	Ratio of length to height	$l \div h$			4.70			4.74			4.62	
9	Ratio of minimum dead to live load	$\frac{W}{L}$			0.541			0.840			1.228	
10	Ratio of minimum dead to total load	$\frac{W}{W+L}$			0.351			0.457			0.551	
	Weight of Parts, using Best Height and Two Thousand Pounds per Running-Foot of Span, as above, for Uniform Live Load.											
11	Top chords, Equation (484)		lbs.	43,769	44,741	45,810	119,228	120,789	122,792	252,874	254,528	257,396
12	Do. from head system, (528)		lbs.	1,325	1,257	1,171	4,289	4,024	3,741	10,369	9,770	9,126
13	Bottom chords, (485)		lbs.	28,089	28,695	29,365	76,460	77,427	78,686	162,168	163,154	164,941
14	Girder diagonals, (491)		lbs.	21,300	23,184	25,050	52,713	57,309	61,865	95,364	104,026	112,713
15	Floor, (492)		lbs.	34,667	34,667	34,667	52,000	52,000	52,000	69,333	69,333	69,333
16	I-Beams, longitudinal, (503)		lbs.	21,733	18,628	16,300	41,914	36,675	32,600	74,514	65,200	57,956
17	I-Beams, transverse, (507), (512)		lbs.	20,409	21,118	21,729	32,136	33,068	33,962	43,632	44,832	46,007
18	Horizontal diagonals, (509)		lbs.	5,692	5,652	5,711	12,710	12,289	12,082	25,162	23,584	22,529
19	Wind chords, (515)		lbs.	23,698	22,631	21,744	77,795	74,847	72,742	188,051	181,776	177,444
20	Vertical supports, (522)		lbs.	14,179	13,740	13,337	32,067	31,394	30,754	60,050	58,984	57,982
21	Head struts, (526)		lbs.	967	1,171	1,365	1,776	2,089	2,389	2,415	2,853	3,278
22	Head diagonals, (524)		lbs.	730	749	757	2,153	2,112	2,061	4,863	4,710	4,531
23	Total bridge weight		lbs.	216,558	216,233	217,006	505,241	504,023	505,674	988,795	982,750	983,236
24	Bridge weight per running-foot		lbs.	1,083	1,081	1,085	1,684	1,680	1,685	2,472	2,457	2,458
25	If live load = 3,000 lbs per running-foot,	h	feet	47.92	46.67	45.21	70.87	69.16	67.56	96.13	94.28	92.51
26	then	nW	tons	139.50	138.20	138.93	321.38	321.42	323.31	621.50	619.60	621.69
27	If live load = 4,000 lbs. per running-foot,	h	feet	50.90	49.45	47.76	75.22	73.17	71.26	102.04	99.75	97.59
28	then	nW	tons	169.18	167.68	168.80	388.50	389.33	392.26	745.09	744.62	748.79
29	Limiting span when $L = 0$, and $l = 10h$,	l	feet			826			828			
30	Limiting span when $W = 5L$, and $l = 5h$,	l	feet			1,146			1,147			
31	Limiting span when $L = 0$, and $l = 5h$,	l	feet			1,587			1,593			

TWO DOUBLE PARABOLIC-BOW OR BRUNEL GIRDERS. HIGHWAY BRIDGE, FIG. 16.

Continued.

1	Span	l	feet		500			600			700	
2	Uniform live load	nL	tons		500			600			700	
3	Number of panels	n		16	18	20	16	18	20	18	20	22
4	Best central height	h	feet	113 258	110.405	109.160	139.891	138.470	137 100	169.875	168 560	167.528
5	Least bridge weight	nW	tons	864.808	863.316	864.930	1,421.468	1,416.762	1,417.381	2,214.140	2,212.012	2,216.630
6	Bridge weight if $l = 10h$	nW	tons		1,490 35			3,126 53			7,555.47	
7	Panel length	$l \div n$	feet	31¼	27$\frac{7}{9}$	25	37½	33⅓	30	38$\frac{8}{9}$	35	31$\frac{9}{11}$
8	Ratio of length to height	$l \div h$			4.52			4.33			4.153	
9	Ratio of minimum dead to live load	$\frac{W}{L}$			1.727			2.361			3.160	
10	Ratio of minimum dead to total load	$\frac{W}{W+L}$			0.633			0.703			0.760	
	Weight of Parts, using Best Height and Two Thousand Pounds per Running-Foot of Span, as above, for Uniform Live Load.											
11	Top chords, Equation (484)		lbs.	467,529	476,345	480,776	810,368	813,582	818,835	1,313,400	1,318,709	1,325,862
12	Do from head system, (508)		lbs.	20,001	18,485	17,272	35,574	33,[illegible]	31,257	55,753	52,274	49,072
13	Bottom chords, (485)		lbs.	299,691	305,245	308,011	519,455	521,[illegible]	524,[illegible]	841,[illegible]	844,841	849,280
14	Girder diagonals, (491)		lbs.	168,985	181,763	196,[illegible]	249,871	272,[illegible]	294,559	387,4[illegible]	420,117	453,065
15	Floor, (492)		lbs	86,667	86,667	86,667	104,000	104,000	104,000	121,333	121,333	121,333
16	I Beams, longitudinal, (503)		lbs	101,875	90,555	81,500	146,700	130,400	117,360	177,48[illegible]	159,740	145,218
17	I-Beams, transverse, (507), (512)		lbs	57,293	58,588	60,046	70,024	71,[illegible]	73,540	85,[illegible]	87,737	89,910
18	Horizontal diagonals, (509)		lbs	42,335	39,131	37,750	69,5[illegible]7	64,233	6[illegible],177	100,919	93,881	88,573
19	Wind chords, (515)		lbs	372,032	359,468	35[illegible]	6[illegible]1,708	649,221	6[illegible]	1,084,07[illegible]	1,067,075	1,055,111
20	(522)		lbs	100,227	97,420	96,003	155,[illegible]	152,824	15[illegible],835	229,047	226,223	224,326
21	(526)		lbs.	3,738	4,251	4,803	4,616	5,331	[illegible],036	6,540	7,417	8,293
22	(524)		lbs	9,242	8,714	8,320	16,053	15,283	14,540	25,005	23,784	22,616
23			lbs	1,729,615	1,726,623	1,729,80[illegible]	2,842,[illegible]	2,833,524	2,834,562	4,428,27[illegible]	4,424,025	4,433,259
24	Bridge weight per running-foot		lbs.	3,459	3,453	3,460	4,7[illegible]8	4,722	4,724	6,126	6,3[illegible]	6,333
25	If live load = 3,000 lbs per running-foot,	h	feet	121 33	119 44	117 71	150 43	148 74	147 15	181 51	17[illegible] 66	177 95
26	then	nW	tons	1,074 [illegible]	1,076 30	1,08[illegible] 73	1,730 00	1,73[illegible]	1,745 99	2,675 53	2,68[illegible] 48	2,6[illegible]7 48
27	If live load = 4,000 lbs per running-foot,	h	feet	128 17	125 84	123 71	159 00	156 53	154 29	190 12	187 79	185 64
28	then	nW	tons	1,278.84	1,284.01	1,294 46	2,047.02	2,053 22	2,066 37	3,125.32	3,142.15	3,168 28
29	Limiting span when $L = 0$, and $l = 10h$,	l	feet			830						832
30	Limiting span when $W = 5L$, and $l = 5h$,	l	feet			1,147						1,144
31	Limiting span when $L = 0$, and $l = 5h$,	l	feet			1,597						1,600

Two Double Parabolic Bow or Brunel Girders. Highway Bridge, Fig. 16.

Concluded.

1	Span	l	feet	800			900			1,000		
2	Uniform live load	nL	tons	800			900			1,000		
3	Number of panels	n		18	20	22	18	20	22	20	22	24
4	Best economic height	h	feet	205.026	203.670	202.445	244.242	242.838	241.582	286.293	285.003	283.857
5	Least bridge weight	nW	tons	3,333.440	3,3[illegible]	3,329.522	4,869.957	4,854.234	4,854.533	6,911.752	6,905.906	6,915.120
6	Bridge weight if $l = 10h$	nW	tons		43,189.70			∞			∞	
7	Panel length	$l \div n$	feet	$44\frac{4}{9}$	40	$36\frac{4}{11}$	50	45	$40\frac{10}{11}$	50	$45\frac{5}{11}$	$41\frac{2}{3}$
8	Ratio of length to height	$l \div h$			3.92			3.70			3.50	
9	Ratio of minimum dead to live load	$\frac{W}{L}$			4.158			5.394			6.906	
10	Ratio of minimum dead to total load	$\frac{W}{W+L}$			0.806			0.844			0.874	
	Weight of Parts, using Best Height and Four Thousand Pounds per Running Foot of Span, as above, for Uniform Live Load											
11	Top chords, Equation (483)		lbs.	2,032,771	2,036,780	2,045,928	3,044,141	3,044,595	3,053,436	4,432,270	4,438,820	4,453,494
12	Do. from head system, (528)		lbs.	87,839	82,493	77,454	132,510	124,491	116,979	181,195	170,375	160,408
13	Bottom chords, (483)		lbs.	1,302,623	1,304,888	1,310,543	1,950,727	1,950,568	1,955,894	2,839,015	2,843,320	2,[illegible]2,328
14	Girder diagonals, (491)		lbs.	531,967	577,667	624,041	710,556	772,122	835,014	1,009,173	1,092,020	1,[illegible]75,913
15	Floor, (492)		lbs.	138,667	138,667	138,667	156,000	156,000	156,000	173,333	173,333	173,333
16	I Beams, longitudinal, (503)		lbs.	231,822	208,639	180,636	291,400	264,060	240,054	326,000	296,364	271,667
17	I Beams, transverse, (507), (512)		lbs.	100,416	102,687	105,229	115,568	118,438	121,407	135,009	138,453	142,003
18	Horizontal diagonals, (509)		lbs.	152,510	140,898	131,856	223,152	205,103	190,917	290,700	269,494	252,423
19	Wind chords, (515)		lbs.	1,708,943	1,685,404	1,665,343	2,570,570	2,543,390	2,515,176	3,701,880	3,663,250	3,630,228
20	Vertical supports, (522)		lbs.	332,153	328,100	325,233	469,535	463,703	459,897	641,900	636,720	631,200
21	Head struts, (526)		lbs.	7,893	8,962	10,021	9,403	10,685	11,959	12,597	14,108	15,612
22	Head diagonals, (524)		lbs.	39,026	36,996	35,076	58,353	55,253	52,334	79,832	75,554	71,631
23	Total bridge weight		lbs.	6,660,480	6,652,186	6,659,043	9,739,915	9,708,468	9,709,067	13,823,505	13,811,811	13,830,240
24	Bridge weight per running-foot		lbs.	8,333	8,315	8,324	10,822	10,787	10,788	13,823	13,812	13,830
25	If live load = 3,000 lbs. per running-foot,	h	feet	217.55	215.64	213.90	257.37	255.38	253.65	299.25	297.45	295.83
26	then	nW	tons	3,966.86	3,972.93	3,991.47	5,712.89	5,715.91	5,737.51	8,031.55	8,055.07	8,096.14
27	If live load = 4,000 lbs. per running-foot,	h	feet	227.00	224.59	222.39	267.48	265.01	262.77	309.31	307.03	304.96
28	then	nW	tons	4,585.33	4,605.67	4,640.24	6,537.20	6,569.48	6,604.32	9,131.09	9,185.38	9,259.46
29	Limiting span when $L = 0$, and $l = 10h$,	l	feet									833
30	Limiting span when $W = 5L$, and $l = 5h$,	l	feet									1,141
31	Limiting span when $L = 0$, and $l = 5h$,	l	feet									1,602

168. Among the inferences which may be legitimately drawn from the table of article 167 in regard to bridges having two lenticular Brunel girders of single system of the same width but of varying span, height, and uniform live load, are the following; viz. (see Fig. 116), —

1st, The best number of panels varies approximately as the cube root of the span,

$$n \propto l^{\frac{1}{3}} \text{ nearly.} \qquad (544)$$

2d, The best central height for a uniform live load of 2,000 pounds per linear foot is about $\frac{1}{4.2} \times$ span,

$$h = \frac{1}{4.2} l \text{ nearly;} \qquad (545)$$

and for different loads the best height for same span and same number of panels varies very nearly as the sixth root of the live load,

$$h \propto (nL)^{\frac{1}{6}} \text{ nearly.} \qquad (546)$$

3d, For spans less than 500 feet, the least bridge weight varies approximately with the product of the best central height of girder multiplied by the span (that is, with the geometrical area of the girder, since the parabolic area is proportional to this product);

$$l < 500, \quad nW \propto lh \text{ nearly.} \qquad (547)$$

4th, For the same span and same number of panels, and best central height of girder, the least bridge weight varies approximately with the square root of the uniform live load;

$$nW \propto (nL)^{\frac{1}{2}} \text{ nearly.} \qquad (548)$$

Or, by reason of (546), the least bridge weight for same span

and best central height of girder varies nearly as the cube of this best central height;

$$nW \propto (nL)^{\frac{1}{2}} \propto h^3 \text{ nearly.} \qquad (549)$$

5th, For each span of 500 feet and over, a large increase of live load, and consequently of best height, causes a diminution in the number of panels corresponding to minimum bridge weight.

6th, Small deviations from the best height and best number of panels, or from either of them, do not greatly affect the bridge weight; but large deviations either way, in this respect, cause a great increase in bridge weight, as shown in sixth line of table, $h = \frac{1}{10}l$, thus rendering the girders only about one-half as high as minimum bridge weight requires.

7th, The limiting span increases slowly with the number of panels, till a maximum value depending upon $\frac{W}{L}$ and $\frac{l}{h}$ is reached.

8th, We cannot, for a given span, assign the best height and the best number of panels till we know the live load which is to be imposed.

169. Example. — Take span $l = 200$ feet, number of panels $n = 14$, central height of double parabola $h = 42.585$ feet, uniform live load $nL = 200$ tons $= 400{,}000$ pounds, width of $2\frac{1}{2}$-inch oak floor $q = 16$ feet, length of transverse I floor beams $q_1 = 18$ feet. (See Fig. 16.) Loads applied at all apices equally by means of struts and suspenders which sustain the floor system in the plane of the axes of girders.

Assume that all cross-sections of members may be strictly adjusted to the developed strains.

Weight of live load	400000 pounds,
Weight of floor, article 167	34667 pounds.
Total load on longitudinal I-beams . .	434667 pounds.

Load on 1 panel length of each longitudinal I-beam spaced 3.2 feet

$$= \frac{3.2}{16} \times \frac{434667}{14} = 6209.53 \text{ pounds.}$$

By (502),

$$\text{Cross-section of beam} = S = 0.00015\left(\frac{520}{3} \times \frac{200}{14} + 2000 \times \frac{200}{14}\right)$$

$$= 4.65714 \text{ square inches.}$$

In order to satisfy the condition in (501), we must have, article 62,

$$I = \tfrac{1}{12}(bd^3 - b_1d_1^3) = \frac{l}{10n}Sd, \qquad (550)$$

$$S = bd - b_1d_1 = 4.65714; \qquad (551)$$

from which equations we find

$$O = (b^3 - bb_1^2)d^3 - 3b^2Sd^2 + \left(3bS^2 + 1.2\frac{l}{n}Sb_1^2\right)d - S^3. \qquad (552)$$

Take $b = 4.00$ inches $=$ breadth of flange.
$b - b_1 = 0.26$ inch $=$ thickness of web.
Then, from (552) and (551),

$$d = 9.080 \text{ inches} = \text{depth of beam,}$$

$$d - d_1 = 0.614 \text{ inches} = \text{depth of two flanges,}$$

$$I = 60.466 = \text{moment of inertia,}$$

$$\frac{I}{d} = \frac{1}{8} \times \frac{12l}{n} \times \frac{6209.53}{2 \times 10000} = \frac{60.466}{9.08} = 6.65,$$

by reason of (502) and (52); the load being uniformly distributed on each panel length of beam, and these beams not being regarded continuous over the transverse beams.

Weight of these 6 longitudinal I-beams, by (503), equals

$$J_1 = 6 \times \tfrac{5}{18} \times 12 \times 200 \times 4.65714 = 18628 \text{ pounds,}$$

as given in preceding table.

It will be noticed that these beams are deep and comparatively thin; but, considering their area of cross-section, it will also be noticed that their moment of inertia is great as compared with ordinary beams of equal area of section.

Supported by the transverse I-beams, we have

Live load	=	400000 pounds,
Floor	=	34667 pounds,
Longitudinal I-beams	=	18628 pounds.
Total for 14 panels	=	453295 pounds.
Load on 1 beam	=	32378 pounds.

From (504), we have

$$\frac{I}{d} = \frac{12 \times 18 \times 32378}{8 \times 2 \times 10000} = 43.7103 = 2S,$$

by (505);

$$\therefore \quad S = 21.855 \text{ square inches for vertical load.}$$

But, in order to resist the assumed wind pressure, $W_1 = 2,500\frac{h}{n} = 7,604$ pounds per panel, we must add to the cross-section due vertical load the areas found from (511), where now

$$Q_2 = \frac{8000}{1 + 0.93312 \times 0.07} = 7509 \text{ pounds per square inch;}$$

$$\therefore \quad S_1 = \frac{2 \times 7604}{3 \times 14 \times 7509}(13^2, 12^2, 11^2, 10^2, 9^2, 8^2, 7^2);$$

$= 8.149$ square inches, 1st and 13th beams;
$= 6.944$ square inches, 2d and 12th beams;
$= 5.835$ square inches, 3d and 11th beams;
$= 4.822$ square inches, 4th and 10th beams;
$= 3.906$ square inches, 5th and 9th beams;
$= 3.086$ square inches, 6th and 8th beams;
$= 2.363$ square inches, 7th beam.

TOTAL SECTIONS.

$S + S_1 =$ 30.004 square inches, 1st and 13th beams;
28.799 square inches, 2d and 12th beams;
27.690 square inches, 3d and 11th beams;
26.677 square inches, 4th and 10th beams;
25.761 square inches, 5th and 9th beams;
24.941 square inches, 6th and 8th beams;
24.218 square inches, 7th beam.

351.962 $= \Sigma S =$ sum of all sections.

To satisfy the condition, (505), we must now have

$$I = \tfrac{1}{12}(bd^3 - b_1d_1^3) = 2Sd = 2d(bd - b_1d_1); \quad (553)$$

whence, eliminating d_1, we find

$$O = b(b^2 - b_1^2)d^3 - 3b^2Sd^2 + (3bS^2 + 24b_1^2S)d - S^3, \quad (554)$$

from which d may be found for each value of total section now called S;

$$d_1 = \frac{bd - S}{b_1}.$$

Taking $b =$ 5.5 inches = width of flange,
$b - b_1 =$ 0.7 inch = thickness of web,
$b_1 =$ 4.8 inches = difference,
$S =$ 30.0 square inches = cross-section,

we find, by (554) and (553),

$d =$ 13.441 inches = depth of beam,
$d_1 =$ 9.152 inches = depth of web,
$d - d_1 =$ 4.289 inches = depth of both flanges,
$\frac{1}{2}(d - d_1) =$ 2.144 inches = depth of one flange,

$$I = 806, \quad \frac{I}{d} = 60.$$

Similarly may the proportions of the other transverse beams be found.

Or, if we choose to assume the thickness of web and of flanges, thus:

$$\left.\begin{aligned} b - b_1 &= a \text{ (say)}, \\ d - d_1 &= c \text{ (say)}, \end{aligned}\right\} \tag{555}$$

then we find, from (553),

$$O = d^3 - \tfrac{3}{2}\left(\frac{S}{a} + c\right)d^2 + \left\{(12 + \tfrac{3}{2}c)\frac{S}{a} + \frac{c^2}{2}\right\}d - \frac{c^2 S}{2a}, \tag{556}$$

from which d is easily found either by trial or by Horner's Method.

Taking $a = 0.7$, $c = 4$, $S = 30$, we find, by (556),

$$d = 13.2, \quad \therefore \quad d_1 = 9.2.$$

But $b = \dfrac{S}{c} + a - \dfrac{ad}{c}$, by (553) and (555),

$= 5.889$;

$$\therefore \quad b_1 = 5.189,$$

$$I = 792, \quad \frac{I}{d} = 60.$$

Or again, by assigning values to d and d_1 in (553), we find

$$b_1 = \frac{(24 - d)dS}{d_1(d^2 - d_1^2)}, \tag{557}$$

$$b = \frac{(24 - d)S}{d^2 - d_1^2} + \frac{S}{d}. \tag{558}$$

Using two 12-inch beams for each panel point, we have

$$d = 12, \quad d_1 = 10, \quad d - d_1 = 2,$$

$b = 5.3416$	$b_1 = 4.9098$	$b - b_1 = 0.4318$ inch.
$= 5.1272$	$= 4.7127$	$= 0.4145$ inch.
$= 4.9296$	$= 4.5311$	$= 0.3985$ inch.
$= 4.7490$	$= 4.3652$	$= 0.3838$ inch.
$= 4.5864$	$= 4.2156$	$= 0.3708$ inch.
$= 4.4404$	$= 4.0814$	$= 0.3590$ inch.
$= 4.3116$	$= 3.9629$	$= 0.3487$ inch.

Whatever be the form of beam section chosen, we have

Weight of 13 × 2 transverse I-beams

$$= 12 \times 18 \times \tfrac{5}{18}\Sigma S = 60 \times 351.962 = 21118 \text{ pounds,}$$

as per table.

Strains on the horizontal diagonals are given by (508), where now $W_1 = 7{,}604$, and $\sin\phi_1 = 0.78329$;

$$\therefore \quad \frac{W_1}{2n \times 15000 \sin\phi_1} = 0.023114.$$

$0.023114 \times 14 \times 13 = 4.206$ square inches = section of 1st diagonal,
$0.023114 \times 13 \times 12 = 3.605$ square inches = section of 2d diagonal,
$0.023114 \times 12 \times 11 = 3.051$ square inches = section of 3d diagonal,
$0.023114 \times 11 \times 10 = 2.543$ square inches = section of 4th diagonal,
$0.023114 \times 10 \times 9 = 2.080$ square inches = section of 5th diagonal,
$0.023114 \times 9 \times 8 = 1.664$ square inches = section of 6th diagonal,
$0.023114 \times 8 \times 7 = 1.295$ square inches = section of 7th diagonal.

$18.444 \times 4 = \Sigma S = 73.776$ square inches.

$$\text{Weight of 28 horizontal diagonals} = 12 \times \frac{5}{18} \times \frac{18}{\sin\phi_1} \times 73.776$$
$$= 5652 \text{ pounds,}$$

as given in the table.

The cross-section of each panel length of a wind chord is shown in (514), thus:

$$\frac{W_1 l}{2nq_1 Q} = \frac{7604 \times 200}{2 \times 14 \times 18 \times 6400} = 0.471478.$$

$0.471478 \times 13 = 6.129$ square inches, 1st panel;
$0.471478 \times 24 = 11.316$ square inches, 2d panel;
$0.471478 \times 33 = 15.559$ square inches, 3d panel;
$0.471478 \times 40 = 18.859$ square inches, 4th panel;
$0.471478 \times 45 = 21.217$ square inches, 5th panel;
$0.471478 \times 48 = 22.631$ square inches, 6th panel;
$0.471478 \times 49 = 23.103$ square inches, 7th panel.

Total, 118.814 square inches for one-half of 1 girder.

$$\therefore \text{ Weight of both wind chords} = 4 \times \frac{5}{18} \times \frac{12 \times 200}{14} \times 118.814$$

$$= 22631 \text{ pounds,}$$

as by (515).

We now have, upon all vertical supports and abutments,

Live load	400000 pounds,
Floor	34667 pounds,
Longitudinal I-beams .	18628 pounds,
Horizontal diagonals . .	5652 pounds,
Wind chords	22631 pounds.

$$\tfrac{1}{28} \times 481578 \text{ pounds} = 17199 \text{ pounds,}$$

$$+\frac{1}{26} \times \text{weight of transverse I-beams} = \frac{21118}{26} \text{ pounds} = 812 \text{ pounds.}$$

$$\text{Load on each vertical, article 163,} = e_n \qquad = 18011 \text{ pounds.}$$

Therefore, by (516),

$$S = \tfrac{18011}{5333} = 3.3771 \text{ square inches} = \text{cross-section of a strut,}$$

due vertical forces; and, by (518),

$$S_1 = \tfrac{18011}{6000} = 3.0018 \text{ square inches} = \text{cross-section of a suspender,}$$

due vertical forces.

From (520),

$$S_2 = \frac{75}{170} \times \frac{200}{n} = 6.3025 \text{ square inches} = \text{cross-section of each vertical,}$$

due bending-moment of assumed wind force;

$$\therefore \quad S + S_2 = 9.6796 \text{ square inches for each strut,}$$

$$S_1 + S_2 = 9.3043 \text{ square inches for each suspender.}$$

From (473), we have length of verticals,

	Suspenders.	Struts.	
$y = 0.43404 \times 13 =$	5.6425 feet.		$r = 1$
$0.43404 \times 24 =$		10.4170 feet.	2
$0.43404 \times 33 =$	14.3233 feet.		3
$0.43404 \times 40 =$		17.3616 feet.	4
$0.43404 \times 45 =$	19.5318 feet.		5
$0.43404 \times 48 =$		20.8340 feet.	6
$0.43404 \times 49 =$	21.2680 feet.		7
Sum required =	200.5268 feet.	194.4504 feet, for	all.

Longest strut, 20.834 feet = 250 inches;

therefore

$$\text{Required radius of gyration} = \tfrac{250}{100} = 2\tfrac{1}{2} \text{ inches.}$$

Each vertical may be made of 4 channels, 6 inches wide, each having an area of

2.4199 square inches for struts,

2.3261 square inches for suspenders,

latticed in pairs, and two pairs in one brace.

Weight of all vertical struts

$$= \tfrac{5}{18} \times 12 \times 194.4504 \times 9.679 = \quad 6273 \text{ pounds,}$$

Weight of all vertical suspenders

$$= \tfrac{5}{18} \times 12 \times 200.5268 \times 9.304 = \quad 6218 \text{ pounds.}$$

Total,	12491 pounds.
Add one-tenth for lattice braces,	1249 pounds.
	13740 pounds,

which accords with (522).

Equation (523) gives the cross-section of each head diagonal thus:

$$S = \frac{0.0208\frac{1}{3} \times 42.585 \times 200}{14 \times 18 \times 0.84609} = 0.831 \text{ square inch,}$$

which requires a round rod 1.056 inches in diameter if the ends are enlarged for cutting threads of screws.

Weight of the 8 head diagonals, by (524), is

$$8 \times \frac{5}{18} \times \frac{12 \times 2 \times 200}{14 \times 0.84609} \times 0.831 = 749 \text{ pounds.}$$

Cross-section of each head strut, by (525), is

$\frac{1}{12} \times 42.585 = 3.549$ square inches.
Add one-tenth for latticing, 0.355 square inch.

3.904 square inches.

Weight of 5 head struts, by (526)	$= 25 \times 42.585 =$	1065 pounds.
Add one-tenth for braces	=	106 pounds.
Total,		1171 pounds.

Since for these head struts we have assumed

$$Q = 2500 = \frac{8000}{1 + \frac{x^2}{20000}} \text{ pounds per square inch,}$$

$$\therefore \quad x = 210 = \frac{q_1}{\rho} = \frac{12 \times 18}{\rho},$$

$$\therefore \quad \rho = \tfrac{216}{210} = 1.03 \text{ inches} = \text{radius of gyration.}$$

We may therefore use, for each head strut, 2 4-inch channels latticed so that the web shall be 4 inches apart.

The increment of section of each top chord due to diagonal strain in head system is given by (527), thus:

$$S = 0.00271267 \times \frac{42.585 \times 200}{14} = 1.65 \text{ square inches.}$$

The total weight thus added along the 4 panel lengths of head system is

$$2 \times 4 \times \frac{5}{18} \times \frac{12 \times 2 \times 200}{14} \times 1.65 = 1257 \text{ pounds,}$$

as by (528).

The strain in the top chords is given by equations (476), (481), and (482), where now

$$W + L = 7.72265 + 14.28571 = 22.00836 \text{ tons.}$$

For the segments of each top chord, the total strains due $n(W + L)$ are

$$\begin{aligned} P_1 &= 101.15 \text{ tons,} \\ P_2 &= 96.55 \text{ tons,} \\ P_3 &= 93.10 \text{ tons,} \\ P_4 &= 91.55 \text{ tons.} \end{aligned}$$

Dividing these strains by the allowed inch strain, $Q = 3.2$ tons, we get cross-sections,

$$\left.\begin{aligned} S_1 &= 31.6075 \text{ square inches } + \\ S_2 &= 30.1704 \text{ square inches} \end{aligned}\right\} \text{for head system,}$$

$$\begin{aligned} S_3 &= 29.0937 \text{ square inches } + 1.65 \text{ square inches,} \\ S_4 &= 28.6078 \text{ square inches } + 1.65 \text{ square inches.} \end{aligned}$$

Now, the longest unsupported segment of top chord is

$$\frac{2l}{n \cos \alpha_2} = 29.405 \text{ feet} = 352.86 \text{ inches,}$$

$$\therefore \quad 3.5286 \text{ inches} = \text{radius of gyration.}$$

Therefore the top chord may be made up of 2 9-inch channels and a plate, or 2 plates 14 inches wide, and having such thickness as is required to complete the area of section.

Weight of top chords due load

$$= \frac{11}{10} \times \frac{5}{18} \times \frac{24l}{n} \Sigma \frac{S}{\cos \alpha} = 44741 \text{ pounds,}$$

1257 pounds, due head system.

Total, 45998 pounds.

Similarly, for the segments of each bottom chord, the total strains due $n(W + L)$ are, from equations (477), (481), and (483),

$$U_1 = 100.128 \text{ tons},$$
$$U_2 = 99.513 \text{ tons},$$
$$U_3 = 92.133 \text{ tons},$$
$$U_4 = 91.374 \text{ tons}.$$

These strains divided by 5, the allowed inch strain in tension, give the cross-sections of the successive segments of bottom chord in each girder,

$$S_1 = 20.0256 \text{ square inches},$$
$$S_2 = 18.9025 \text{ square inches},$$
$$S_3 = 18.4265 \text{ square inches},$$
$$S_4 = 18.2748 \text{ square inches},$$

from which the links can easily be made up according to specified forms of body and head.

No change is here made on account of longitudinal component of lateral diagonal strain, since in the present case there is no lateral system between bottom chords, by reason of gravity.

Weight of all bottom chords increased by $\frac{1}{10}$

$$= \frac{11}{10} \times \frac{5}{18} \times \frac{24l}{n} \Sigma \frac{S}{\cos \beta} = 28695 \text{ pounds}.$$

The equations (490), (491), and (478) give cross-sections of alternate girder diagonals, thus:

$$S_1 = 5.115 \text{ square inches},$$
$$S_2 = 7.212 \text{ square inches},$$
$$S_3 = 8.664 \text{ square inches},$$
$$S_4 = 8.881 \text{ square inches},$$
$$S_5 = 7.750 \text{ square inches},$$
$$S_6 = 5.132 \text{ square inches},$$

for each of the two girders, the alternate set being the same inverted; and the weight of all is, calling $Q_1 = \frac{5}{8}$, and multiplying by $1.8 \times \frac{11}{10}$, as specified,

$$4 \times \frac{5}{18} \times \frac{3 \times 1.8 \times 11}{8 \times 10} \times \frac{12 \times 200}{14}\Sigma\frac{S}{\cos\theta} = \frac{990}{7}\Sigma\frac{S}{\cos\theta}$$
$$= 23184 \text{ pounds.}$$

Now, since the longest unsupported length of any girder diagonal is

$$\frac{1}{2} \times \frac{l}{n\cos\theta_4} = 22.253 \text{ feet} = 267 \text{ inches,}$$

we have radius of gyration = 2.67 inches; and therefore 2 8-inch channels latticed 8 inches between webs will suffice for the longest diagonals.

We have thus determined the size and weight of all parts of this bridge, and find the total 216,233 pounds, as by table.

STRAIN SHEET.

STRAINS, TONS; CROSS-SECTIONS, SQUARE INCHES.

For each of Two Girders.

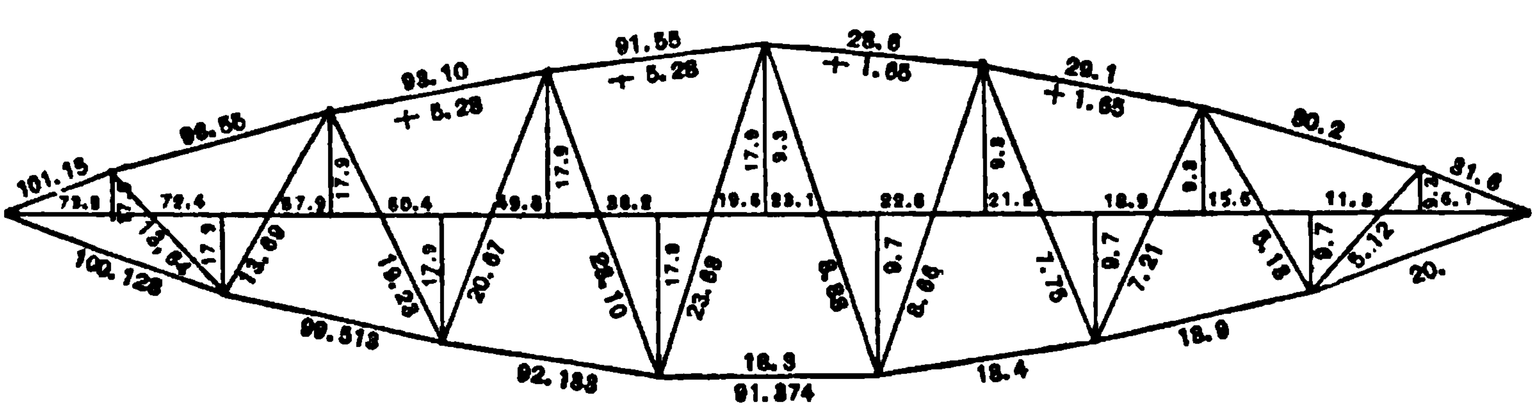


FIG. 118.

Span, 200 feet; central height, 42.585 feet (best); uniform live load, 1 ton = 2,000 pounds per linear foot, applied at all apices by vertical members.

Regarding the greatest strains upon the chord pins as acting in "quadruple shear," and allowing 6,000 pounds as the inch strain in shearing, these pins will require a diameter of $3\frac{1}{4}$ inches.

It remains to compute the deflection of this girder under the allowed chord strain of

$$B_1 = \tfrac{1}{2}(3.2 + 5) = 4.1 \text{ tons per square inch.}$$

For this purpose, use equations (318) and (319) combined thus:

$$D = \frac{B_1 a}{E h_1}\{1.386295a - 2.302585[(a + x)\log(a + x) + (a - x)\log(a - x) - 2a\log a]\}, \quad (559)$$

where $a = \frac{1}{2}l = 100$ feet = half-span, and x is measured from the centre. Also $h_1 = h = 42.585$ feet, and we will call $E = 24{,}000{,}000$ pounds $= 12{,}000$ tons per square inch.

We have then, from (559),

$$\begin{aligned}
\text{Deflection } D_1 &= 1.3347 \text{ inches for } x = 0, \text{ centre};\\
D_2 &= 1.3150 \text{ inches for } x = \tfrac{100}{7};\\
D_3 &= 1.2549 \text{ inches for } x = \tfrac{200}{7};\\
D_4 &= 1.1521 \text{ inches for } x = \tfrac{300}{7};\\
D_5 &= 1.0006 \text{ inches for } x = \tfrac{400}{7};\\
D_6 &= 0.7911 \text{ inch for } x = \tfrac{500}{7};\\
D_7 &= 0.4954 \text{ inch for } x = \tfrac{600}{7};\\
D_8 &= 0 \text{ inch for } x = \tfrac{700}{7}, \text{ ends.}
\end{aligned}$$

The proper camber may be given to the girders by equation (366), thus:

$$\lambda = \frac{3.2 + 5}{12000} \times \text{length of one parabolic chord.}$$

This length is given by equation (140); viz.,

$$S = \tfrac{1}{2}(200^2 + 4 \times 42.585^2)^{\frac{1}{2}}$$
$$+ 0.287823 \times \frac{200^2}{\frac{42.585}{2}} \log \frac{2h + (l^2 + 4h^2)^{\frac{1}{2}}}{200} = 205.5 \text{ feet,}$$

$$\therefore \quad \lambda = \frac{8.2 \times 12}{12000} \times 205.5 = 1.685 \text{ inches}$$
$$= \text{length to be added to top chord.}$$

Or, each segment should be lengthened by

$$\frac{1.685}{7} = 0.241 = \tfrac{1}{4} \text{ inch nearly.}$$

Section 3.

The Brunel Girder of Double System.

170. We now take the girder shown in Fig. 22; but we will apply the dead and live loads at all apices by means of verticals whose upper half will act in tension, and lower half in compression. These verticals must also resist bending-moment due wind. Each girder has two equal parabolic chords, and the floor system is in the plane of girders' axes; each panel length of chord is straight, and the number of panels may be odd or even; each system will be assumed to do one-half of the work.

The height between the two parabolic arcs at the centre being h, the height at any apex is given by (474). Equation (473) gives y.

$$\tan \alpha_r = -\tan \beta_r = \frac{(y_r - y_{r-1})n}{l} = \frac{2h}{nl}(n - r - r_{-1}), \qquad (560)$$

$$\frac{1}{\cos^2 \alpha_r} = \frac{1}{\cos^2 \beta_r} = 1 + \frac{4h^2}{n^2 l^2}(n - r - r_{-1})^2, \qquad (561)$$

$$\tan\phi = -\tan\theta = (y_r + y_{r+1})\frac{n}{l}$$

$$= \frac{2h}{nl}[r(n - r) + r_{+1}(n - r_{+1})], \qquad (562)$$

$$\frac{1}{\cos^2\phi} = \frac{1}{\cos^2\theta} = 1 + \frac{4h^2}{n^2l^2}[r(n - r) + r_{+1}(n - r_{+1})]^2. \qquad (563)$$

171. Moments at all apices due total dead and live uniform loads are given by (65),

$$M_r = \frac{W + L}{2n}l(n - r)r;$$

and the horizontal component of chord strain is

$$H_r = \frac{M_r}{h_r} = (W + L)\frac{nl}{8h}; \qquad (564)$$

that is, this component is uniform throughout the girder under uniform load.

Strain in top chords $= P = \dfrac{H}{\cos\alpha}$,

Strain in bottom chords $= U = \dfrac{H}{\cos\beta}$,

Cross-section of top chords $= P \div Q, \quad Q = 3.7647;$

Cross-section of bottom chords $= U \div T, \quad T = 5.0000.$

Volume of a segment of top chord is, therefore,

$$\frac{12lH}{nQ\cos^2\alpha} = \tfrac{3}{2}(W + L)\frac{l^2}{Qh\cos^2\alpha} \text{ cubic inches.} \qquad (565)$$

Weight of top chords, in pounds,

$$= \frac{3}{2}\frac{m(W + L)l^2}{Qh}\Sigma\sec^2\alpha$$

$$= \frac{3 \times 5(W + L)l^2}{2 \times 18Qh}\left\{n + \frac{4h^2}{3l^2}\left(n - \frac{1}{n}\right)\right. \qquad (566)$$

$$= \frac{W + L}{h}\left\{0.1106771nl^2 + 0.14757\left(n - \frac{1}{n}\right)h^2\right\}$$

$$= \frac{W+L}{h}$$

	n
$0.442709l^2 + 0.55338h^2$	4
$0.553386l^2 + 0.70833h^2$	5
$0.664063l^2 + 0.86082h^2$	6
$0.774740l^2 + 1.01190h^2$	7
$0.885418l^2 + 1.16211h^2$	8
$0.996095l^2 + 1.31172h^2$	9
$1.106772l^2 + 1.46093h^2$	10
$1.217449l^2 + 1.60984h^2$	11
$1.328126l^2 + 1.75853h^2$	12
$1.438804l^2 + 1.90705h^2$	13
$1.549481l^2 + 2.05542h^2$	14
$1.660158l^2 + 2.20370h^2$	15
$1.770835l^2 + 2.35188h^2$	16
$1.881512l^2 + 2.50000h^2$	17
$1.992190l^2 + 2.64804h^2$	18
$2.102867l^2 + 2.79610h^2$	19
$2.213544l^2 + 2.94400h^2$	20
$2.324221l^2 + 3.09192h^2$	21
$2.434898l^2 + 3.23981h^2$	22
$2.545576l^2 + 3.38767h^2$	23
$2.656253l^2 + 3.53554h^2$	24

Similarly, we have

Weight of bottom chords, in pounds,

$$= \frac{3m(W+L)l^2}{2Th}\Sigma \sec^2\theta = \frac{(W+L)l^2}{12h}\left\{n + \frac{4h^2}{3l^2}\left(n - \frac{1}{n}\right)\right\} \quad (567)$$

$$= \frac{W+L}{h}\left\{\tfrac{1}{12}nl^2 + \frac{1}{9}\left(n - \frac{1}{n}\right)h^2\right\}$$

$$= \frac{W+L}{h}$$

	n
$0.333333l^2 + 0.41667h^2$	4
$0.416667l^2 + 0.53333h^2$	5
$0.500000l^2 + 0.64815h^2$	6
$0.583333l^2 + 0.76191h^2$	7
$0.666667l^2 + 0.87500h^2$	8
$0.750000l^2 + 0.98765h^2$	9
$0.833333l^2 + 1.10000h^2$	10

		n
$= \dfrac{W+L}{h}$	$0.916667l^2 + 1.21212h^2$	11
	$1.000000l^2 + 1.32407h^2$	12
	$1.083333l^2 + 1.43590h^2$	13
	$1.166667l^2 + 1.54762h^2$	14
	$1.250000l^2 + 1.65926h^2$	15
	$1.333333l^2 + 1.77083h^2$	16
	$1.416667l^2 + 1.88235h^2$	17
	$1.500000l^2 + 1.99383h^2$	18
	$1.583333l^2 + 2.10526h^2$	19
	$1.666667l^2 + 2.21667h^2$	20
	$1.750000l^2 + 2.32804h^2$	21
	$1.833333l^2 + 2.43939h^2$	22
	$1.916667l^2 + 2.55072h^2$	23
	$2.000000l^2 + 2.66204h^2$	24

172. For the advancing uniform live load of $\frac{1}{2}L$ at each upper and lower apex, or of L at each vertical section through apices, we have at foremost end, by (64) and (474),

$$(H_L)_r = \frac{M_r}{h_r} = \frac{\frac{Ll}{2n^2}r(r+1)(n-r)}{\frac{4h}{n^2}r(n-r)} = \frac{Ll}{8h}(r+1), \quad (568)$$

and at one interval before the foremost end of live load, by (68) and (474),

$$(H_L)_{r+1} = \frac{M_{r+1}}{h_{r+1}} = \frac{\frac{Ll}{2n^2}r(r+1)(n-r-1)}{\frac{4h}{n^2}(r+1)(n-r-1)} = \frac{Ll}{8h}r. \quad (569)$$

Therefore

$$\Delta H = (H_L)_{r+1} - (H_L)_r = -\frac{Ll}{8h}, \quad (570)$$

which is the horizontal component of strain on both diagonals of a panel, on the present assumption that the two diagonals do equal work, and that the whole load is on one girder.

Hence, for each of two girders, we shall have

$$\text{Cross-section of a girder diagonal} = \frac{\frac{1}{4}Ll \times 1.8}{8hQ_1 \cos\theta}, \quad (571)$$

according to our specifications for members alternately in compression and tension. Putting $m = \frac{5}{18}$, $Q_1 = \frac{8}{3}$, we find

$$\text{Weight of girder diagonals, pounds,} = \frac{4 \times 12ml \times 0.45Ll}{8nhQ_1}\Sigma \sec^2\theta$$

$$= \frac{0.28125Ll^2}{h}\left\{1 - \frac{2}{n} + \frac{4h^2}{15l^2}\left(2n^2 - 5 - \frac{30}{n} + \frac{63}{n^2} - \frac{30}{n^3}\right)\right\} \quad (572)$$

$$= \frac{L}{h}\left\{0.28125\left(1 - \frac{2}{n}\right)l^2 + 0.075\left(2n^2 - 5 - \frac{30}{n} + \frac{63}{n^2} - \frac{30}{n^3}\right)h^2\right\}$$

$= \frac{L}{h}$

	n
$0.140625l^2 + 1.72266h^2$	4
$0.168750l^2 + 3.09600h^2$	5
$0.187500l^2 + 4.77083h^2$	6
$0.200892l^2 + 6.74344h^2$	7
$0.210937l^2 + 9.01318h^2$	8
$0.218750l^2 + 11.58025h^2$	9
$0.225000l^2 + 14.44500h^2$	10
$0.230113l^2 + 17.60781h^2$	11
$0.234375l^2 + 21.06901h^2$	12
$0.237980l^2 + 24.82886h^2$	13
$0.241071l^2 + 28.88759h^2$	14
$0.243750l^2 + 33.24533h^2$	15
$0.246093l^2 + 37.90228h^2$	16
$0.248161l^2 + 42.85854h^2$	17
$0.250000l^2 + 48.11420h^2$	18
$0.251644l^2 + 53.66934h^2$	19
$0.253125l^2 + 59.52403h^2$	20
$0.254463l^2 + 65.67833h^2$	21
$0.255682l^2 + 72.13228h^2$	22
$0.256803l^2 + 78.88592h^2$	23
$0.257812l^2 + 85.93179h^2$	24

All girder diagonals must be so constructed as to transmit stresses of tension or compression.

173. Collecting the weights now found for top and bottom chords and girder diagonals, we find

Weight of girders due to loads, pounds,

				n
$=\frac{W}{h}$	$0.776042l^2 + 0.97005h^2$	$+\frac{L}{h}$	$0.916667l^2 + 2.69271h^2$	4
	$0.970053l^2 + 1.24166h^2$		$1.138803l^2 + 4.33766h^2$	5
	$1.164063l^2 + 1.50897h^2$		$1.351563l^2 + 6.27980h^2$	6
	$1.358073l^2 + 1.77381h^2$		$1.558965l^2 + 8.51725h^2$	7
	$1.552085l^2 + 2.03711h^2$		$1.763022l^2 + 11.05029h^2$	8
	$1.746095l^2 + 2.29937h^2$		$1.964845l^2 + 13.87962h^2$	9
	$1.940105l^2 + 2.56093h^2$		$2.165105l^2 + 17.00593h^2$	10
	$2.134116l^2 + 2.82196h^2$		$2.364229l^2 + 20.42977h^2$	11
	$2.328126l^2 + 3.08260h^2$		$2.562501l^2 + 24.15161h^2$	12
	$2.522137l^2 + 3.34295h^2$		$2.760117l^2 + 28.17181h^2$	13
	$2.716148l^2 + 3.60304h^2$		$2.957219l^2 + 32.49063h^2$	14
	$2.910158l^2 + 3.86296h^2$		$3.153908l^2 + 37.10829h^2$	15
	$3.104168l^2 + 4.12271h^2$		$3.350261l^2 + 42.02499h^2$	16
	$3.298179l^2 + 4.38235h^2$		$3.546340l^2 + 47.24089h^2$	17
	$3.492190l^2 + 4.64187h^2$		$3.742190l^2 + 52.75607h^2$	18
	$3.686200l^2 + 4.90136h^2$		$3.937844l^2 + 58.57070h^2$	19
	$3.880211l^2 + 5.16067h^2$		$4.133336l^2 + 64.68470h^2$	20
	$4.074221l^2 + 5.41996h^2$		$4.328684l^2 + 71.09829h^2$	21
	$4.268231l^2 + 5.67920h^2$		$4.523913l^2 + 77.81148h^2$	22
	$4.462243l^2 + 5.93839h^2$		$4.719046l^2 + 84.82431h^2$	23
	$4.656253l^2 + 6.19758h^2$		$4.914065l^2 + 92.12937h^2$	24

This weight of girders is to be increased by one-tenth of itself, as explained in article 165. Also, it will be augmented to meet the strain brought upon the top chords by the head system.

174. Make the floor of $2\frac{1}{2}$-inch oak planks, 52 pounds per cubic foot, and of the width of q feet. Then, if $q = 16$ feet,

$$\text{Weight of floor} = \frac{2.5}{12} \times 52ql = \frac{520}{3}l \text{ pounds.} \quad (573)$$

175. Longitudinal I Floor Beams; Conditions and Weights given in Article 157. — We may further explain the assumption in (501) thus: Taking an analytical table of ordinary wrought-iron I-beams, we may easily see, that, for depths of 8 inches and upwards, we have approximately

$$\frac{r^2}{d^2} = 0.15, \qquad (574)$$

r being the radius of gyration, and d the depth of beam. Now, by equation (184),

$$r^2 = \frac{I}{S};$$

hence if we take, as we manifestly may, $d = \frac{2}{3}\frac{l}{n}$, and eliminate r, we shall find

$$\frac{I}{d} = 0.15Sd = \frac{l}{10n}S,$$

as in (501), where the same notation is used, d being in inches, and l in feet.

We obtain, from (503),

Weight of 6 wrought-iron longitudinal I-beams, in pounds,

$$= 6Ll + l^2\frac{0.52}{n} = J_1 \qquad (575)$$

		n			n
$= 6Ll + l^2$	0.1300000	4	$= 6Ll + l^2$	0.0346667	15
	0.1040000	5		0.0325000	16
	0.0866667	6		0.0305882	17
	0.0742857	7		0.0288889	18
	0.0650000	8		0.0273632	19
	0.0577778	9		0.0260000	20
	0.0520000	10		0.0247619	21
	0.0472727	11		0.0236364	22
	0.0433333	12		0.0226087	23
	0.0400000	13		0.0216667	24
	0.0371429	14			

176. Also, let the transverse I-beams be conditioned as in article (158); then (507) yields, taking S from (506),

Weight of $(n - 1)$ transverse I-beams due load, pounds,

$$= (n - 1) \times \tfrac{5}{18} \times 12 \times 18S \quad (576)$$

					n
$= l$	$5.2650 + l^2$	$0.003949 + L^3$	$0.18225 + L$	243	4
	5.6160	0.003369	0.19440	324	5
	5.8500	0.002925	0.20250	405	6
	6.0171	0.002579	0.20828	486	7
	6.1425	0.002303	0.21262	567	8
	6.2400	0.002080	0.21600	648	9
	6.3180	0.001895	0.21870	729	10
	6.3817	0.001740	0.22091	810	11
	6.4350	0.001609	0.22275	891	12
	6.4800	0.001495	0.22431	972	13
	6.5186	0.001397	0.22564	1053	14
	6.5520	0.001304	0.22680	1134	15
	6.5812	0.001234	0.22781	1215	16
	6.6071	0.001166	0.22871	1296	17
	6.6300	0.001105	0.22950	1377	18
	6.6505	0.001050	0.23021	1458	19
	6.6690	0.001000	0.23085	1539	20
	6.6857	0.000955	0.23143	1620	21
	6.7009	0.000914	0.23195	1701	22
	6.7148	0.000876	0.23244	1782	23
	6.7275	0.000841	0.23287	1863	24

177. Equation (512) becomes, when n is odd,

Weight of iron to be added to transverse I-beams on account of wind, in pounds,

$$= h\left(\frac{25}{24} + 0.972\frac{n}{l}\right)\left(7n - 12 + \frac{5}{n}\right) \quad (577)$$

			n				n
$= h$	$25.000 + \frac{h}{l}$	116.65	5	$= h$	$97.222 + \frac{h}{l}$	1360.80	15
	39.285	256.62	7		111.765	1772.93	17
	53.703	451.01	9		126.315	2239.49	19
	68.182	699.85	11		140.873	2760.48	21
	82.692	1003.11	13		155.435	3335.91	23

178. When n is odd, we use $\frac{n-1}{2}$ terms in summing (509), adding $4 \times \frac{1}{8}(n^2 - 1) = 2\left(n - \frac{n+1}{2}\right)\left(n - \frac{n-1}{2}\right)$ for the two diagonals of the middle panel, and find, as in (509),

Weight of horizontal diagonals, in pounds,

$$= 4 \times \frac{12q_1}{\sin^2\phi_1} \times m \times \frac{W_1}{2nT_1}\left\{\begin{array}{r} n(n-1) + (n-1)(n-2) \\ + (n-2)(n-3) \\ + (n-3)(n-4) \\ \ldots \frac{n-1}{2} \text{ terms} + \frac{1}{8}(n^2-1) \end{array}\right\}$$

$$= \tfrac{35}{8}h\left(n - \frac{1}{n}\right)\left(1 + \frac{l^2}{324n^2}\right) \tag{578}$$

			n
$= h$	$28.000 + hl^2$	0.0034568	5
	40.000	0.0025195	7
	51.852	0.0019758	9
	63.636	0.0016232	11
	75.385	0.0013767	13
	87.111	0.0011949	15
	98.823	0.0010554	17
	110.526	0.0009450	19
	122.222	0.0008554	21
	133.913	0.0007813	23

179. In summing (515) for odd values of n, we use $\frac{n-1}{2}$ terms of the series, and add $\frac{1}{8}(n^2 - 1)$ for middle panel, then multiply the sum by 4, since the two wind chords are to be alike.

Weight of wind chords, in pounds,

$$= 4 \times \frac{5}{18} \times \frac{12l}{n} \times \frac{W_1 l}{2nq_1 Q}\left\{\begin{array}{l} n - 1 + 2(n-2) + 3(n-3) \\ \ldots \frac{n-1}{2} \text{ terms} + \frac{n^2-1}{8} \end{array}\right\}$$

$$= 0.006028164\left(2 + \frac{3}{n} - \frac{2}{n^2} - \frac{3}{n^3}\right)hl^2 \tag{579}$$

$= hl^2$		$n =$
	0.0150463	5
	0.0143414	7
	0.0138920	9
	0.0135871	11
	0.0133679	13
	0.0132030	15
	0.0130747	17
	0.0129721	19
	0.0128882	21
	0.0128183	23

180. We shall now have, at the centre of each vertical, the load, ε_n, as defined in article 163. Therefore

Cross-section of lower half of vertical due ε_n, in square inches,

$$= S = \frac{\frac{1}{2}\varepsilon_n}{Q_3} = \frac{\varepsilon_n}{10667}; \tag{580}$$

Weight of all lower halves of vertical due ε_n, in pounds,

$$= 2 \times \frac{5}{18} \times \frac{\varepsilon_n}{10667} \times 12\Sigma y = 0.0002083\tfrac{1}{3}\left(n - \frac{1}{n}\right)h\varepsilon_n. \tag{581}$$

Cross-section of upper half of vertical due ε_n, in square inches,

$$= S_1 = \frac{\frac{1}{2}\varepsilon_n}{T_1} = \frac{\varepsilon_n}{12000}; \tag{582}$$

Weight of all upper halves of verticals due ε_n, in pounds,

$$= 2 \times \frac{5}{18} \times \frac{\varepsilon_n}{12000} \times 12\Sigma y = 0.000\dot{1}85\left(n - \frac{1}{n}\right)h\varepsilon_n. \tag{583}$$

As in the second part of article 163, so here, suiting the expression to the changed length of chord segments, we have, from the assumed wind pressure, the moment

$$62.5\frac{l}{n}y = \tfrac{1}{2}S_2 \times \tfrac{1}{10}yB_1, \quad B_1 = 5667.$$

Therefore

Cross-section of any vertical due bending-moment, in square inches,

$$= S_2 = \frac{15l}{68n}; \tag{584}$$

Weight of verticals required to resist bending-moment due wind, in pounds,

$$= 4 \times \frac{5}{18} \times \frac{15l}{68n} \times 12\Sigma y = 0.9803922\left(1 - \frac{1}{n^2}\right)hl. \tag{585}$$

Adding together the three expressions, (581), (583), and (585), the sum is

Weight of all verticals, pounds,

$$= h\left\{0.000393518\left(n - \frac{1}{n}\right)\varepsilon_n + 0.9803922\left(1 - \frac{1}{n^2}\right)l\right\} \tag{586}$$

$$= h\left|\begin{array}{l|r}
 & n \\
0.001152Ll + 1.5425L + 0.000000744l^3 + 0.00002495l^2 + 0.95631l + 0.0032 & 4 \\
0.001179Ll + 1.9654L + 0.000000699l^3 + 0.00002002l^2 + 0.97747l + 0.0055 & 5 \\
0.001194Ll + 2.3885L + 0.000000677l^3 + 0.00001725l^2 + 0.99043l + 0.0081 & 6 \\
0.001203Ll + 2.8077L + 0.000000650l^3 + 0.00001490l^2 + 0.99821l + 0.0115 & 7 \\
0.001209Ll + 3.2245L + 0.000000633l^3 + 0.00001310l^2 + 1.00383l + 0.0152 & 8 \\
0.001213Ll + 3.6396L + 0.000000616l^3 + 0.00001168l^2 + 1.00770l + 0.0197 & 9 \\
0.001216Ll + 4.0536L + 0.000000606l^3 + 0.00001054l^2 + 1.01060l + 0.0245 & 10 \\
0.001218Ll + 4.4668L + 0.000000593l^3 + 0.00000960l^2 + 1.01289l + 0.0300 & 11 \\
0.001220Ll + 4.8793L + 0.000000585l^3 + 0.00000881l^2 + 1.01475l + 0.0359 & 12 \\
0.001221Ll + 5.2914L + 0.000000577l^3 + 0.00000814l^2 + 1.01637l + 0.0425 & 13 \\
0.001222Ll + 5.7031L + 0.000000570l^3 + 0.00000756l^2 + 1.01767l + 0.0494 & 14 \\
0.001223Ll + 6.1145L + 0.000000564l^3 + 0.00000706l^2 + 1.01885l + 0.0571 & 15 \\
0.001224Ll + 6.5257L + 0.000000559l^3 + 0.00000663l^2 + 1.01992l + 0.0651 & 16 \\
0.001224Ll + 6.9367L + 0.000000555l^3 + 0.00000623l^2 + 1.02090l + 0.0739 & 17
\end{array}\right.$$

$$
= h \left| \begin{array}{l} 0.001225Ll + 7.3474L + 0.000000549l^3 + 0.00000590l^2 + 1.02178l + 0.0829 \\ 0.001225Ll + 7.5811L + 0.000000546l^3 + 0.00000553l^2 + 1.02263l + 0.0928 \\ 0.001225Ll + 8.1686L + 0.000000543l^3 + 0.00000531l^2 + 1.02342l + 0.1029 \\ 0.001226Ll + 8.5797L + 0.000000539l^3 + 0.00000505l^2 + 1.02418l + 0.1138 \\ 0.001226Ll + 8.9894L + 0.000000537l^3 + 0.00000483l^2 + 1.02491l + 0.1250 \\ 0.001226Ll + 9.3997L + 0.000000533l^3 + 0.00000462l^2 + 1.02560l + 0.1370 \\ 0.001226Ll + 9.8092L + 0.000000532l^3 + 0.00000443l^2 + 1.02627l + 0.1492 \end{array} \right| \begin{array}{c} n \\ 18 \\ 19 \\ 20 \\ 21 \\ 22 \\ 23 \\ 24 \end{array}
$$

since for the even values of n we have ε_n, given in article 163, and for the odd values

$$
\begin{aligned}
\varepsilon_5 &= 0.62430Ll + 1040.5L + 0.010821l^2 + 18.0353l + 0.001850hl^2 + 5.925h + 14.58\frac{h}{l}, \\
\varepsilon_7 &= 0.44593Ll + 1040.5L + 0.005521l^2 + 12.8823l + 0.001204hl^2 + 6.131h + 21.38\frac{h}{l}, \\
\varepsilon_9 &= 0.34683Ll + 1040.5L + 0.003340l^2 + 10.0196l + 0.000881hl^2 + 6.237h + 28.19\frac{h}{l}, \\
\varepsilon_{11} &= 0.28378Ll + 1040.5L + 0.002236l^2 + 8.1978l + 0.000691hl^2 + 6.302h + 34.99\frac{h}{l}, \\
\varepsilon_{13} &= 0.24012Ll + 1040.5L + 0.001600l^2 + 6.9366l + 0.000567hl^2 + 6.345h + 41.80\frac{h}{l}, \\
\varepsilon_{15} &= 0.20810Ll + 1040.5L + 0.001202l^2 + 6.0117l + 0.000480hl^2 + 6.376h + 48.60\frac{h}{l}, \\
\varepsilon_{17} &= 0.18362Ll + 1040.5L + 0.000935l^2 + 5.3045l + 0.000416hl^2 + 6.399h + 55.40\frac{h}{l}, \\
\varepsilon_{19} &= 0.16429Ll + 1040.5L + 0.000742l^2 + 4.7461l + 0.000366hl^2 + 6.417h + 62.21\frac{h}{l}, \\
\varepsilon_{21} &= 0.14864Ll + 1040.5L + 0.000613l^2 + 4.2940l + 0.000327hl^2 + 6.432h + 69.01\frac{h}{l}, \\
\varepsilon_{23} &= 0.13571Ll + 1040.5L + 0.000511l^2 + 3.9205l + 0.000295hl^2 + 6.444h + 75.82\frac{h}{l},
\end{aligned}
$$

= the load applied at centre of each support.

Here, as in article 163, we have put $\frac{1}{6}l$ for h in the last three terms of the value of ε_n; a substitution introducing no practical error in the small resulting terms, but enabling us to keep our final equation down to the second degree with respect to h.

181. For the head lateral system, we proceed as in article 164, now having a pair of diagonals and a strut for each panel so far as the head system extends, say to $n - 6$ of the central panels when head room is sufficient.

Hence, by (480), the moment at $\left(\frac{n-1}{2}\right)^{\text{th}}$ panel point is, since $W_1 = 2{,}500\frac{h}{n}$,

$$M = \frac{\frac{1}{4}W_1}{2n}l\left(n - \frac{n-1}{2}\right)\frac{n-1}{2} = \frac{W_1 l}{32}\frac{n^2-1}{n},$$

$$= 78.125\frac{n^2-1}{n^2}hl \ (n \text{ odd});$$

and at the $\left(\frac{n}{2}\right)^{\text{th}}$ panel point,

$$M = \frac{\frac{1}{4}W_1}{2n}l\left(n - \frac{n}{2}\right)\frac{n}{2} = \frac{W_1 ln}{32},$$

$$= 78.125hl \ (n \text{ even});$$

$$H = \frac{M}{q_1} = 78.125\frac{n^2-1}{n^2}\frac{hl}{q_1} \qquad (n \text{ odd}),$$

$$= 78.125\frac{hl}{q_1} \qquad (n \text{ even});$$

$$\Delta H = H \times \frac{2}{n} = 156.25\frac{n^2-1}{n^3}\frac{hl}{q_1} \qquad (n \text{ odd}),$$

$$= 156.25\frac{hl}{nq_1} \qquad (n \text{ even});$$

requiring each diagonal tie to resist

$$\frac{156.25}{\cos\alpha\cos\phi_1} \times \frac{hl}{q_1} \times \frac{n^2-1}{n^3} \text{ pounds } (n \text{ odd}),$$

or

$$\frac{156.25}{\cos\alpha\cos\phi_1} \times \frac{hl}{nq_1} \text{ pounds } (n \text{ even}),$$

and to have a cross-section

$$S = \frac{156.25hl(n^2-1)}{15000\cos\alpha\cos\phi_1 n^3 q_1},$$

$$= \frac{0.0104\frac{1}{6}(n^2-1)hl}{n^3 q_1 \cos\phi_1} \text{ square inches } (n \text{ odd}), \qquad (587)$$

or

$$S = \frac{0.0104\frac{1}{6}hl}{nq_1 \cos\phi_1} \text{ square inches } (n \text{ even}); \qquad (588)$$

calling, as before, $\cos\alpha = 1$.

Length of each head diagonal $= \dfrac{l}{n\cos\alpha\cos\phi_1} = \dfrac{l}{n\cos\phi_1}$ practically.

Weight of 2 $(n - 6)$ wrought-iron head diagonals, in pounds,

$$= 2(n-6) \times \frac{5}{18} \times \frac{12l}{n\cos\phi_1} \times \frac{0.0104\frac{1}{6}(n^2-1)hl}{n^3 q_1 \cos\phi_1}$$

$$= 0.003858\frac{(n^2-1)(n-6)}{n^4\cos^2\phi_1}hl^2,$$

$$= 0.003858(n^2-1)(n-6)\left(\frac{hl^2}{n^4} + \frac{324h}{n^2}\right) \ (n \text{ odd}), \qquad (589)$$

$$= 0.003858(n-6)\left(\frac{hl^2}{n^2} + 324h\right) \qquad (n \text{ even}), \qquad (590)$$

			n				n
$= h$	– $+ hl^2$	–	4	$= h$	11.1999 $+ hl^2$	0.00015363	15
	–	–	5		12.5000	0.00015070	16
	–	0	6		13.7023	0.00014634	17
	1.2245	0.000007713	7		15.0000	0.00014289	18
	2.5000	0.00012056	8		16.2049	0.00013855	19
	3.7037	0.00014113	9		17.5000	0.00013503	20
	5.0000	0.00015432	10		18.7074	0.00013093	21
	6.1983	0.00015811	11		20.0000	0.00012754	22
	7.5000	0.00016075	12		21.2092	0.00012375	23
	8.6982	0.00015885	13		22.5000	0.00012056	24
	10.0000	0.00015747	14				

since $q_1 = 18$ feet, and $\dfrac{1}{\cos^2\phi_1} = 1 + \left(\dfrac{18n}{l}\right)^2 = 1 + \tan^2\phi_1$.

Multiplying the cross-section, (587), (588), of head diagonal by $2 \times 10{,}000 \cos \phi_1 \tan \phi_{11}$, we find, after dividing by 2,500 pounds inch strain,

$$S = \frac{0.0104\frac{1}{6} \times 2 \times 10000(n^2 - 1)hl \tan \phi_1}{2500n^3q_1},$$

$$= \frac{1}{12}\frac{n^2 - 1}{n^2}h \text{ square inches } (n \text{ odd}), \qquad (591)$$

$$= \frac{0.0104\frac{1}{6} \times 2 \times 10000hl \tan \phi_1}{2500nq_1},$$

$$= \tfrac{1}{12}h \text{ square inches } (n \text{ even}), \qquad (592)$$

= cross-section of head strut.

Weight of $(n - 5)$ head struts, in pounds,

$$= (n - 5) \times \frac{5}{18} \times 12 \times 18\frac{(n^2 - 1)}{n^2}\frac{h}{12}, \qquad (593)$$

$$= 5\frac{(n - 5)(n^2 - 1)}{n^2}h \qquad (n \text{ odd}),$$

$$= \tfrac{5}{18}(n - 5)\frac{h}{12} \times 12 \times 18, \qquad (594)$$

$$= 5(n - 5)h \qquad (n \text{ even}),$$

		n			n
$= h$	–	4	$= h$	49.7777	15
	0	5		55.0000	16
	5.0000	6		59.7924	17
	9.7959	7		65.0000	18
	15.0000	8		69.8061	19
	19.7531	9		75.0000	20
	25.0000	10		79.8186	21
	29.7521	11		85.0000	22
	35.0000	12		89.8299	23
	39.7633	13		95.0000	24
	45.0000	14			

Calling the compression along each segment of top chord due head diagonals equal to

$$\Delta H = 156.25\frac{n^2 - 1}{n^3}\frac{hl}{q_1} \quad \text{or} \quad 156.25\frac{hl}{nq_1}$$

according as n is odd or even, and taking the allowed inch strain in compression, as above, viz.,

$$Q = \frac{8000}{1 + \frac{50^2}{40000}} = 7529 \text{ pounds} = 3.764 \text{ tons},$$

we have

Cross-section of iron to be added to segments of top chord in head system, square inches,

$$= \frac{156.25}{7529} \cdot \frac{n^2 - 1}{n^3} \cdot \frac{hl}{18} \qquad (n \text{ odd}),$$

$$= S = 0.001115295\frac{n^2 - 1}{n^3}hl, \qquad (595)$$

$$= \frac{156.25}{7529}\,\frac{hl}{18n} \qquad (n \text{ even}),$$

$$= 0.001115295\frac{hl}{n}. \qquad (596)$$

Weight of added iron in $(n - 6)$ panels for top chords, pounds,

$$= 2(n - 6) \times \frac{5}{18} \times \frac{12l}{n}S, \qquad (597)$$

$$= 0.0076863\frac{(n - 6)(n^2 - 1)}{n^4}hl^2 \qquad (n \text{ odd}),$$

$$= 0.0076863\frac{n - 6}{n^2}hl^2 \qquad (n \text{ even}),$$

		n			n
$= hl^2$	-	4	$= hl^2$	0.00030609	15
	-	5		0.00030024	16
	0	6		0.00029155	17
	0.00015366	7		0.00028468	18
	0.00024020	8		0.00027603	19
	0.00028116	9		0.00026902	20
	0.00030745	10		0.00026085	21
	0.00031427	11		0.00025409	22
	0.00032026	12		0.00024654	23
	0.00031648	13		0.00024020	24
	0.00031373	14			

182. As explained in article 165, we shall here augment, by one-tenth of itself, each of the following expressions just found; viz., —

The girders proper,
The vertical supports, and
The lateral head struts.

Then, adding together all the parts of the complete bridge, and putting the sum $= 2{,}000nW$, the weight of any bridge in pounds, we derive the following values of W, in terms of L, l, and h, for the different values of n.

Then, by assigning values to L and l, differentiating, and putting $\frac{dW}{dh} = 0$, we get W a *minimum*, and h best, as in article 140, equations (469) and (470).

$$n = 4.$$

$$W = \frac{\begin{array}{l} h[L(6.18225l + 243) + 0.133949l^2 + 178.5983l] + 1.008333Ll^2 \\ \quad + h^2[L(0.001267l + 4.65868) + 0.000000818l^3 + 0.0201916l^2 + 1.05194l + 40.692 + 64.15l^{-1}] \end{array}}{-0.8536462l^2 + 8000h - 1.067055h^2}, \quad (598)$$

$$= \frac{3.1510417 + 1.616619h + 0.02041786h^2}{-0.2134115 + 0.8h - 0.0001067055h^2} \text{ if } l = nL = 50, \qquad = 2.7156 \text{ tons, a minimum for } h = 13.4222 \text{ feet.}$$

$$n = 5.$$

$$W = \frac{\begin{array}{l} h[L(6.1944l + 324) + 0.107369l^2 + 178.9493l] + 1.252683Ll^2 \\ \quad + h^2[L(0.001297l + 6.9333) + 0.000000769l^3 + 0.0185239l^2 + 1.07522l + 53.006 + 116.65l^{-1}] \end{array}}{-1.067058l^2 + 10000h - 1.36583h^2}, \quad (599)$$

$$= \frac{3.1317075 + 1.5553087h + 0.02254876h^2}{-0.2667645 + h - 0.000136583h^2} \text{ if } l = nL = 50, \qquad = 2.1368 \text{ tons, a minimum for } h = 12.742 \text{ feet.}$$

$$n = 6.$$

$$W = \frac{\begin{array}{l} h[L(6.2025l + 405) + 0.0895917l^2 + 179.1833l] + 1.486719Ll^2 \\ \quad + h^2[L(0.001313l + 9.53508) + 0.000000745l^3 + 0.0177074l^2 + 1.08947l + 66.05 + 176.91l^{-1}] \end{array}}{-1.280469l^2 + 12000h - 1.65987h^2}, \quad (600)$$

$$= \frac{3.09733125 + 1.51425h + 0.02484294h^2}{-0.320117 + 1.2h - 0.000165987h^2} \text{ if } l = nL = 50, \qquad = 1.7676 \text{ tons, a minimum for } h = 12.072 \text{ feet.}$$

$$n = 7.$$

$$W = \frac{\begin{array}{l} h[L(6.20828l + 486) + 0.0768647l^2 + 179.3504l] + 1.714861Ll^2 \\ \quad + h^2[L(0.0013233l + 12.45747) + 0.000000715l^3 + 0.0168778l^2 + 1.09803l + 79.2976 + 256.62l^{-1}] \end{array}}{-1.49388l^2 + 14000h - 1.95119h^2}, \quad (601)$$

$$= \frac{3.062252 + 1.484835h + 0.0271069h^2}{-0.37347 + 1.4h - 0.000195119h^2} \text{ if } l = nL = 50, \qquad = 1.5111 \text{ tons, a minimum for } h = 11.504 \text{ feet.}$$

$n = 8$ (without head system).

$$W = \frac{\begin{array}{l} h[L(6.21262l + 567) + 0.067303l^2 + 179.4758l] + 1.939324Ll^2 \\ \quad + h^2[L(0.001330l + 15.7022) + 0.000000696l^3 + 0.016503l^2 + 1.10421l + 92.378 + 344.09l^{-1}] \end{array}}{-1.707293l^2 + 16000h - 2.24082h^2}, \quad (602)$$

$$= \frac{3.030194 + 1.462724h + 0.02943693h^2}{-0.426823 + 1.6h - 0.000224082h^2} \text{ if } l = nL = 50, \quad = 1.3351 \text{ tons, a minimum for } h = 10.996 \text{ feet.}$$

$n = 8$ (with head system).

$$W = \frac{\begin{array}{l} h[L(6.21262l + 567) + 0.067303l^2 + 179.4758l] + 1.939324Ll^2 \\ \quad + h^2[L(0.001330l + 15.7022) + 0.000000696l^3 + 0.0167341l^2 + 1.10421l + 111.378 + 344.09l^{-1}] \end{array}}{-1.707293l^2 + 16000h - 2.24082h^2}, \quad (603)$$

$$= \frac{24.24155 + 3.347388h + 0.059122h^2}{-1.707293 + 1.6h - 0.000224082h^2} \text{ if } l = nL = 100, \quad = 3.7885 \text{ tons, a minimum for } h = 22.629 \text{ feet.}$$

$n = 9$.

$$W = \frac{\begin{array}{l} h[L(6.216l + 648) + 0.0598578l^2 + 179.5733l] + 2.161329Ll^2 \\ \quad + h^2[L(0.001334l + 19.27118) + 0.000000678l^3 + 0.0163029l^2 + 1.10847l + 131.0088 + 451.01l^{-1}] \end{array}}{-1.920704l^2 + 18000h - 2.52931h^2}, \quad (604)$$

$$= \frac{24.01477 + 3.2662h + 0.062568h^2}{-1.920704 + 1.8h - 0.000252931h^2} \text{ if } l = nL = 100, \quad = 3.3586 \text{ tons, a minimum for } h = 21.918 \text{ feet.}$$

$n = 10$.

$$W = \frac{\begin{array}{l} h[L(6.2187l + 729) + 0.053895l^2 + 179.6513l] + 2.381616Ll^2 \\ \quad + h^2[L(0.0013376l + 23.16548) + 0.000000667l^3 + 0.01600766l^2 + 1.11166l + 151.152 + 565.71l^{-1}] \end{array}}{-2.134116l^2 + 20000h - 2.81702h^2}, \quad (605)$$

$$= \frac{23.81616 + 3.201278h + 0.06617111h^2}{-2.134116 + 2h - 0.000281702h^2} \text{ if } l = nL = 100, \quad = 3.0248 \text{ tons, a minimum for } h = 21.252 \text{ feet.}$$

$$n = 11.$$

$$W = \frac{h[L(6.22091l + 810) + 0.0490127l^2 + 179.715l] + 2.600652Ll^2 + h^2[L(0.001339l + 27.58623) + 0.000000652l^3 + 0.0156933l^2 + 1.11418l + 170.777 + 699.85l^{-1}]}{-2.337523l^2 + 22000h - 3.10416h^2}, \quad (606)$$

$$\therefore \; \frac{7.992732 + 0.51879h + 0.0107404619h^2}{-0.5281995 + 0.22h - 0.0000310416h^2} \text{ if } l = nL = 150, \quad = 5.4878 \text{ tons, a minimum for } h = 31.575 \text{ feet.}$$

$$n = 12.$$

$$W = \frac{h[L(6.22275l + 891) + 0.0449423l^2 + 179.7683l] + 2.818751Ll^2 + h^2[L(0.001342l + 31.93397) + 0.000000643l^3 + 0.015465l^2 + 1.11622l + 190.9345 + 841.75l^{-1}]}{-2.560939l^2 + 24000h - 3.39086h^2}, \quad (607)$$

$$\frac{7.027737 + 0.507816h + 0.0111158054h^2}{-0.5762113 + 0.24h - 0.0000339086h^2} \text{ if } l = nL = 150, \quad = 5.0347 \text{ tons, a minimum for } h = 30.917 \text{ feet.}$$

$$n = 13.$$

$$W = \frac{h[L(6.22431l + 972) + 0.0414951l^2 + 179.8123l] + 3.036129Ll^2 + h^2[L(0.001343l + 36.80949) + 0.000000635l^3 + 0.0152288l^2 + 1.11801l + 210.5615 + 1003.11l^{-1}]}{-2.774351l^2 + 26000h - 3.67724h^2}, \quad (608)$$

$$= \frac{18.6838 + 0.7172783h + 0.01623834h^2}{-1.10974 + 0.26h - 0.0000367724h^2} \text{ if } l = nL = 200, \quad = 7.9452 \text{ tons, a minimum for } h = 40.773 \text{ feet;}$$

$$nW = 126.072 \text{ tons if } 10h = l = nL = 200.$$

$$n = 14.$$

$$W = \frac{h[L(6.22564l + 1053) + 0.0385399l^2 + 179.8519l] + 3.252941Ll^2 + h^2[L(0.001344l + 42.01309) + 0.000000627l^3 + 0.0150484l^2 + 1.11944l + 230.7163 + 1172.18l^{-1}]}{-2.987763l^2 + 28000h - 3.963344h^2}, \quad (609)$$

$$= \frac{62.73529 + 1.200104h + 0.0285066h^2}{-2.6889867 + 0.28h - 0.0000396334h^2} \text{ if } l = nL = 300, \quad = 16.9912 \text{ tons, a minimum for } h = 60.957 \text{ feet.}$$

$$n = 15.$$

$$W = \frac{h[L(6.22680l + 1134) + 0.035971l^2 + 179.8853l] + 3.469299Ll^2 + h^2[L(0.0013453l + 47.545) + 0.00000062l^3 + 0.01486539l^2 + 1.12073l + 250.351 + 1360.8l^{-1}]}{-3.201174l^2 + 30000h - 4.24926h^2}, \quad (610)$$

$$= \frac{62.447382 + 1.1724378h + 0.029047043h^2}{-2.8810566 + 0.3h - 0.0000424926h^2} \text{ if } l = nL = 300, \quad = 15.85998 \text{ tons, a minimum for } h = 60.320 \text{ feet.}$$

$$n = 16.$$

$$W = \frac{h[L(6.22781l + 1215) + 0.033734l^2 + 179.9145l] + 3.685287Ll^2 + h^2[L(0.0013464l + 53.40576) + 0.00000061 5l^3 + 0.0147202l^2 + 1.12191l + 270.4936 + 1557.15l^{-1}]}{-3.414585l^2 + 32000h - 4.53498h^2}, \quad (611)$$

$$= \frac{147.4115 + 1.7001634h + 0.044663 56h^2}{-5.46334 + 0.32h - 0.0000453498h^2} \text{ if } l = nL = 400, \quad = 28.6796 \text{ tons, a minimum for } h = 81.339 \text{ feet.}$$

$$n = 17.$$

$$W = \frac{h[L(6.22871l + 1296) + 0.0317542l^2 + 179.9404l] + 3.900974Ll^2 + h^2[L(0.001346l + 59.5954) + 0.00000061l^3 + 0.01457484l^2 + 1.12299l + 290.1432 + 1772.93l^{-1}]}{-3.627997l^2 + 34000h - 4.82058h^2}, \quad (612)$$

$$= \frac{286.83632 + 2.2762507h + 0.063477 44h^2}{-9.07 + 0.34h - 0.0000482058h^2} \text{ if } l = nL = 500, \quad = 46.9617 \text{ tons, a minimum for } h = 104.126 \text{ feet.}$$

$$n = 18.$$

$$W = \frac{h[L(6.2295l + 1377) + 0.029994l^2 + 179.9633l] + 4.116409Ll^2 + h^2[L(0.0013475l + 66.11378) + 0.000000604l^3 + 0.0144564l^2 + 1.124l + 310.272 + 1996.5l^{-1}]}{-3.841409l^2 + 36000h - 5.106057h^2}, \quad (613)$$

$$= \frac{285.86174 + 2.2225h + 0.064210 43h^2}{-9.60352 + 0.36h - 0.000005106057h^2} \text{ if } l = nL = 500, \quad = 44.4024 \text{ tons, a minimum for } h = 103.511 \text{ feet.}$$

$$n = 19.$$

$$W = \frac{\begin{aligned}&h[L(6.23021l + 1458) + 0.028413l^2 + 179.9838l] + 4.331628Ll^2\\ &+ h^2[L(0.0013475l + 72.767) + 0.000000601l^3 + 0.0143378l^2 + 1.12489l + 329.9347 + 2239.49l^{-1}]\end{aligned}}{-4.05482l^2 + 38000h - 5.391496h^2}, \quad (614)$$

$$= \frac{781.9728 + 3.5430075h + 0.11067843h^2}{-19.86862 + 0.38h - 0.000005391496h^2} \text{ if } l = nL = 700, \quad = 105.7218 \text{ tons, a minimum for } h = 157.380 \text{ feet.}$$

$$n = 20.$$

$$W = \frac{\begin{aligned}&h[L(6.23085l + 1539) + 0.027l^2 + 180.0023l] + 4.54667Ll^2\\ &+ h^2[L(0.0013475l + 80.1386) + 0.000000597l^3 + 0.0142392l^2 + 1.12576l + 350.05 + 2490.27l^{-1}]\end{aligned}}{-4.268232l^2 + 40000h - 5.676737h^2}, \quad (615)$$

$$= \frac{1163.0475 + 4.22220h + 0.13021166h^2}{-27.31068 + 0.4h - 0.000005676737h^2} \text{ if } l = nL = 800, \quad = 149.9221 \text{ tons, a minimum for } h = 188.687 \text{ feet;}$$

$$nW = 124.521 \text{ tons if } l = 10h = nL = 200.$$

$$n = 21.$$

$$W = \frac{\begin{aligned}&h[L(6.23143l + 1620) + 0.0257169l^2 + 180.019l] + 4.761552Ll^2\\ &+ h^2[L(0.0013480l + 87.6458) + 0.000000592l^3 + 0.0141409l^2 + 1.126598l + 369.728 + 2760.48l^{-1}]\end{aligned}}{-4.481643l^2 + 42000h - 5.961956h^2}, \quad (616)$$

$$= \frac{1652.93877 + 4.926315h + 0.17081376h^2}{-36.301308 + 0.42h - 0.000005961956h^2} \text{ if } l = nL = 900, \quad = 206.7529 \text{ tons, a minimum for } h = 223.620 \text{ feet;}$$

$$nW = 124.630 \text{ tons if } l = 10h = nL = 200.$$

$$n = 22.$$

$$W = \frac{\begin{aligned}&h[L(6.23195l + 1701) + 0.0245504l^2 + 180.0342l] + 4.976304Ll^2\\ &+ h^2[L(0.001349l + 95.4809) + 0.000000591l^3 + 0.014529l^2 + 1.1274l + 335.8275 + 3038.47l^{-1}]\end{aligned}}{-4.695054l^2 + 44000h - 6.24712h^2}, \quad (617)$$

$$= \frac{2261.95636 + 5.651732h + 0.20987644h^2}{46.95054 + 0.44h - 0.00000624712h^2} \text{ if } l = nL = 1000, \quad = 283.4025 \text{ tons, a minimum for } h = 261.545 \text{ feet;}$$

$$nW = 124.350 \text{ tons if } l = 10h = nL = 200.$$

$$n = 23.$$

$$W = \frac{h[L(6.23244l + 1782) + 0.0234847l^2 + 180.0481l] + 5.190951Ll^2 + h^2[L(0.0013486l + 103.6464) + 0.0000000586l^3 + 0.0139751l^2 + 1.12816l + 275.608 + 3335.91l^{-1}]}{-4.908467l^2 + 46000h - 6.53223h^2}, \quad (618)$$

$$= \frac{18.05548 + 0.632837h + 0.01985225h^2}{-1.9633868 + 0.46h - 0.0000653223h^2} \text{ if } l = nL = 200, \quad = 4.5864 \text{ tons, a minimum for } h = 36.645 \text{ feet;}$$

$$nW = 105.487 \text{ tons,}$$

$$nW = 123.296 \text{ tons if } l = 10h = nL = 200.$$

$$n = 24.$$

$$W = \frac{h[L(6.23287l + 1863) + 0.0225077l^2 + 180.0608l] + 5.405471Ll^2 + h^2[L(0.0013486l + 112.1324) + 0.0000000585l^3 + 0.0139039l^2 + 1.1289l + 429.612 + 3641.11l^{-1}]}{-5.121878l^2 + 48000h - 6.81734h^2}, \quad (619)$$

$$= \frac{18.01824 + 0.6282558h + 0.02171117h^2}{-2.0487512 + 0.48h - 0.0000681734h^2} \text{ if } l = nL = 200, \quad = 4.5407 \text{ tons, a minimum for } h = 35.224 \text{ feet;}$$

$$nW = 108.978 \text{ tons,}$$

$$nW = 124.969 \text{ tons if } l = 10h = nL = 200.$$

In equation (619),

W = infinity when $l = 2h = 3516$ feet,
= infinity when $l = 4h = 2163$ feet,
= infinity when $l = 6h = 1506$ feet,
= infinity when $l = 8h = 1148$ feet,
= infinity when $l = 10h = 925$ feet,

W = infinity when $l = 12h = 774$ feet,
= infinity when $l = 14h = 665$ feet,
= infinity when $l = 16h = 583$ feet,
= infinity when $l = 100h = 94$ feet,

which are limiting spans when the number of panels is $24 = n$, and h and l are related as above. These limiting values of l are found by putting the denominator of (619) equal to zero, as usual.

TWO DOUBLE PARABOLIC-BOW OR BRUNEL GIRDERS. HIGHWAY BRIDGE, FIG. 22.

Uniform Live and Dead Loads applied at all Apices. Height of Girder and Number of Panels yielding Minimum Bridge Weight. Double Web System. Limiting Span.

1	$nL \div l$	Span	l	feet	50			100			150		
2	1.0 ton	Number of panels, best	n		6	7	8	8	9	10	10	11	12
3	1.0 "	Best central height	h	feet	12.072	11.504	10.996	22.629	21.918	21.252	32.272	31.575	30.917
4	1.0 "	Least bridge weight	nW	tons	10.606	10.578	10.680	30.308	30.227	30.248	60.464	60.366	60.416
5	1.0 "	Top chords, Eq. (566)		lbs.	1,643	1,714	1,787	7,482	7,691	7,906	18,956	19,319	19,692
6	1.0 "	Do. for head system, (597)		lbs.				54	61	65	223	223	223
7	1.0 "	Bottom chords, (567)		lbs	1,237	1,293	1,345	5,633	5,790	5,953	14,273	14,443	14,827
8	1.0 "	Girder diagonals, (572)		lbs.	884	952	1,012	4,086	4,322	4,541	10,280	10,799	11,302
9	1.0 "	Floor, (573)		lbs.	8,667	8,667	8,667	17,333	17,333	17,333	26,000	26,000	26,000
10	1.0 "	I-Beams, longitudinal, (575)		lbs.	2,717	2,329	2,038	8,150	7,244	6,520	14,670	13,336	12,225
11	1.0 "	I-Beams, transverse, (576), (577)		lbs	4,183	4,364	4,506	9,112	9,360	9,568	14,496	14,784	15,054
12	1.0 "	Horizontal diagonals, (509), (578)		lbs.	505	533	758	1,551	1,569	1,613	3,172	3,207	3,195
13	1.0 "	Wind chords, (515), (579)		lbs.	444	412	389	3,197	3,045	2,921	9,980	9,653	9,377
14	1.0 "	Vertical supports, (586)		lbs.	930	893	858	3,559	3,452	3,356	7,719	7,557	7,404
15	1.0 "	Head struts, (593), (594)		lbs.				373	476	585	887	1,103	1,190
16	1.0 "	Head diagonals, (589), (590)		lbs.				84	112	139	273	308	344
17	1.0 "	Total, least bridge weight		lbs.	21,210	21,157	21,360	60,614	60,455	60,500	120,929	120,732	120,833
18	1.0 "	Bridge weight per running-foot		lbs.	424	423	427	606	604	605	806	805	806
19	1.0 "	Panel length	$l \div n$	feet	$8\frac{1}{3}$	$7\frac{1}{7}$	$6\frac{1}{4}$	$12\frac{1}{2}$	$11\frac{1}{9}$	10	15	$13\frac{7}{11}$	$12\frac{1}{2}$
20	1.0 "	Ratio of length to central height,	$l \div h$		4.142	4.346	4.547	4.419	4.562	4.705	4.648	4.751	4.852
21	1.0 "	Ratio of minimum dead to live load	$W \div L$		0.2121	0.2115	0.2136	0.3031	0.3023	0.3025	0.4031	0.4024	0.4028
22	1.0 "	Ratio of minimum dead to total load	$\frac{W}{W+L}$		0.1750	0.1746	0.1760	0.2326	0.2321	0.2322	0.2873	0.2869	0.2871
23	1.0 "	If $l =$ { panels,	n		7	8	9	14	15	16	16	17	18
24	1.0 "		nW	tons	11.802	11.713	11.722	34.556	34.545	34.565	71.016	70.995	71.021
25	1.5 "	Best n simultaneous,	n		7	8	9	9	10	11	10	11	12
26	1.5 "	Best h	h	feet	12.863	12.279	11.389	24.351	23.566	22.877	35.757	33.784	33.603
27	1.5 "	Least nW	nW	tons	12.955	12.949	12.977	37.979	37.978	38.046	76.799	76.708	76.757
28	2.0 "	Best n simultaneous,	n		7	8	9	9	10	11	10	11	12
29	2.0 "	Best h	h	feet	13.704	13.157	12.244	26.005	25.140	24.372	38.149	37.180	35.770
30	2.0 "	Least nW	nW	tons	15.301	15.282	15.310	45.596	45.577	45.648	92.761	92.693	92.792
			n		6	7	8	9	10	11	12	13	14
31	$L = 0$	Limiting span, $l = 10h$	l	feet	925.16	925.07	925.02	924.98	924.95	924.93	924.91	924.90	924.89
32	$L = \frac{1}{2}W$	Limiting span, $l = 5h$	l	feet	1,414	1,438	1,434	1,465	1,473	1,478	1,482	1,483	1,484
33	$L = 0$	Limiting span, $l = 5h$	l	feet	1,781.96	1,781.96	1,780.82	1,780.53	1,780.31	1,780.15	1,780.04	1,779.95	1,779.87

TWO DOUBLE PARABOLIC-BOW OR BRUNEL GIRDERS. HIGHWAY BRIDGE, FIG. 22.

Uniform Live and Dead Loads applied at all Apices. Height of Girder and Number of Panels yielding Minimum Bridge Weight. Double Web System. Limiting Span. — *Continued.*

1	$nL \div l$	Span	l	feet	200			300			400		
2	1.0 ton	Number of panels, best	n		**12**	**13**	**14**	**13**	**14**	**15**	**14**	**15**	**16**
3	1.0 "	Best central height	h	feet	41.437	40.788	40.147	61.689	60.957	60.320	82.637	81.990	81.339
4	1.0 "	Least bridge weight	nW	tons	103.304	103.288	103.460	238.086	237.877	237.900	459.480	458.782	458.874
5	1.0 "	Top chords, Eq. (566)		lbs.	37,671	38,206	38,777	100,930	101,980	102,952	214,066	215,408	216,981
6	1.0 "	Do. for head system, (597)		lbs.	531	516	504	1,757	1,721	1,662	4,148	4,015	3,907
7	1.0 "	Bottom chords, (567)		lbs.	28,364	28,767	29,197	75,994	76,785	77,516	161,178	162,182	163,174
8	1.0 "	Girder diagonals, (572)		lbs.	20,153	21,088	21,999	47,694	49,896	52,119	89,695	93,909	98,092
9	1.0 "	Floor, (573)		lbs.	34,667	34,667	34,667	52,000	52,000	52,000	69,333	69,333	69,333
10	1.0 "	I-Beams, longitudinal, (575)		lbs.	21,733	20,062	18,629	45,138	41,914	39,120	74,515	69,546	65,200
11	1.0 "	I-Beams, transverse, (576), (577)		lbs.	20,233	20,577	20,885	31,370	31,804	32,262	43,153	43,738	44,283
12	1.0 "	Horizontal diagonals, (509), (578)		lbs.	5,366	5,321	5,328	12,294	11,999	11,742	23,684	22,817	22,183
13	1.0 "	Wind chords, (515), (579)		lbs.	22,343	21,810	21,336	74,219	72,891	71,677	175,675	173,202	171,000
14	1.0 "	Vertical supports, (586)		lbs.	13,373	13,165	12,957	30,660	30,273	29,937	56,514	55,994	55,496
15	1.0 "	Head struts, (593), (594)		lbs.	1,595	1,784	1,987	2,698	3,017	3,303	4,091	4,488	4,921
16	1.0 "	Head diagonals, (589), (590)		lbs.	577	614	654	1,419	1,473	1,510	2,908	2,933	2,978
17	1.0 "	Total least bridge weight		lbs.	206,606	206,577	206,920	476,173	475,753	475,800	918,960	917,565	917,748
18	1.0 "	Bridge weight per running-foot		lbs.	1,033	1,032	1,035	1,587	1,585	1,586	2,297	2,294	2,294
19	1.0 "	Panel length	$l \div n$	feet	$16\frac{2}{3}$	$15\frac{5}{13}$	$14\frac{2}{7}$	$23\frac{1}{13}$	$21\frac{3}{7}$	20	$28\frac{4}{7}$	$26\frac{2}{3}$	25
20	1.0 "	Ratio of length to central height,	$l \div h$		4.827	4.905	4.982	4.863	4.921	4.973	4.840	4.879	4.918
21	1.0 "	Ratio of minimum dead to live load	$W \div L$		0.5165	0.5164	0.5173	0.7936	0.7929	0.7930	1.1487	1.1469	11.472
22	1.0 "	Ratio of minimum dead to total load	$\frac{W}{W+L}$		0.3406	0.3405	0.3409	0.4425	0.4422	0.4423	0.5346	0.5342	0.5343
23	1.0 "	If $l = 10h$ { Best number of panels,	n		**18**	**19**	**20**	**22**	**23**	**24**	**22**	**23**	**24**
24	1.0 "	If $l = 10h$ { Bridge weight	nW	tons	124.482	124.436	124.521	301.902	299.398	302.713	625.739	623.321	626.786
25	1.5 "	Best number of panels } simultaneous,	n		**11**	**12**	**13**	**12**	**13**	**14**	**14**	**15**	**16**
26	1.5 "	Best central height } simultaneous,	h	feet	46.646	45.747	44.914	68.585	67.926	67.014	90.681	89.773	88.877
27	1.5 "	Least bridge weight } simultaneous,	nW	tons	132.035	131.777	131.826	303.570	302.873	302.886	578.976	578.844	579.662
28	2.0 "	Best number of panels } simultaneous,	n		**11**	**12**	**13**	**12**	**13**	**14**	**13**	**14**	**15**
29	2.0 "	Best central height } simultaneous,	h	feet	49.734	48.685	47.725	73.430	71.396	71.198	97.589	96.377	95.249
30	2.0 "	Least bridge weight } simultaneous,	nW	tons	159.786	159.668	159.796	366.528	366.081	366.358	695.465	695.184	695.744
			n		**15**	**16**	**17**	**18**	**19**	**20**			
31	$L = 0$	Limiting span, $l = 10h$	l	feet	924.88								
32	$L = \frac{1}{3}W$	Limiting span, $l = 5h$	l	feet	1,483	1,481	1,479	1,477	1,474	1,470			
33	$L = 0$	Limiting span, $l = 5h$	l	feet	1,779.81	1,779.76	1,779.72	1,779.69	1,779.67	1,779.64			

Two Double Parabolic-Bow or Brunel Girders. Highway Bridge, Fig. 22.

Uniform Live and Dead Loads applied at all Apices. Height of Girder and Number of Panels yielding Minimum Bridge Weight Double Web System. Limiting Span. — *Continued.*

1	$nL \div l$	Span	l	feet	500			600			700		
2	1.0 ton	Number of panels, best	n		16	17	18	16	17	18	18	19	20
3	1.0 "	Best central height	h	feet	104 738	104 126	103 511	130 503	129.897	129.786	157 921	157.380	156.787
4	1.0 "	Least bridge weight	nW	tons	798 445	798.349	799 244	1,298.631	1,297 507	1,298.089	2,008 775	2,008.714	2,012.428
5	1.0 "	Top chords, Eq. (566)		lbs.	399,309	401,380	403,790	677,700	680,105	683,167	1,092,471	1,095,746	1,100,896
6	1.0 "	Do. for head system, (597)		lbs.	7,862	7,589	7,367	14,105	13,613	13,250	22,029	21,286	20,668
7	1.0 "	Bottom chords, (567)		lbs.	300,650	302,215	304,030	510,268	512,078	514,383	822,566	825,030	828,909
8	1.0 "	Girder diagonals, (572)		lbs.	156,654	163,657	170,627	232,040	242,838	251,010	358,219	374,054	389,762
9	1.0 "	Floor. (573)		lbs.	86,667	86,667	86,667	104,000	104,000	104,000	121,333	121,333	121,333
10	1.0 "	(575)		lbs.	101,875	95,882	90,556	146,700	138,070	130,400	177,489	168,145	159,740
11	1.0 "	576), (577)		lbs.	56,377	57,083	57,746	69,030	69,870	70,003	84,202	85,206	86,158
12	1.0 "	509), (578)		lbs.	39,146	37,764	36,689	64,889	62,190	60,024	93,818	90,269	87,324
13	1.0 "	515), (579)		lbs.	344,052	340,353	337,028	617,307	611,410	606,167	1,007,797	1,000,356	993,388
14	1.0 "	Vertical supports, (586)		lbs.	92,701	92,029	91,339	144,617	143,654	142,681	212,908	210,674	210,626
15	1.0 "	Head struts, (593), (594)		lbs.	6,337	6,849	7,401	7,895	8,543	9,244	11,291	12,095	12,935
16	1.0 "	Head diagonals, (589), (590)		lbs.	5,255	5,236	5,250	8,711	8,623	8,590	13,426	13,235	13,118
17	1.0 "	Total least bridge weight		lbs.	1,596,891	1,596,698	1,598,490	2,597,262	2,595,014	2,596,179	4,017,549	4,017,429	4,024,857
18	1.0 "	Bridge weight per running-foot		lbs.	3,193	3,193	3,197	4,343	4,325	4,327	5,739	5,739	5,749
19	1.0 "	Panel length	$l \div n$	feet	$31\frac{1}{4}$	$29\frac{7}{17}$	$27\frac{7}{9}$	$37\frac{1}{2}$	$35\frac{5}{17}$	$33\frac{1}{3}$	$38\frac{8}{9}$	$36\frac{16}{19}$	35
20	1.0 "	Ratio of length to central height,	$l \div h$		4.774	4.802	4.830	4.598	4.619	4.641	4.433	4.448	4.465
21	1.0 "	Ratio of minimum dead to live load	$W \div L$		1.5969	1.5967	1.5985	2.1644	2.1625	2.1635	2.8697	2.8696	2.8749
22	1.0 "	Ratio of minimum dead to total load	$\frac{W}{W+L}$		0.6149	0.6149	0.6152	0.6840	0.6838	0.6839	0.7416	0.7416	0.7419
23	1.0 "	If $l =$ { panels	n										
24	1.0 "		nW	tons									
25	1.5 "	Best: } simultaneous,	n		14	15	16	15	16	17	15	16	17
26	1.5 "	Best	h	feet	115.875	114.037	114.003	142 222	141.268	140.376	172 090	171 127	170.205
27	1.5 "	Least bridge weight	nW	tons	995.876	994.685	995.249	1,595 130	1,595.034	1,596 133	2,435 363	2,433.813	2,434.103
28	2.0 "	Best number of panels } simultaneous,	n		14	15	16	14	15	16	14	15	16
29	2.0 "	Best central height	h	feet	122.999	121.805	119.529	150 413	150.356	149 135	182.594	181.331	180.081
30	2.0 "	Least bridge weight	nW	tons	1,184 732	1,184.657	1,189.604	1,882 195	1,880.760	1,882.939	2,845.514	2,842.088	2,844.075
31	$L = 0$	Limiting span, $l = 10h$	l	feet									
32	$L = \frac{1}{2} W$	Limiting span, $l = 5h$	l	feet									
33	$L = 0$	Limiting span, $l = 5h$	l	feet									

TWO DOUBLE PARABOLIC-BOW OR BRUNEL GIRDERS. HIGHWAY BRIDGE, FIG. 22.

Uniform Live and Dead Loads applied at all Apices. Height of Girder and Number of Panels yielding Minimum Bridge Weight. Double Web System. Limiting Span. — *Concluded.*

1	$nL \div l$	Span	l	feet	800			900			1000		
2	1.0 ton	Number of panels, best	n		**18**	**19**	**20**	**18**	**19**	**20**	**18**	**19**	**20**
3	1.0 "	Best central height	h	feet	189.793	189.265	188.687	225.202	224.687	224.125	264.381	263.874	263.332
4	1.0 "	Least bridge weight	nW	tons	2,995.681	2,994.244	2,998.442	4,337.270	4,333.528	4,337.932	6,127.960	6,120.546	6,124.522
5	1.0 "	Eq. (566)		lbs.	1,674,835	1,678,264	1,684,587	2,484,205	2,487,316	2,494,739	3,587,317	3,589,403	3,597,617
6	1.0 "	(597)		lbs.	34,580	33,435	32,487	51,929	50,236	49,838	75,264	72,837	70,841
7	1.0 "	(567)		lbs.	1,261,052	1,263,630	1,268,394	1,870,459	1,872,798	1,878,388	2,701,038	2,702,601	2,708,793
8	1.0 "	(572)		lbs.	487,656	509,875	531,959	645,403	675,596	705,648	835,150	875,112	914,971
9	1.0 "	[illegible], (573)		lbs.	138,667	138,667	138,667	156,000	156,000	156,000	173,333	173,333	173,333
10	1.0 "	I-Beams, longitudinal, (575)		lbs.	231,822	219,617	208,640	293,400	277,954	264,060	362,223	343,152	326,000
11	1.0 "	I-Beams, transverse, (576), (577)		lbs.	98,405	99,572	100,688	113,308	114,654	115,944	128,928	130,475	131,955
12	1.0 "	Horizontal diagonals, (509), (578)		lbs.	141,179	135,385	130,533	205,744	196,820	189,299	291,696	278,525	267,387
13	1.0 "	Wind chords, (515), (579)		lbs.	1,581,972	1,571,304	1,561,470	2,375,716	2,360,875	2,347,410	3,443,249	3,422,991	3,404,988
14	1.0 "	Vertical supports, (586)		lbs.	307,418	304,359	304,285	432,831	428,698	428,613	597,074	591,567	591,269
15	1.0 "	Head struts, (593), (594)		lbs.	13,570	14,532	15,567	16,102	17,253	18,490	18,904	20,261	21,724
16	1.0 "	Head diagonals, (589), (590)		lbs.	20,204	19,849	19,608	29,443	28,856	28,436	41,744	40,835	40,166
17	1.0 "	Total least bridge weight		lbs.	5,991,360	5,988,489	5,996,885	8,674,540	8,667,056	8,675,865	12,255,920	12,241,092	12,249,044
18	1.0 "	Bridge weight per running-foot		lbs.	7,489	7,485	7,496	9,638	9,630	9,639	12,256	12,241	12,249
19	1.0 "	Panel length	$l \div n$	feet	44$\frac{4}{9}$	42$\frac{2}{19}$	40	50	47$\frac{7}{19}$	45	55$\frac{5}{9}$	52$\frac{12}{19}$	50
20	1.0 "	Ratio of length to central height,	$l \div h$		4.215	4.227	4.240	3.996	4.006	4.015	3.782	3.790	3.797
21	1.0 "	Ratio of minimum dead to live load	$W \div L$		3.7446	3.7428	3.7480	4.8192	4.8150	4.8199	6.1280	6.1205	6.1245
22	1.0 "	Ratio of minimum dead to total load	$\frac{W}{W+L}$		0.7892	0.7891	0.7894	0.8282	0.8280	0.8282	0.8597	0.8596	0.8597
23	1.0 "	If $l = 10h$ { Best number of panels,	n										
24	1.0 "	If $l = 10h$ { Bridge weight	nW	tons									
25	1.5 "	Best number of panels } simultaneous,	n		**16**	**17**	**18**	**16**	**17**	**18**	**16**	**17**	**18**
26	1.5 "	Best central height } simultaneous,	h	feet	203.969	203.061	202.173	240.059	239.174	238.295	279.688	278.814	277.951
27	1.5 "	Least bridge weight } simultaneous,	nW	tons	3,580.560	3,578.825	3,582.758	5,114.320	5,110.989	5,114.813	7,136.005	7,128.995	7,132.081
28	2.0 "	Best number of panels } simultaneous,	n		**14**	**15**	**16**	**16**	**17**	**18**	**15**	**16**	**17**
29	2.0 "	Best central height } simultaneous,	h	feet	216.367	215.091	213.824	250.668	249.488	248.326	292.131	290.863	289.690
30	2.0 "	Least bridge weight } simultaneous,	nW	tons	4,145.736	4,139.019	4,140.448	5,867.203	5,861.532	5,873.936	8,097.810	8,095.976	8,099.568
31	$L = 0$	Limiting span, $l = 10h$	l	feet									
32	$L = \frac{1}{4}W$	Limiting span, $l = 5h$	l	feet									
33	$L = 0$	Limiting span, $l = 5h$	l	feet									

183. Among the deductions to be drawn from this table, for the Brunel double-bow bridge of double web system, are the following: —

1st, For a given uniform live load,

$$n \propto l^{\frac{1}{8}} \text{ nearly;} \qquad (620)$$

and generally

$$n \propto \left(\frac{l}{nL}\right)^{\frac{1}{5}} \times l^{\frac{1}{8}} \text{ nearly.} \qquad (621)$$

2d, For spans less than 400 feet,

$$h = \frac{l}{4.8}\left(\frac{nL}{l}\right)^{\frac{1}{5}} \text{ nearly.} \qquad (622)$$

For spans of 400 feet and upwards,

$$h = \frac{l}{4.3}\left(\frac{nL}{l}\right)^{\frac{1}{4}} \text{ nearly.} \qquad (623)$$

3d, For different spans with same live load per running-foot,

$$W \propto lh \text{ nearly;} \qquad (624)$$

and for the same span under different uniform live loads,

$$nW \propto \left(\frac{nL}{l}\right)^{\frac{1}{2}} \text{ nearly.} \qquad (625)$$

Many other conclusions may be drawn from this table, and weights of intermediate spans may be derived by interpolation; but the equations (598) to (619), inclusive, cover the whole case.

184. EXAMPLE. — We now proceed to find the strain sheet for the 200-feet span of the table in article 182 in a manner similar to that employed in article 169.

We now have

$$\begin{aligned}
l &= 200 \text{ feet}, & q &= 16 \text{ feet;} \\
h &= 40.788 \text{ feet}, & q_1 &= 18 \text{ feet.} \\
n &= 13. & & \\
nL &= 200 \text{ tons}, & L &= 15\tfrac{5}{13} \text{ tons;} \\
nW &= 103.288 \text{ tons}, & W &= \underline{7.945} \text{ tons;} \\
& & W + L &= 23.330 \text{ tons.}
\end{aligned}$$

Weight of floor, by article 174, $\frac{520}{3} \times 200$	$=$ 34667 pounds;
Total live load, nL,	$=$ 400000 pounds.
Total load on longitudinal I-beams	$=$ 434667 pounds.

Load on each panel length of every longitudinal I-beam spaced 3.2 feet

$$= \frac{3.2}{16} \times \frac{434667}{13} = 6687 \text{ pounds.}$$

Then, by (502),

Cross-section of beam

$$= S = 0.00015 \times \frac{434667}{13} = 5.0154 \text{ square inches.}$$

Take $b = 4$ inches $=$ breadth of flange.
$b - b_1 = 0.26$ inch $=$ thickness of web.
Then, from (552), (551), and (550),

$$\begin{aligned}
d &= 9.774 \text{ inches} &&= \text{depth of beam,} \\
d - d_1 &= 0.661 \text{ inch} &&= \text{depth of two flanges,} \\
I &= 75.416 &&= \text{moment of inertia of section,}
\end{aligned}$$

which is larger than I for the sections given by ordinary beams of the same area of section.

By (503),

Weight of longitudinal I-beams, 6 in number,

$$= 6 \times \tfrac{5}{18} \times 12 \times 200 \times 5.01538 = 20062 \text{ pounds.}$$

Upon the transverse I-beams we have

Live load,	400000 pounds,
Floor,	34667 pounds,
Longitudinal I-beams,	20062 pounds.
Total for 13 panels,	454729 pounds.
Load on 1 beam,	34979 pounds.

From (504),

$$\frac{I}{d} = \frac{12 \times 18 \times 34979}{8 \times 2 \times 10000} = 47.2216 = 2S,$$

by (505);

$$\therefore \quad S = 23.6108 \text{ square inches for vertical load,}$$

and for the wind pressure,

$$W_1 = 2500 \times \frac{40.788}{13} = 7844 \text{ pounds per panel,}$$

$$Q_2 = \frac{8000}{1 + 0.93312 \times \frac{18}{200}} = 7542.5 \text{ pounds per square inch,}$$

$$S = \frac{2 \times 7844}{3 \times 13 \times 7542.5}(12^2, 11^2, 10^2, 9^2, 8^2, 7^2) \text{ by (511);}$$

$= 7.6798$ square inches, 1st and 12th beams;
$= 6.4532$ square inches, 2d and 11th beams;
$= 5.3332$ square inches, 3d and 10th beams;
$= 4.3199$ square inches, 4th and 9th beams;
$= 3.4132$ square inches, 5th and 8th beams;
$= 2.6133$ square inches, 6th and 7th beams.

The total cross-sections for each half-span are

$S = 31.2906$ square inches,
$= 30.0640$ square inches,
$= 28.9440$ square inches,
$= 27.9307$ square inches,
$= 27.0240$ square inches,
$= 26.2241$ square inches.

Satisfying the condition (505), we may assign values to d and d_1, and use (557) and (558) in finding the thickness of each transverse beam.

Put 2 light 12-inch beams at each panel point, the section of each being $\frac{1}{2}S$. Then we have

$$d = 12 \text{ inches} = \text{depth of beam},$$
$$d - d_1 = 2 \text{ inches} = \text{depth of 2 flanges};$$

and (558) becomes, for breadth of flanges,

$$b = \left(\frac{24 - 12}{2 \times 22} + \frac{1}{12}\right) \times \tfrac{1}{2}S = 0.17803S = 5.5707 \text{ inches},$$
$$= 5.3523 \text{ inches},$$
$$= 5.1529 \text{ inches},$$
$$= 4.9725 \text{ inches},$$
$$= 4.8111 \text{ inches},$$
$$= 4.6687 \text{ inches};$$

and (557) gives

$$b_1 = \frac{(24 - 12)12}{10 \times 2 \times 22} \times \tfrac{1}{2}S = 0.16363S = 5.1203 \text{ inches},$$
$$= 4.9083 \text{ inches},$$
$$= 4.7363 \text{ inches},$$
$$= 4.5705 \text{ inches},$$
$$= 4.4221 \text{ inches},$$
$$= 4.2912 \text{ inches}.$$

$$\text{Thickness of web} = 0.4504 \text{ inch} = b - b_1,$$
$$= 0.4440 \text{ inch},$$
$$= 0.4166 \text{ inch},$$
$$= 0.4020 \text{ inch},$$
$$= 0.3890 \text{ inch},$$
$$= 0.3775 \text{ inch}.$$

The weight of these 24 transverse I-beams is

$$12 \times 18 \times \tfrac{5}{18}\Sigma S = 20577 \text{ pounds},$$

since ΣS (= sum of all the cross-sections) is 342.9548 square inches.

Cross-sections of horizontal diagonals are found by dividing the strains in (508) by 15,000, where we now have

$$W_1 = 7844, \quad \frac{W_1}{15000} = 0.52293,$$

$$\sin\phi_1 = 0.76017,$$

$$S = \frac{0.52293}{2 \times 13 \times 0.76017}(13 \times 12, 12 \times 11,$$
$$11 \times 10, 10 \times 9, 9 \times 8, 8 \times 7, 7 \times 6)$$
$$= 0.026458 \times 156 = 4.1274 \text{ square inches,}$$
$$= 0.026458 \times 132 = 3.4925 \text{ square inches,}$$
$$= 0.026458 \times 110 = 2.9104 \text{ square inches,}$$
$$= 0.026458 \times 90 = 2.3812 \text{ square inches,}$$
$$= 0.026458 \times 72 = 1.9050 \text{ square inches,}$$
$$= 0.026458 \times 56 = 1.4816 \text{ square inches,}$$
$$= 0.026458 \times 42 = 1.1112 \text{ square inches,}$$

for the respective panels.

$$\therefore \quad \Sigma S = 67.4150 \text{ square inches,}$$

and weight of 26 horizontal diagonals is

$$\frac{12 \times 18}{\sin\phi_1} \times \frac{5}{18} \times 67.415 = 5321 \text{ pounds.}$$

The cross-section of each panel length of each wind chord is given by (514), thus,

$$S = \frac{7844 \times 200}{2 \times 13 \times 18 \times 6400}(1 \times 12, 2 \times 11,$$
$$3 \times 10, 4 \times 9, 5 \times 8, 6 \times 7, 7 \times 6)$$
$$= 0.52377 \times 12 = 6.2852 \text{ square inches,}$$
$$= 0.52377 \times 22 = 11.5229 \text{ square inches,}$$
$$= 0.52377 \times 30 = 15.7131 \text{ square inches,}$$
$$= 0.52377 \times 36 = 18.8557 \text{ square inches,}$$
$$= 0.52377 \times 40 = 20.9508 \text{ square inches,}$$
$$= 0.52377 \times 42 = 21.9983 \text{ square inches,}$$
$$= 0.52377 \times 42 = 21.9983 \text{ square inches.}$$

$$\Sigma S = 106.3253 \text{ square inches.}$$

These sections can easily be made up of channels and plates, or of beams and plates, with the required radius of gyration given in article 162.

In summing these sections for the weight formula, all are to be taken four times, except the last, which is taken twice only.

$$\text{Weight of wind chords} = \frac{12 \times 200 \times 106.3253}{13} \times \frac{5}{18} = 21810 \text{ pounds.}$$

Supported by verticals, we have

Live load,	400000 pounds,	
Floor,	34667 pounds,	
Longitudinal I-beams,	20062 pounds,	
Horizontal diagonals,	5321 pounds,	
Wind chords,	21810 pounds.	
	481860 ÷ 26 =	18533
Transverse I-beams,	20577 ÷ 24 =	857
Weight on each vertical = ε_n	=	19390

Therefore we have the cross-sections, by (580),

$$S = \frac{19390}{10667} = 1.81776 \text{ square inches;}$$

by (582),

$$S_1 = \frac{19390}{12000} = 1.61583 \text{ square inches;}$$

for the lower and upper halves respectively of the verticals due load; and, for the bending-moment due wind, (584) gives

$$S_2 = \frac{15 \times 200}{68 \times 13} = 3.39367 \text{ square inches,}$$

Section of compressed half = 5.21143 square inches,
Section of extended half = 5.00950 square inches.

And, since the upper and lower halves of the girder are symmetrical, and the sum of the lengths of the verticals $= \Sigma y = \frac{1}{3}h\left(n - \frac{1}{n}\right)$ by (521), $= \frac{1}{3} \times 40.788 \times \frac{168}{13}$, we have

Weight of lower halves

$$= 2 \times \frac{5}{18} \times 5.21143 \times \frac{12}{3} \times 40.788 \times \frac{168}{13} = 6103 \text{ pounds,}$$

Weight of upper halves

$$= 2 \times \frac{5}{18} \times 5.0095 \times \frac{12}{3} \times 40.788 \times \frac{168}{13} = 5866 \text{ pounds.}$$

Total weight $= 11969 \times \frac{11}{10}$

$= 13165$ pounds.

after adding $\frac{1}{10}$ for braces, etc.

The sections may be made up of 2 channels, the one vertical, the other inclined at an angle whose tangent is $\frac{1}{10}$.

According to the principles of article 165, the bars in the bracing of these supports should have a cross-section of about $\frac{1}{2}$ inch; that is, about $\frac{1}{10}$ of $(S + S_2)$.

Equation (587) gives the cross-section of each head diagonal thus:

$$S = \frac{0.0625 \times 168 \times 40.788 \times 200}{6 \times 13^3 \times 18 \times 0.64972} = 0.5556 \text{ square inch.}$$

From (589) comes the weight of 14 head diagonals in the seven central panels equal to

$$14 \times \frac{5}{18} \times \frac{12}{13} \times \frac{200}{0.64972} \times 0.5556 = 614 \text{ pounds.}$$

Cross-section of head struts, by (591),

$$= \frac{168 \times 40.788}{12 \times 169} = 3.3789 \text{ square inches,}$$

requiring 2 light 4-inch channels latticed not less than 4 inches apart.

Weight of 8 head struts,

$$\frac{11}{10} \times 8 \times \frac{5}{18} \times 12 \times 18 \times 3.3789 = 1784 \text{ pounds,}$$

after adding $\frac{1}{10}$ for bracing.

Increment of section of top chord due to head diagonal strain is given by (595), thus:

$$S = \frac{156.25 \times 168 \times 40.788 \times 200}{7529 \times 13^3 \times 18} = 0.7192 \text{ square inch,}$$

the strain being $= 0.7192 \times \frac{7529}{2000} = 2.707$ tons.

Weight added to top chords

$$= 2 \times 7 \times \frac{5}{18} \times \frac{12 \times 200}{13} \times 0.7192 = 516 \text{ pounds.}$$

For each of two girders, the horizontal component of chord strain is, by (564),

$$H = \frac{1}{2} \times 23.330 \times \frac{13 \times 200}{8 \times 40.788} = 92.947 \text{ tons;}$$

and the chord strains are

$$\begin{aligned} P = \frac{92.947}{\cos\alpha} = U = \frac{92.947}{\cos\beta} &= 99.317 \text{ tons, 1st panel;} \\ &= 97.415 \text{ tons, 2d panel;} \\ &= 95.830 \text{ tons, 3d panel;} \\ &= 94.580 \text{ tons, 4th panel;} \\ &= 93.672 \text{ tons, 5th panel;} \\ &= 93.130 \text{ tons, 6th panel;} \\ &= 92.947 \text{ tons, 7th panel.} \end{aligned}$$

Cross-section of top chord due load

$$\begin{aligned} = \frac{P}{3.7647} &= 26.381 \text{ square inches, 1st panel;} \\ &= 25.876 \text{ square inches, 2d panel;} \\ &= 25.455 \text{ square inches, 3d panel;} \\ &= 25.123 \text{ square inches, 4th panel;} \\ &= 24.882 \text{ square inches, 5th panel;} \\ &= 24.738 \text{ square inches, 6th panel;} \\ &= 24.689 \text{ square inches, 7th panel.} \end{aligned}$$

Augment 4th, 5th, 6th, 7th, 8th, 9th, 10th, by 0.7192.

$$\text{Cross-section of bottom chord} = \frac{U}{5} = 19.863 \text{ square inches, 1st panel;}$$
$$= 19.483 \text{ square inches, 2d panel;}$$
$$= 19.166 \text{ square inches, 3d panel;}$$
$$= 18.916 \text{ square inches, 4th panel;}$$
$$= 18.734 \text{ square inches, 5th panel;}$$
$$= 18.626 \text{ square inches, 6th panel;}$$
$$= 18.589 \text{ square inches, 7th panel.}$$

The top chord may be composed of 2 9-inch channels and 1 plate; the bottom chord, of 3 bars and 2 bars in alternate panels.

From (561), we find

$$\Sigma \sec^2 \alpha = 27.4329;$$

and (566) gives, adding $\frac{1}{10}$,

$$\text{Weight of top chords} = \frac{11}{10} \times \frac{3}{2} \times \frac{5}{18} \times \frac{11.665}{3.7647} \times \frac{200^2}{40.788} \times 27.4329$$
$$= 38206 \text{ pounds.}$$

From (567),

$$\text{Weight of bottom chords} = \frac{3.7647}{5} \times 38206 = 28767 \text{ pounds.}$$

From (571), the strain on a girder diagonal is called

$$Z = \frac{\frac{1}{4} \times 15\frac{5}{13} \times 200 \times 1.8}{2 \times 8 \times 40.788} \times \sec\theta = 2.12166 \sec\theta,$$

$= 3.1023$ tons, 2d panel. Section $= \frac{3}{8}Z = 1.16$ square inches.
$= 4.0600$ tons, 3d panel. $= 1.52$ square inches.
$= 4.8790$ tons, 4th panel. $= 1.83$ square inches.
$= 5.4861$ tons, 5th panel. $= 2.06$ square inches.
$= 5.8564$ tons, 6th panel. $= 2.20$ square inches.
$= 5.9807$ tons, 7th panel. $= 2.24$ square inches.
$= 5.8564$ tons, 8th panel. $= 2.20$ square inches.
$= 5.4861$ tons, 9th panel. $= 2.06$ square inches.
$= 4.8790$ tons, 10th panel. $= 1.83$ square inches.
$= 4.0603$ tons, 11th panel. $= 1.52$ square inches.
$= 3.1023$ tons, 12th panel. $= 1.16$ square inches.

These sections, being in alternate tension and compression, may be made up of 4 angle irons, $1\frac{1}{2} \times 1\frac{1}{2}$, latticed at such a distance apart that the unsupported length may be not more than one hundred times the radius of gyration of the section.

The weight of these girder diagonals is, from (572), equal to

$$\frac{4 \times 12 \times 5 \times 0.45 \times 200^3 \times 3}{8 \times 13^2 \times 18 \times 40.788 \times 8} \times \Sigma \sec^2\theta \times \frac{11}{10} = 326.41 \Sigma \sec^2\theta \times \frac{11}{10}$$
$$= 21088 \text{ pounds,}$$

since $\Sigma \sec^2\theta = 58.7324$ by (563), and we increase by one-tenth for latticing and attachments.

STRAINS AND CROSS-SECTIONS.

TONS. SQUARE INCHES.

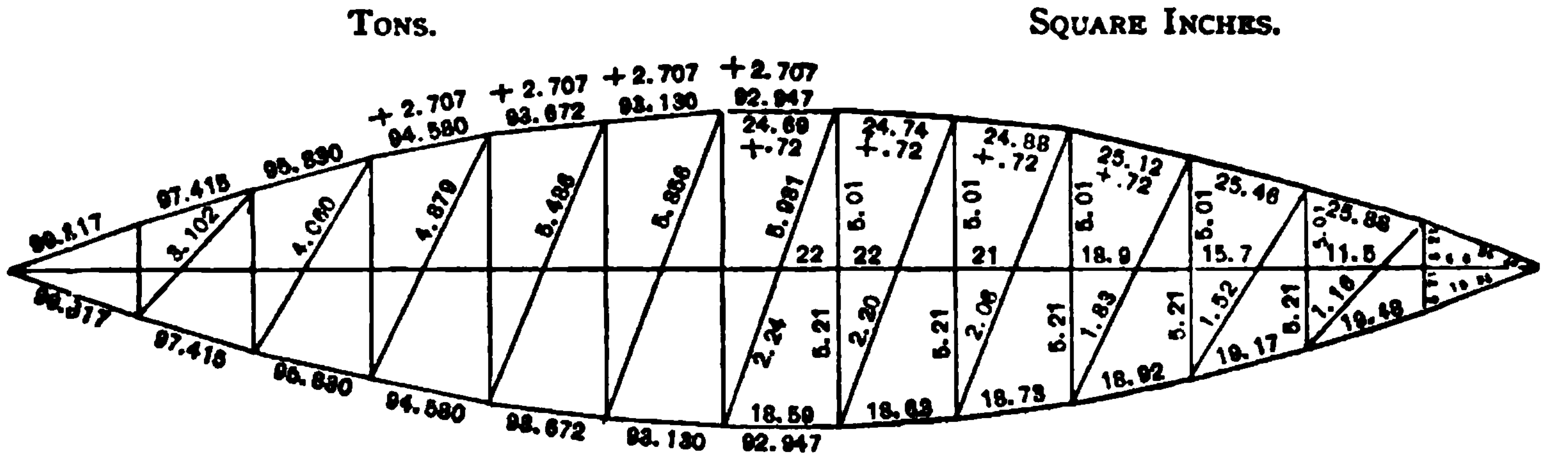


FIG. 119.

For each of two girders. Span, 200 feet. Central height, 40.788 feet. Uniform live load, 1 ton = 2,000 pounds per linear foot, applied at centres of verticals; the verticals in this case acting merely as struts in the lower half, and as suspenders in the upper half. The diagonals, only one-half of them being shown in the figure, are alternately in tension and compression; the greatest strains being the same on each of the two diagonals of a panel. Bridge weight = 103.288 tons.

The deflection due to full load is found for any point by equation (559), having now

$$B_1 = 4.380 \text{ tons per square inch,}$$
$$E = 12000 \text{ tons per square inch,}$$
$$h_1 = h = 40.788 \text{ feet,}$$
$$a = \tfrac{1}{2}l = 100 \text{ feet,}$$

and x being measured from centre of span;

$$
\begin{aligned}
\therefore \quad \text{Deflection } D_1 &= 1.490 \text{ inches for } x = 0, \quad \text{centre;} \\
D_2 &= 1.483 \text{ inches for } x = 100 \div 13; \\
D_3 &= 1.432 \text{ inches for } x = 300 \div 13; \\
D_4 &= 1.327 \text{ inches for } x = 500 \div 13; \\
D_5 &= 1.162 \text{ inches for } x = 700 \div 13; \\
D_6 &= 0.815 \text{ inch for } x = 900 \div 13; \\
D_7 &= 0.583 \text{ inch for } x = 1100 \div 13; \\
D_8 &= 0 \text{ inch for } x = 1300 \div 13, \text{ ends.}
\end{aligned}
$$

Equation (366) yields the excess of length required in top chord to give the proper camber,

$$\lambda = \frac{3.76 + 5}{12000} \times 205.34 \times 12 = 1.8 \text{ inches,}$$

since length of polygonal top chord is equal to

$$\frac{200}{13} \times \Sigma \sec \alpha = \frac{200}{13} \times 13.3466 = 205.34 \text{ feet;}$$

$$\therefore \quad \text{Mean excess per panel} = \frac{1.8}{13} = 0.138 \text{ inch,}$$

or a little more than $\frac{1}{8}$ inch.

CHAPTER XI.

BRIDGES OF CLASS II. — BEST NUMBER OF PANELS AND BEST HEIGHT DETERMINED FOR A GIVEN SPAN UNDER A GIVEN UNIFORM LIVE LOAD. — LEAST BRIDGE WEIGHT AND LIMITING SPAN FOUND.

SECTION I.

The Parabolic Bowstring Girder of Double Triangular System (Fig. 35), with the Extreme Diagonals omitted, and a Vertical Suspender at Extreme Panel Point.

185. Let l = span, in feet.
h = height of girder at centre, in feet.
n = number of panels.
L = panel weight of uniform live load, in tons.
W = panel weight of bridge, in tons.

The height of girder at any point, x, is given by (472), and at all vertices by (473), if we make $r = 1$ for the first point, and put $2h$ for h throughout, thus:

$$y_r = \frac{4h}{n^2}r(n - r), \tag{626}$$

$$y_{r+1} = \frac{4h}{n^2}(r + 1)(n - r - 1),$$

$$\Delta y = y_{r+1} - y_r = \frac{4h}{n^2}(n - 2r - 1), \tag{627}$$

$$\tan\alpha = \Delta y \div \frac{l}{n} = \frac{4h}{nl}(n - 2r - 1), \tag{628}$$

$$\sec^2\alpha = 1 + \tan^2\alpha = 1 + \frac{16h^2}{n^2l^2}(n - 2r - 1)^2, \tag{629}$$

$$\tan\theta_r = -\tan\phi_{r-1} = -y_r \div \frac{l}{n} = -\frac{4h}{nl}r(n-r), \quad (630)$$

$$\sec^2\theta = 1 + \tan^2\theta = 1 + \frac{16h^2}{n^2l^2}r^2(n-r)^2. \quad (631)$$

α, θ, and ϕ are defined in article 49.

The live load and a large part of the dead load are applied at the panel points of the bottom chord, and are transmitted by the diagonals to the parabolic arch, which is equilibrated by the uniform load, leaving only a tensile strain on the diagonals from full uniform load. We shall assume that the two diagonals which support any panel weight of uniform dead load carry each one-half of the same.

186. Moments at all panel points due $n(W+L)$, the total load, are, from equation (65),

$$M_r = \frac{W+L}{2n}l(n-r)r;$$

and the horizontal component of chord strain under same load is equal to

$$H = \frac{M}{y} = \tfrac{1}{8}(W+L)\frac{nl}{h},$$

as in (564), and is uniform throughout for maximum.

$$\left.\begin{array}{ll}\text{Greatest strains in top chord} & = P = \dfrac{H}{\cos\alpha} \\ \text{Greatest strains in bottom chord} & = U = H\end{array}\right\}, \quad (632)$$

$$\left.\begin{array}{ll}\text{Cross-section of top chord, } S & = P \div Q \\ \text{Cross-section of bottom chord, } S_1 & = U \div T\end{array}\right\}, \quad (633)$$

Take

$$\left.\begin{array}{ll} Q = \dfrac{4}{1 + \dfrac{50^2}{40000}} = 3.7647 \text{ tons} \\ T = \qquad\qquad 5 \quad \text{tons}\end{array}\right\} \quad (634)$$

as the allowed inch strains on top and bottom chords respectively.

$$\text{Length of segment of top chord} = \frac{12l}{n\cos\alpha} \text{ inches,}$$

Volume of segment of top chord

$$= \frac{12lH}{nQ\cos^2\alpha} = \tfrac{3}{2}(W+L)\frac{l^2}{Qh\cos^2\alpha} \text{ cubic inches,}$$

$$\Sigma\sec^2\alpha = n + \frac{16h^2}{3l^2}\left(n - \frac{1}{n}\right), \qquad (635)$$

by summing (629) for values of r from 0 to $n - 1$, inclusive.

Therefore, calling weight of a cubic inch of wrought-iron, as in all cases, $\frac{5}{18}$ pound, we find

Weight of top chords, in pounds,

$$= \tfrac{3}{2} \times \tfrac{5}{18}(W+L)\frac{l^2}{Qh}\Sigma\sec^2\alpha, \qquad (636)$$

$$= \frac{5(W+L)l^2}{12 \times 3.7647h}\left\{n + \frac{16h^2}{3l^2}\left(n - \frac{1}{n}\right)\right\}$$

$$= \frac{W+L}{h}\left\{0.1106771nl^2 + 0.59028\left(n - \frac{1}{n}\right)h^2\right\}$$

$= \dfrac{W+L}{h}$ ×

	n
$0.885417l^2 + 4.64844h^2$	8
$0.996095l^2 + 5.24688h^2$	9
$1.106772l^2 + 5.84372h^2$	10
$1.217449l^2 + 6.43936h^2$	11
$1.328126l^2 + 7.03412h^2$	12
$1.438804l^2 + 7.62820h^2$	13
$1.549481l^2 + 8.22168h^2$	14
$1.660158l^2 + 8.81480h^2$	15
$1.770835l^2 + 9.40752h^2$	16
$1.881512l^2 + 10.00000h^2$	17
$1.992190l^2 + 10.59216h^2$	18
$2.102867l^2 + 11.18440h^2$	19
$2.213544l^2 + 11.77600h^2$	20

Weight of bottom chords, in pounds, $= \frac{5}{18} \times \frac{12}{8} \frac{W + L}{Th} \times nl^2$, (637)

$$= \frac{W + L}{h} \times \frac{l^2 n}{12}$$

$= \frac{W + L}{h}$		n
	$0.666667l^2$	8
	$0.750000l^2$	9
	$0.833333l^2$	10
	$0.916667l^2$	11
	$1.000000l^2$	12
	$1.083333l^2$	13
	$1.166667l^2$	14
	$1.250000l^2$	15
	$1.333333l^2$	16
	$1.416667l^2$	17
	$1.500000l^2$	18
	$1.583333l^2$	19
	$1.666667l^2$	20

187. **The Girder Diagonals.** — Separating the double system into the single web systems of Fig. 35*a* and Fig. 27, let us consider first that of Fig. 35*a*, and find the difference, ΔH, of horizontal strains at the foremost end of the advancing uniform discontinuous load, nL, and for the same instant at the next two forward panel points of *the double system.*

Putting L for W in equation (60), and taking $r_1 = -\frac{1}{2}$, we find the moment at any point, x, at or before the foremost end, to be

$$M_x = \frac{Lc}{2l}(r + \tfrac{1}{2})^2(l - x), \tag{638}$$

where c = length of whole interval in the single system $= \frac{2l}{n}$ in the double system, $r + \frac{1}{2}$ = number of panel points of bottom chord loaded, x = distance from left end to the point where moment is taken.

From (472), putting $2h$ for h,

$$y = \frac{4h}{l^2}x(l - x). \tag{639}$$

Therefore the simultaneous horizontal strains due live load at the three consecutive panel points, of which the foremost end of live load is at the rear one, are

$$H = \frac{M_x}{y} = \frac{Llc}{8h} \cdot \frac{(r + \frac{1}{2})^2}{x}, \tag{640}$$

$$= \frac{Ll}{8h}\frac{(r + \frac{1}{2})^2}{r} \text{ if } x = rc, \text{ foremost end;}$$

$$= \frac{Ll}{8h}(r + \tfrac{1}{2}) \quad \text{if } x = (r + \tfrac{1}{2})c, \text{ next panel point;}$$

$$= \frac{Ll}{8h}\frac{(r + \frac{1}{2})^2}{r + 1} \text{ if } x = (r + 1)c, \text{ next panel point;}$$

where r takes the successive values $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \ldots \frac{n - 1}{2}$.

Then we find the greatest differences, that is, the greatest horizontal component of diagonal strain due live load, thus:

$$\left.\begin{aligned} \Delta_{\frac{1}{2}}H &= \frac{Ll}{8h}\left\{r + \tfrac{1}{2} - \frac{(r + \frac{1}{2})^2}{r}\right\} = -\frac{Ll}{16h}\frac{r + \frac{1}{2}}{r} \\ \Delta_1 H &= \frac{Ll}{8h}\left\{\frac{(r + \frac{1}{2})^2}{r + 1} - (r + \tfrac{1}{2})\right\} = -\frac{Ll}{16h}\frac{r + \frac{1}{2}}{r + 1} \end{aligned}\right\} \tag{641}$$

for the first single system; r taking the successive values $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}$, etc.

Similarly, for the second single system, we get greatest horizontal component of diagonal strain due live load,

$$\left.\begin{aligned} \Delta_{\frac{1}{2}}H &= -\frac{Ll}{16h}\frac{r + 1}{r + \frac{1}{2}} \\ \Delta_1 H &= -\frac{Ll}{16h}\frac{r}{r + \frac{1}{2}} \end{aligned}\right\}, \tag{642}$$

where r becomes 1, 2, 3, 4, etc.

Now, if we consider the horizontal components of the two diagonals in the same panel of the double system, Fig. 35, with its extreme tie made vertical, we see that $\Delta_1 H$ of (641) and $\Delta_{\frac{1}{2}} H$ of (642) belong to the odd panels, and $\Delta_{\frac{1}{2}} H$ of (641) and $\Delta_1 H$ of (642) belong to the even panels. Also, for the odd panels, r of (641) is less by $\frac{1}{2}$ than r of (642); and, for the even panels, r of (641) is greater by $\frac{1}{2}$ than r of (642).

Therefore, reducing so that r belongs to the second system, we find

$$\left.\begin{array}{ll} \text{Compression,} & \Delta_1 H = -\dfrac{Ll}{16h}\dfrac{r}{r+\frac{1}{2}} \\ \text{Tension,} & \Delta_{\frac{1}{2}} H = -\dfrac{Ll}{16h}\dfrac{r+1}{r+\frac{1}{2}} \end{array}\right\} \text{odd panels,} \qquad (643)$$

$$\left.\begin{array}{ll} \text{Tension,} & \Delta_{\frac{1}{2}} H = -\dfrac{Ll}{16h}\dfrac{r+1}{r+\frac{1}{2}} \\ \text{Compression,} & \Delta_1 H = -\dfrac{Ll}{16h}\dfrac{r}{r+\frac{1}{2}} \end{array}\right\} \text{even panels,} \qquad (644)$$

which expressions are identical; and the sum of either pair is

$$\Delta_1 H + \Delta_{\frac{1}{2}} H = -\frac{Ll}{16h}\frac{2(r+\frac{1}{2})}{r+\frac{1}{2}} = -\frac{Ll}{8h}, \qquad (645)$$

as already given by (570) for the total horizontal component of diagonal strain due live load in any panel.

Since these diagonals are to be alternately in tension and compression, the load travelling either way, and our specifications would multiply the compressive strains by 1.8, we shall, for convenience, take

$$\Delta H = -\frac{Ll}{8h}$$

due live load for each diagonal in a panel, instead of multiply-

ing by 1.8, and shall treat all diagonals as in compression under the inch strain,

$$Q = \frac{4}{1 + \frac{100^2}{20000}} = \frac{8}{3} \text{ tons.}$$

This procedure varies a little from the specifications, but on the safe side, since

$$2r + 1 > 1.8r.$$

Moreover, since the dead load, with the exception of the top chords and head system and wind braces, is suspended at all times on two diagonals which transmit it to the equilibrated top chord, and since the tensile section will, for practical spans, not be greater than the compressive section due to live load as above augmented, and to be provided for in compression, we may, as appears below, leave the tensile strain which will come upon the diagonals acting as suspenders, almost entirely to the material put into them to resist maximum compression.

It is to be observed, that, when the live load is fully on the bridge, there is no compression on the girder diagonals, but each one acts simply as a suspender to transmit $\frac{1}{2}(W_2 + L)$ to the equilibrated top chord; W_2 being that part of the dead load at any lower apex. Now, from the results tabulated in Chap. X., we may doubtless, in the present case, for spans not over 600 feet, consider W_2 as ranging from $\frac{4}{5}W$ to $\frac{2}{3}W$, while W ranges from $\frac{1}{8}L$ to $2L$ approximately. Therefore $\frac{1}{2}(W_2 + L)$ ranges from $\frac{19}{30}L$ to $\frac{7}{6}L$ nearly, in spans from 100 feet to 600 feet.

Taking the greater, $\frac{7}{6}L$, as the vertical component of tension on each girder diagonal, we have

$$\tfrac{7}{6}L \cot\theta = \frac{7}{24} \times \frac{Ll}{h} \times \frac{n}{r(n-r)}$$

as the horizontal component of tension on diagonals acting as

suspenders; the greatest value of which is found when $r = 1$, or $r = n - 1$. That is,

$$(\Delta H)_{\max} = \frac{7}{24} \cdot \frac{Ll}{h} \cdot \frac{n}{n-1} \text{ for tension.}$$

But

$$\Delta H = -\frac{Ll}{8h} \text{ for compression.}$$

Dividing $\Delta H_{\max}$ and ΔH by 5 and by $\frac{8}{3}$, the allowed inch strains in tons respectively for tension and compression, and multiplying by $\sec\theta$, we find these resulting cross-sections for comparison:

$$\left.\begin{aligned} S_1 &= \frac{1}{17} \times \frac{Ll}{h\cos\theta} \times \frac{n}{n-1} \text{ nearly} \\ S &= \frac{3}{64} \times \frac{Ll}{h\cos\theta} \end{aligned}\right\} . \quad (646)$$

Now, S_1 will be greater than S only when $r = 1$, or $r = n - 1$; that is, the girder diagonals which meet at the first and second and at the $(n-2)^{\text{th}}$ and $(n-1)^{\text{th}}$ lower apices, will need additional section under full load to the extent of the difference between S_1 and S. And this we shall supply in the vertical braces at these points.

$$\therefore \text{ Cross-section of a girder diagonal} = S = \frac{3Ll}{64h\cos\theta}; \quad (647)$$

Weight of $2(n-2)$ girder diagonals, in pounds,

$$= 2 \times \frac{12l^2}{n} \times \frac{5}{18} \times \frac{3}{64} \times \frac{L}{h}\Sigma\sec^2\theta, \quad (648)$$

$$= \frac{5Ll^2}{16nh}\Sigma\sec^2\theta$$

$$= \frac{5Ll^2}{16nh}\left\{n - 2 + \frac{16h^2}{n^2l^2}\left(\frac{n^5}{30} - n^2 + \frac{59}{30}n - 1\right)\right\}$$

$$= \frac{L}{h}\left\{\frac{5(n-2)l^2}{16n} + \left(\frac{n^2}{6} - \frac{5}{n} + \frac{59}{6n^2} - \frac{5}{n^3}\right)h^2\right\}$$

		n
$= \frac{L}{h}$	$0.234375l^2 + 10.18555h^2$	8
	$0.243056l^2 + 13.05899h^2$	9
	$0.250000l^2 + 16.26000h^2$	10
	$0.255682l^2 + 19.78964h^2$	11
	$0.260417l^2 + 23.64873h^2$	12
	$0.264423l^2 + 27.83796h^2$	13
	$0.267857l^2 + 32.35788h^2$	14
	$0.270833l^2 + 37.20889h^2$	15
	$0.273437l^2 + 42.39136h^2$	16
	$0.275735l^2 + 47.90556h^2$	17
	$0.277778l^2 + 53.75172h^2$	18
	$0.279605l^2 + 59.93002h^2$	19
	$0.281250l^2 + 66.44062h^2$	20

Adding together (636), (637), and (648), we find

Weight of girders due to loads, pounds,

				n
$= \frac{L}{h}$	$1.786459l^2 + 14.83399h^2$	$+ \frac{W}{h}$	$1.552084l^2 + 4.64844h^2$	8
	$1.989151l^2 + 18.30587h^2$		$1.746095l^2 + 5.24688h^2$	9
	$2.190105l^2 + 22.10372h^2$		$1.940105l^2 + 5.84372h^2$	10
	$2.389798l^2 + 26.22900h^2$		$2.134116l^2 + 6.43936h^2$	11
	$2.588543l^2 + 30.68285h^2$		$2.328126l^2 + 7.03412h^2$	12
	$2.786560l^2 + 35.46616h^2$		$2.522137l^2 + 7.62820h^2$	13
	$2.984005l^2 + 40.57956h^2$		$2.716148l^2 + 8.22168h^2$	14
	$3.180991l^2 + 46.02369h^2$		$2.910158l^2 + 8.81480h^2$	15
	$3.377605l^2 + 51.79888h^2$		$3.104168l^2 + 9.40752h^2$	16
	$3.573914l^2 + 57.90556h^2$		$3.298179l^2 + 10.00000h^2$	17
	$3.769968l^2 + 64.34388h^2$		$3.492190l^2 + 10.59216h^2$	18
	$3.965805l^2 + 71.11442h^2$		$3.686200l^2 + 11.18440h^2$	19
	$4.161461l^2 + 78.21662h^2$		$3.880211l^2 + 11.77600h^2$	20

To be augmented by one-tenth.

188. Floor to be the same as in article 174.

$$\text{Weight of floor} = \frac{520}{3} l \text{ pounds.} \qquad (649)$$

189. Take longitudinal I floor beams, as in articles 157, 175, equation (575).

190. The transverse I-beams supporting live load, floor, and longitudinal beams are here conditioned as in article 158, and their cross-section due vertical forces is given by equation (506). Their weight due same forces is given by (576).

The cross-sections of transverse I-beams due to wind are expressed in (511).

The weight of iron to be added to the transverse I-beams on account of wind, is given by (512) and (577) for n even and n odd.

The whole effect of wind pressure is to be transferred to the horizontal system, in the plane of the bottom chords, by means of vertical braces connecting each transverse beam with both top chords.

It may be observed, that in this and all like cases a shearing-stress is generated throughout the transverse beam by the wind pressure transmitted through these vertical braces; but there will be sufficient reserve material in the web of the beam to resist this shearing-stress, as becomes evident on reflection.

191. If we divide equation (508) by 15,000, we have the cross-section of any horizontal diagonal in the floor system. And equations (509) and (578) give us the weight of the horizontal diagonals, in pounds, for the even and odd values of n.

192. For wind chords, let us use the bottom chords of the girders, augmenting their cross-sections by the quantity in (513) divided by the tensile inch strain, 10,000 pounds.

Although this augment will only resist tension, while the compressive chord strain due to wind will sometimes be greater than the tensile chord strain due to dead load, yet, as the

excess of compression is not great, it may be left safely to the outside longitudinal I-beams, which, it will be remembered, are otherwise only half loaded.

We may compare the chord strains due dead load and due wind by means of Fig. 112, thus:

$$\text{For } W,\ N = \frac{Wl}{2nh};$$

$$\text{For } W_1,\ N = \frac{W_1 l}{2nq_1}.$$

These co-efficients of strain will be equal when

$$q_1 = \frac{W_1 h}{W}; \qquad (650)$$

and this might be made a condition determining the width, q_1, of the bridge, so that no compressive strain would prevail in a bottom chord. But we shall not now change the uniform value of $q_1 = 18$, assumed at first.

Therefore the increase of section of each bottom chord due to wind, as derived from (513), is, in square inches,

$$S = \frac{W_1 l}{20000nq_1}\left\{(n-1),\ 2(n-2),\ 3(n-3),\ \text{etc.}\right\}. \qquad (651)$$

The weight of this wind augment to bottom chords is found by putting 10,000, instead of 6,400, for Q in equations (515) and (579), thus:

Weight of bottom chords due to wind, pounds,

$$\left.\begin{aligned} &= 0.00385802496\left(2 + \frac{3}{n} - \frac{2}{n^2}\right)hl^2 \qquad (n \text{ even}) \\ &= 0.00385802496\left(2 + \frac{3}{n} - \frac{2}{n^2} - \frac{3}{n^3}\right)hl^2 \quad (n \text{ odd}) \end{aligned}\right\} \qquad (652)$$

$= hl^2$	0.0090422	$n =$ 8	$= hl^2$	0.0084499	$n =$ 15
	0.0088909	9		0.0084093	16
	0.0087963	10		0.0083678	17
	0.0086957	11		0.0083352	18
	0.0086272	12		0.0083021	19
	0.0085555	13		0.0082755	20
	0.0085034	14			

193. Assuming that a wind pressure per panel of $125\frac{l}{n}$ pounds (l being in feet) acts in a direction normal to the plane of girder at each apex in each top chord, we have the moment of each brace due wind equal to

$$125\frac{l}{n} \times y = \tfrac{1}{2}S \times \tfrac{1}{10}B_1 y \qquad (653)$$

if S = cross-section of flanges of a brace, B_1 = inch strain allowed for bending-moment $= \frac{1}{2}(5,333 + 6000) = \frac{17000}{3}$, and if the length of brace is to its width at broadest end in the ratio of 10 to 1, as in article 163, where the flanges of each brace meet at one end, and diagonal lattice work forms the web. From (653),

Cross-section of a brace due wind, in square inches,

$$= S = \frac{2500l}{nB_1} = \frac{15l}{34n}. \qquad (654)$$

Increase this section by 50 per cent for the first and second braces, to take a part of load, as explained in article 187. See value of Σy.

Weight of vertical braces, pounds,

$$= 2 \times \frac{5}{18} \times \frac{15l}{34n} \times 12\Sigma y$$

$$= 1.9607844\left(1 + \frac{17}{n^2} - \frac{30}{n^3}\right)hl \qquad (655)$$

$= hl$	2.366729	$n =$ 8	$= hl$	2.091502	$n =$ 15
	2.291617	9		2.076631	16
	2.235294	10		2.064152	17
	2.192072	11		2.053578	18
	2.158224	12		2.044544	19
	2.131248	13		2.036764	20
	2.109415	14			

since we now have the special value

$$\Sigma y = \frac{4h}{n^2}\left\{\begin{array}{l}(n-1) + 2(n-2) + 3(n-3) + \ldots (n-1) \text{ terms} \\ \qquad\qquad + (n-1) + 2(n-2) \text{ for suspenders}\end{array}\right\}$$

$$= \tfrac{2}{3}h\left(n + \frac{17}{n} - \frac{30}{n^2}\right).$$

194. **The Head Lateral System.** — Cross-section of head diagonals is given by equations (587), (588). Weight of head diagonals found in (589) and (590).

Cross-section of head strut expressed in (591) and (592).

Weight of head struts is given by (593) and (594).

Cross-section of iron to be added to segments of top chord shown in (595), (596).

Weight of added iron in $(n - 6)$ panels of top chords is to be found in equation (597).

195. We may now collect the weights of all the parts of the bridge, and, after augmenting by one-tenth of itself the weight of the girders, the vertical braces, and the head struts, as explained in article 165, we may equate the weight so found to $2000nW$, and so determine W in terms of L, l, and h, for different values of n, and from $\frac{dW}{dh} = 0$ we find h rendering W a minimum.

$$n = 8.$$

$$W = \frac{h[L(6.21262l + 567) + 0.067303l^2 + 179.4758l] + 1.965105Ll^2 + h^2(16.31739L + 0.0116334l^2 + 2.603402l + 111.361 + 344.09l^{-1})}{-1.707292 4l^2 + 16000h - 5.113284h^2}; \quad (656)$$

Limiting span, $l = 2345.84$ feet if $h = l$, $\quad l = 1673.77$ feet if $h = l \div 5$, $\quad l = 909.91$ feet if $h = l \div 10$.

$$n = 9.$$

$$W = \frac{h[L(6.2160l + 648) + 0.059857l^2 + 179.5733l] + 2.188066Ll^2 + h^2(20.13646L + 0.0112890l^2 + 2.520779l + 130.987 + 451.01l^{-1})}{-1.9207045l^2 + 18000h - 5.771568h^2}; \quad (657)$$

Limiting span, $l = 2340.01$ feet if $h = l$, $\quad l = 1673.2$ feet if $h = l \div 5$, $\quad l = 909.81$ feet if $h = l \div 10$.

$$n = 10.$$

$$W = \frac{h[L(6.2187l + 729) + 0.053895l^2 + 179.6513l] + 2.4091155Ll^2 + h^2(24.31409L + 0.0110482l^2 + 2.458823l + 151.125 + 565.71l^{-1})}{-2.1341155l^2 + 20000h - 6.428092h^2}; \quad (658)$$

Limiting span, $l = 2335.85$ feet if $h = l$, $\quad l = 1672.78$ feet if $h = l \div 5$, $\quad l = 909.76$ feet if $h = l \div 10$.

$$n = 11.$$

$$W = \frac{h[L(6.2209l + 810) + 0.0490127l^2 + 179.7150l] + 2.628778Ll^2 + h^2(28.8519L + 0.0107913l^2 + 2.411279l + 170.7436 + 699.85l^{-1})}{-2.3475276l^2 + 22000h - 7.083296h^2}; \quad (659)$$

Limiting span, $l = 2332.78$ feet if $h = l$, $\quad l = 1672.46$ feet if $h = l \div 5$, $\quad l = 909.71$ feet if $h = l \div 10$.

$$n = 12.$$

$$W = \frac{h[L(6.2227l + 891) + 0.044942l^2 + 179.7683l] + 2.847397Ll^2 + h^2(33.75113L + 0.0106026l^2 + 2.374046l + 190.895 + 841.75l^{-1})}{-2.5609386l^2 + 24000h - 7.737532h^2}; \quad (660)$$

Limiting span, $l = 2330.44$ feet if $h = l$, $\quad l = 1672.22$ feet if $h = l \div 5$, $\quad l = 909.67$ feet if $h = l \div 10$.

$$n = 13.$$

$$W = \frac{\begin{array}{r} h[L(6.2243l + 972) + 0.041495l^2 + 179.8133l] + 3.065216Ll^2 \\ + h^2(39.01278L + 0.0104075l^2 + 2.344373l + 210.5148 + 1003.11l^{-1}) \end{array}}{-2.7743507l^2 + 26000h - 8.39102h^2}; \quad (661)$$

Limiting span, $l = 2328.63$ feet if $h = l$, $l = 1672.03$ feet if $h = l \div 5$, $l = 909.65$ feet if $h = l \div 10$.

$$n = 14.$$

$$W = \frac{\begin{array}{r} h[L(6.2256l + 1053) + 0.0385399l^2 + 179.8519l] + 3.282405Ll^2 \\ + h^2(44.6375L + 0.0102569l^2 + 2.320356l + 230.662 + 1172.18l^{-1}) \end{array}}{-2.9877628l^2 + 28000h - 9.043848h^2}; \quad (662)$$

Limiting span, $l = 2327.2$ feet if $h = l$, $l = 1671.89$ feet if $h = l \div 5$, $l = 909.62$ feet if $h = l \div 10$.

$$n = 15.$$

$$W = \frac{\begin{array}{r} h[L(6.2268l + 1134) + 0.035971l^2 + 179.8853l] + 3.49909Ll^2 \\ + h^2(50.62606L + 0.0101045l^2 + 2.30065l + 250.288 + 1360.80l^{-1}) \end{array}}{-3.2011738l^2 + 30000h - 9.69628h^2}; \quad (663)$$

Limiting span, $l = 2326.04$ feet if $h = l$, $l = 1671.77$ feet if $h = l \div 5$, $l = 909.61$ feet if $h = l \div 10$.

$$n = 16.$$

$$W = \frac{\begin{array}{r} h[L(6.2278l + 1215) + 0.033734l^2 + 179.9145l] + 3.715365Ll^2 \\ + h^2(56.9788L + 0.0099827l^2 + 2.28429l + 270.422 + 1557.15l^{-1}) \end{array}}{-3.4145848l^2 + 32000h - 10.348272h^2}; \quad (664)$$

Limiting span, $l = 2325.1$ feet if $h = l$, $l = 1671.67$ feet if $h = l \div 5$, $l = 909.59$ feet if $h = l \div 10$.

$$n = 17.$$

$$W = \frac{\begin{array}{r} h[L(6.2287l + 1296) + 0.031754l^2 + 179.9404l] + 3.93105Ll^2 \\ + h^2(63.6961L + 0.009861l^2 + 2.27057l + 290.062 + 1772.93l^{-1}) \end{array}}{-3.6279969l^2 + 34000h - 11h^2}; \quad (665)$$

Limiting span, $l = 2324.31$ feet if $h = l$, $l = 1671.58$ feet if $h = l \div 5$, $l = 909.58$ feet if $h = l \div 10$.

$$n = 18.$$

$$W = \frac{h[L(6.2295l + 1377) + 0.029994l^2 + 179.9633l] + 4.146965Ll^2 + h^2(70.7783L + 0.0097613l^2 + 2.258936l + 310.181 + 1996.49l^{-1})}{-3.841409l^2 + 36000h - 11.65138h^2}; \quad (666)$$

Limiting span, $l = 2323.66$ feet if $h = l$, $l = 1671.52$ feet if $h = l \div 5$, $l = 909.57$ feet if $h = l \div 10$.

$$n = 19.$$

$$W = \frac{h[L(6.2302l + 1458) + 0.028413l^2 + 179.9838l] + 4.362385Ll^2 + h^2(78.2259L + 0.0096616l^2 + 2.248998l + 329.833 + 2239.49l^{-1})}{-4.05482l^2 + 38000h - 12.30284h^2}; \quad (667)$$

Limiting span, $l = 2323.08$ feet if $h = l$, $l = 1671.46$ feet if $h = l \div 5$, $l = 909.56$ feet if $h = l \div 10$.

$$n = 20.$$

$$W = \frac{h[L(6.2308l + 1539) + 0.027l^2 + 180.0023l] + 4.577607Ll^2 + h^2(86.03828L + 0.0095784l^2 + 2.24044l + 349.938 + 2490.27l^{-1})}{-4.268232l^2 + 40000h - 12.9536h^2}; \quad (668)$$

Limiting span, $l = 2322.64$ feet if $h = l$, $l = 1671.41$ feet if $h = l \div 5$, $l = 909.55$ feet if $h = l \div 10$.

The limiting spans are found by placing each denominator equal to zero, and replacing h by its assigned value.

If a relation between W and L be assumed, we may substitute for L its value in terms of W, and find W and the limiting span as before.

HIGHWAY BRIDGE. TWO PARABOLIC BOWSTRING GIRDERS. UNIFORM LIVE LOAD. FIG. 35.

1	$nL \div l$ Tons.	Span	l	feet	200			400			600		
2	1.0	Best number of panels	n		**12**	**13**	**14**	**14**	**15**	**16**	**15**	**16**	**17**
3	1.0	Best central height	h	feet	40.325	39.726	39.134	83.584	82.871	82.154	134 292	133.514	132.781
4	1.0	Least bridge weight	nW	tons	106.234	106.157	106.285	455.119	454.685	455.032	1,245.722	1,244.937	1,245.175
5	1.0	Top chords, Equation (636)		lbs.	44,943	45,380	45,857	245,456	246,683	248,165	762,604	764,945	767.584
6	1.0	Do. head system, (597)		lbs.	517	503	491	4,195	4,059	3,944	14,798	14,429	13,934
7	1.0	Bottom chords, (637)		lbs.	27,844	28,257	28,697	150,049	151,264	152,647	453,556	456,003	458,580
8	1 0	Do. for wind, (652)		lbs.	13,917	13,595	13,311	113,720	112,041	110,537	408,511	404,194	399,991
9	1.0	Girder diagonals, (648)		lbs.	22,219	23,220	24,201	101,117	105,789	110,417	251,807	263,882	275,978
10	1 0	Floor, (649)		lbs.	34,667	34,667	34,667	69,333	69,333	69,333	104,000	104,000	104,000
11	1.0	I-Beams, longitudinal, (575)		lbs.	21,733	20,062	18,628	74,514	69,547	65,200	156,480	146,700	138,070
12	1.0	I-Beams, transverse, (576)		lbs.	16,944	17,000	17,047	35,496	35,488	35,483	55,204	55,081	54,968
13	1 0	Do. for wind, (512), (577)		lbs.	3,201	3,484	3,741	7,745	8,338	8,888	13,361	14,272	15,232
14	1.0	Horizontal diagonals, (509), (578)		lbs.	5,222	5,182	5,194	23,955	23,063	22,406	69,466	66,387	63,571
15	1.0	Braces, (655)		lbs.	19,147	18,627	18,161	77,578	76,263	75,066	185,375	182,991	180,897
16	1.0	Head diagonals, (589), (590)		lbs.	562	598	638	2,942	2,965	3,008	8,930	8,912	8,812
17	1.0	Head struts, (593), (594)		lbs.	1,552	1,738	1,937	4,138	4,537	4,970	7,353	8,078	8,733
18	1.0	Bridge weight per running-foot		lbs.	1,062	1,061	1,063	2,275	2,273	2,275	4,152	4,149	4,150
19	1.0	Panel length	$l \div n$	feet	$16\frac{2}{3}$	$15\frac{5}{13}$	$14\frac{2}{7}$	$28\frac{4}{7}$	$26\frac{2}{3}$	25	40	$37\frac{1}{2}$	$35\frac{5}{17}$
20	1.0	Ratio of length to central height	$l \div h$		4.960	5.034	5.111	4.786	4.827	4.869	4.468	4.494	4.519
21	1.0	Ratio of minimum load to live load	$W \div L$		0.5311	0.5307	0.5314	1.1378	1.1367	1.1376	2.0762	2.0749	2.0753
22	1.0	Ratio of minimum dead to total load,	$\frac{W}{W+L}$		0.3469	0.3467	0.3470	0.4765	0.4763	0.4765	0.6749	0.6748	0.6748
23	1.0	Best number of panels } $l = 10h$	n		**18**	**19**	**20**						
24	1.0	Bridge weight } $l = 10h$	nW	tons	126.598	126.529	126.828						
25	1.5	Best number of panels } simultaneous,	n		**12**	**13**	**14**	**13**	**14**	**15**	**14**	**15**	**16**
26	1.5	Best central height } simultaneous,	h	feet	44.461	43.694	42.940	92.190	91.134	90.140	145.819	144.694	143.607
27	1.5	Least bridge weight } simultaneous,	nW	tons	135.660	135.647	135 922	579.238	578.833	579.187	1,555.091	1,554.032	1,557.373
28	2.0	Best number of panels } simultaneous,	n		**12**	**13**	**14**	**12**	**13**	**14**	**13**	**14**	**15**
29	2.0	Best central height } simultaneous,	h	feet	47.273	46.375	45 496	98.993	97.661	96.323	155.062	153.512	152.085
30	2.0	Least bridge weight } simultaneous,	nW	tons	164.603	164.555	164.998	700.735	699.353	699.692	1,855.156	1,853.540	1,854.922

196. A few simple relations may be stated here, as resulting from our investigation of this bridge having two parabolic bowstring girders with double triangular web.

1st, For a given uniform live load,

$$n \propto l^{\frac{1}{2}} \text{ nearly,} \tag{669}$$

$$W \propto lh \text{ nearly.} \tag{670}$$

2d, For different live loads,

$$hW \propto nL \text{ nearly.} \tag{671}$$

197. EXAMPLE. — Specifications for the bridge of 200 feet span, as tabulated in article 194. We have found

$$\begin{array}{llll} l = 200 \text{ feet,} & n = 13, & L = & 15.38460 \text{ tons;} \\ h = 39.726 \text{ feet,} & & W = & 8.16590 \text{ tons;} \\ q = 16 \text{ feet,} & & \frac{1}{2}(W + L) = & 11.77525 \text{ tons;} \\ q_1 = 18 \text{ feet.} & & & \end{array}$$

Maximum horizontal component of chord strain, each of 2 girders,

$$= H = \frac{M}{y} = \frac{11.77525 \times 13 \times 200}{8 \times 39.726} = 96.333 \text{ tons,}$$

by equation (564). Therefore, by (632),

Strains in top chord due loads

$$\begin{array}{lll} = P = \dfrac{H}{\cos\alpha} = 119.464 \text{ tons,} & \therefore \text{ Sections} = & 31.739 \text{ square inches;} \\ \qquad\qquad = 110.330 \text{ tons,} & = & 29.312 \text{ square inches;} \\ \qquad\qquad = 107.231 \text{ tons,} & = & 28.489 \text{ square inches;} \\ \qquad\qquad = 102.605 \text{ tons,} & = & 27.260 \text{ square inches;} \\ \qquad\qquad = 99.170 \text{ tons,} & = & 26.347 \text{ square inches;} \\ \qquad\qquad = 97.050 \text{ tons,} & = & 25.783 \text{ square inches;} \\ = U \qquad\quad = 96.333 \text{ tons,} & = & 25.593 \text{ square inches;} \end{array}$$

for each half-span of each girder.

Additional cross-section of $n - 6$ central panels of top chords to resist lateral displacement, by (595), is

$$S = 0.70049 \text{ square inch.}$$

Corresponding chord strain $= 0.70049 \times 3.764 = 2.637$ tons.

$\therefore$ Total top chord strains $= 119.464$ tons, 1st and 13th panels;
$= 110.330$ tons, 2d and 12th panels;
$= 107.231$ tons, 3d and 11th panels;
$= 105.242$ tons, 4th and 10th panels;
$= 101.807$ tons, 5th and 9th panels;
$= 99.687$ tons, 6th and 8th panels;
$= 98.970$ tons, 7th panel.

From (651), the varying sections of a bottom chord due wind are

$$S = \frac{2500 \times 39.7265 \times 200}{20000 \times 13^2 \times 18}(12, 2 \times 11, 3 \times 10, \text{etc.})$$

$= 3.918$ square inches,	$\therefore$ Strains $= 19.590$ tons;
$= 7.183$ square inches,	$= 35.915$ tons;
$= 9.794$ square inches,	$= 48.970$ tons;
$= 11.753$ square inches,	$= 58.765$ tons;
$= 13.059$ square inches,	$= 65.295$ tons;
$= 13.713$ square inches,	$= 68.565$ tons;
Add 19.267 square inches,	Add 96.333 tons,

for total sections and for total strains.

Putting $\frac{1}{2}L$ for L in (647), we have

The cross-section of any girder diagonal

$$= \frac{3 \times 40000 \sec\theta}{64 \times 13 \times 39.7265} = 1.8153 \sec\theta.$$

	1st System.	2nd System.
2d panel	$= 2.250$ square inches	$= 3.041$ square inches,
3d panel	$= 3.791$ square inches	$= 3.041$ square inches,
4th panel	$= 3.791$ square inches	$= 4.387$ square inches,
5th panel	$= 4.795$ square inches	$= 4.387$ square inches,
6th panel	$= 4.795$ square inches	$= 5.001$ square inches,
7th panel	$= 5.001$ square inches	$= 5.001$ square inches.

The actual strains on these girder diagonals given in the diagram below, Fig. 120, where compressive strains are marked negative, have been derived from equations (641) and (642), using the proper value of r, and dividing by the proper value of $\cos\theta$.

The floor and the longitudinal I-beams will be the same here as in article 184; viz., —

Floor of $2\frac{1}{2}$-inch oak.

I-beams, depth $d = 9.7740$ inches.

$d - d_1 = 0.6610$ inch.

$d_1 = 9.1130$ inches.

$S = 5.0154$ square inches.

$I = 75.4160$.

Also, the cross-section of the transverse I-beams due load will be, as in article 184,

$$S = 23.6108 \text{ square inches.}$$

But, for the wind pressure,

$$W_1 = \frac{2500}{13} \times 39.7265 = 7639.7 \text{ pounds};$$

$$(511), \quad S = \frac{2 \times 7639.7}{3 \times 13 \times 7542.5}(12^2, 11^2, 10^2, 9^2, 8^2, 7^2)$$

$= 7.480$ square inches, $\therefore$ Total $= 31.091$ square inches:
$= 6.285$ square inches, $= 29.896$ square inches:
$= 5.194$ square inches, $= 28.805$ square inches:
$= 4.207$ square inches, $= 27.818$ square inches:
$= 3.324$ square inches, $= 26.935$ square inches;
$= 2.545$ square inches, $= 26.156$ square inches;

for each half-span.

Take depth of beam $d = 12$ inches.

$d - d_1 = 2$ inches.

$d_1 = 10$ inches.

Use 2 I-beams at each joint.

Then, by (558) and (557),

$$
\begin{aligned}
\text{Flange, } b &= 0.17803S = 5.5351 \text{ inches;} \\
&= 5.3224 \text{ inches;} \\
&= 5.1282 \text{ inches;} \\
&= 4.9525 \text{ inches;} \\
&= 4.7953 \text{ inches;} \\
&= 4.6566 \text{ inches;} \\
b_1 &= 0.16363S = 5.0874 \text{ inches;} \\
&= 4.8919 \text{ inches;} \\
&= 4.7134 \text{ inches;} \\
&= 4.5519 \text{ inches;} \\
&= 4.4074 \text{ inches;} \\
&= 4.2799 \text{ inches;} \\
\text{Web, } b - b_1 &= 0.01440S = 0.4477 \text{ inch;} \\
&= 0.4305 \text{ inch;} \\
&= 0.4148 \text{ inch;} \\
&= 0.4006 \text{ inch;} \\
&= 0.3879 \text{ inch;} \\
&= 0.3767 \text{ inch.}
\end{aligned}
$$

From (508), the cross-section of each horizontal diagonal in floor system is found, thus:

$$\sin \phi_1 = 0.76017,$$

$$
\begin{aligned}
S &= \frac{7639.7}{15000 \times 2 \times 13 \times \sin \phi_1}(13 \times 12, 12 \times 11, 11 \times 10, \text{etc.}), \\
&= 4.020 \text{ square inches, 1st and 13th panels;} \\
&= 3.402 \text{ square inches, 2d and 12th panels;} \\
&= 2.835 \text{ square inches, 3d and 11th panels;} \\
&= 2.319 \text{ square inches, 4th and 10th panels;} \\
&= 1.855 \text{ square inches, 5th and 9th panels;} \\
&= 1.443 \text{ square inches, 6th and 8th panels;} \\
&= 1.082 \text{ square inches, 7th panel.}
\end{aligned}
$$

Cross-section of head diagonals is, from (587),

$$S = \frac{0.0625 \times 168 \times 39.7265}{6 \times 13^3 \times 18 \times \cos \phi_1} = 0.54115 \text{ square inch,}$$

since $\cos \phi_1 = 0.64972$.

Cross-section of each head strut is given by (591),

$$S = \frac{168 \times 39.7265}{12 \times 169} = 3.291 \text{ square inches.}$$

Section of a brace, by (654), $= S = \dfrac{15 \times 200}{34 \times 13} = 6.7873$ square inches.

Add 50 per cent, 3.3936 square inches.

For 2 end braces, 10.1809 square inches.

RESULTS.

Use, in each top chord, 2 10-inch channels, 21 square inches; 1 15-inch plate by 0.76 to 0.36 inch; 1 $15 \times 18 \times \frac{1}{4}$ inch plate in every 3 feet, riveted to the bottom flanges. Make bottom chords of flat bars.

1st panel, 4 bars, 6×0.966 inch;
2d panel, 5 bars, 6×0.882 inch;
3d panel, 6 bars, 6×0.807 inch;
4th panel, 5 bars, 6×1.034 inches;
5th panel, 6 bars, 6×0.898 inch;
6th panel, 5 bars, 6×1.100 inches;
7th panel, 6 bars, 6×0.916 inch;

and proportion eyes as already specified.

In girder diagonals, use 4 angle irons $2\frac{1}{4} \times 2\frac{1}{4}$ inches, latticed both ways by diagonal strips of wrought-iron $1\frac{1}{4} \times \frac{1}{4}$ inch, and placed so far apart that the ratio of length to radius of gyration shall be 100, as already provided. This will require, for a strut of 40 feet length, a diameter of about $\dfrac{40 \times 12}{100} \times 2 = 9.6$ inches, since by this arrangement of the material the radius of gyration is nearly one-half of the diameter.

In this case, where two struts intersect in a panel, the smaller one may pass within the flanges of the larger one at the intersection, involving thereby a little riveting in place, and perhaps a little irregularity of the lattice work. The pin

bearings at the ends of these diagonals are to be formed of wrought-iron plates affording a bearing-surface equal to

$$S = \frac{P_1}{12000} = 2t \times d, \qquad (672)$$

where $t = \dfrac{S}{2d}$ = thickness of plate, in inches; P_1 = whole pressure on strut, in pounds; d = diameter of pin, in inches; as by general specifications.

It is here assumed, as in previous examples, that all cross-sections can be made exactly as the calculations require; hence we need only notice further the head struts and side braces.

For head struts, use 2 light 4-inch channels latticed, giving the required section 3.291. For wind braces, use 2 7-inch channels latticed with a slope of 1 to 10; and, if a clear roadway of more than 10 (= 18 − 4 − 4) feet is required, these braces must have their broad end at the top, or else they must have a bearing at the bottom beyond the ends of the transverse I-beams. Hence this mode of bracing high girders on narrow bridges is objectionable, and we shall henceforth either use a different style of brace, or provide it a head bearing, or increase the space between girders.

PARABOLIC BOW. — TWO GIRDERS.

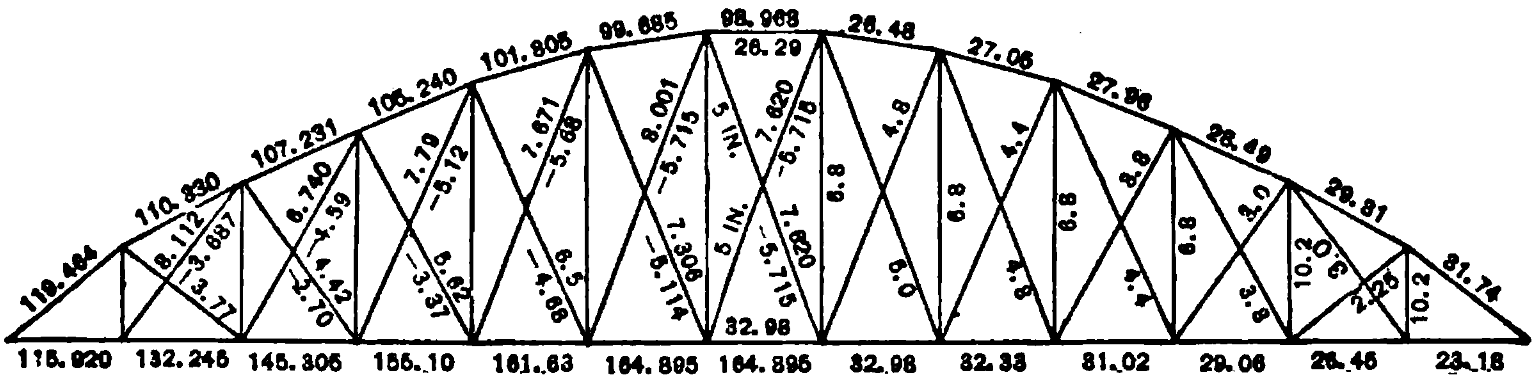


FIG. 120.

Span, 200 feet. Central height, 39.726 feet (best). Maxima strains in each girder. Cross-sections in square inches. Live load, 1 ton to the running-foot. Bridge weight, 106.157 tons (minimum). Section of bottom chord pins for shearing = 3.3 to 3 inches. Section of bottom chord pins for bending = 5.2 to 4 inches. Diameter of bottom chord pins = 3⅞ to 3 inches. Diameter of top chord pins = 3½ inches. By equations (169), (46), and (52).

The deflection is derived from (559), putting $B_1 = \frac{3.7646+5}{2}$ $= 4.3823$ tons, $E = 12,000$ tons, $h_1 = h = 39.7265$ feet, $a = \frac{1}{2}l$ $= 100$ feet, and measuring x from centre of span.

$$
\begin{aligned}
\text{Deflection } D_1 &= 1.529 \text{ inches for } x = 0 \text{ at centre;}\\
D_2 &= 1.522 \text{ inches for } x = 100 \div 13;\\
D_3 &= 1.469 \text{ inches for } x = 300 \div 13;\\
D_4 &= 1.361 \text{ inches for } x = 500 \div 13;\\
D_5 &= 1.192 \text{ inches for } x = 700 \div 13;\\
D_6 &= 0.836 \text{ inch for } x = 900 \div 13;\\
D_7 &= 0.598 \text{ inch for } x = 1100 \div 13;\\
D_8 &= 0 \text{ inch for } x = 100 \text{ at end.}
\end{aligned}
$$

$$\text{Length of top chord} = \frac{200}{13} \times \Sigma \sec \alpha = 219.297 \text{ feet.}$$

Contraction due strain, by (336),

$$= \lambda = \frac{3.7646 + 5}{12000} \times 219.297 \times 12 = 1.922 \text{ inches.}$$

$$\text{Mean excess per panel} = \frac{1.922}{13} = 0.1478 = \tfrac{1}{7} \text{ inch nearly.}$$

SECTION 2.

The Post Truss with Parabolic Top Chord (Fig. 36).

198. Let our previous notation be continued as far as applicable; viz., —

l = span, in feet.

h = height of girder at centre, in feet.

n = number of panels, counting on the bottom chord, and odd.

L = panel weight of uniform live load, in tons, given.

W = panel weight of bridge, in tons, to be determined.

Symmetry here requires an odd number of panels for the bottom chord, and an even number for the top chord. As the

live load will here be applied to the bottom chord, we shall take n odd, and ranging from 9 to 21, inclusive. Each upper apex is in the middle of a panel's length. There is but a single system of counter diagonals, while there are two systems of mains.

We shall here assume the difference of level between the centre and end of the top chord to be one-tenth of the whole central height, h; and consequently the height at the end of a top chord is $\frac{9}{10}h$. This, of course, is wholly arbitrary, except, possibly, in some cases where the head room at the ends would be too little. The top chord is to be polygonal (that is, straight from joint to joint), and we will take it parabolic in this case.

Putting $2 \times \frac{1}{10}h$ for h, and $\frac{n-1}{n}l$ for l, and $\frac{rl}{n}$ for x, in (472), we have the height of any upper apex,

$$y = h\left\{0.9 + \frac{0.4r(n-r-1)}{(n-1)^2}\right\} = \varepsilon h \text{ (say)}; \quad (673)$$

r to be counted on top chord from 0 to $\frac{n-1}{2}$, inclusive (that is, to the centre).

VALUES OF ε IN (673).

$n =$	9	11	13	15	17	19	21
$r =$ 0	0.90000	0.900	0.900000	0.900000	0.900000	0.900000	0.900
1	0.94375	0.936	0.930556	0.926531	0.923439	0.920988	0.919
2	0.97500	0.964	0.955556	0.948980	0.943750	0.939506	0.936
3	0.99375	0.984	0.975000	0.967347	0.960938	0.955556	0.951
4	1.00000	0.996	0.988889	0.981633	0.975000	0.969136	0.964
5		1.000	0.997222	0.991837	0.985938	0.980247	0.975
6			1.000000	0.997959	0.993750	0.988889	0.984
7				1.000000	0.998437	0.995062	0.991
8					1.000000	0.998765	0.996
9						1.000000	0.999
10							1.000

Since the top chord for every panel length slopes uniformly, we have manifestly the height of girder, if measured in the vertical through any lower apex, equal to the mean of the two heights at the adjacent upper apices just found.

If, then, we put $r + 1$ for r in (673), and add the resulting equation to (673), we have, after dividing by 2,

$$y = h\left\{0.9 + \frac{0.4r(n - r - 2) + 0.2(n - 2)}{(n - 1)^2}\right\} = \varepsilon_1 h, \quad (674)$$

which is the height through any lower apex; r taking the values $0, 1, 2, 3, \ldots \frac{n - 3}{2}$, counted on upper apices.

VALUES OF ε_1 IN (674).

$n =$	9	11	13	15	17	19	21
$r = 0$	0.921875	0.918	0.915278	0.913265	0.911718	0.910494	0.9095
1	0.959375	0.950	0.943056	0.937755	0.933595	0.930247	0.9275
2	0.984375	0.974	0.965278	0.958163	0.952344	0.947531	0.9435
3	0.996875	0.990	0.981944	0.974490	0.967969	0.962346	0.9575
4		0.998	0.993055	0.986735	0.980469	0.974691	0.9695
5			0.998611	0.994898	0.989844	0.984568	0.9795
6				0.998980	0.996094	0.991975	0.9875
7					0.999218	0.996913	0.9935
8						0.999382	0.9975
9							0.9995

Calling α the slope of any segment of the top chord, we have

$$\tan \alpha = \Delta y \div \frac{l}{n} = \frac{0.4hn}{(n - 1)^2 l}[n - 2(r + 1)],$$

since $\Delta y = y_{r+1} - y_r = \frac{0.4h}{(n - 1)^2}[n - 2(r + 1)]$.

$$\sec^2 \alpha = 1 + \frac{0.16h^2n^2}{(n-1)^4 l^2}[n - 2(r+1)]^2, \qquad (675)$$

$$= 1 + \frac{h^2}{l^2}\varepsilon_2,$$

where r is to be counted 0, 1, 2, 3, etc., and

$$\varepsilon_2 = \frac{0.16n^2}{(n-1)^4}[n - 2(r+1)]^2.$$

VALUES OF ε_2 IN (675).

$n =$	9	11	13	15	17	19	21
$r = 0$	0.155039	0.156816	0.157785	0.158372	0.158752	0.159014	0.159201
1	0.079102	0.094864	0.105625	0.113390	0.119241	0.123799	0.127449
2	0.028477	0.048400	0.063896	0.075906	0.085374	0.092987	0.099225
3	0.003164	0.017424	0.032600	0.045918	0.057151	0.066577	0.074529
4		0.001936	0.011736	0.023428	0.034573	0.044568	0.053361
5			0.001304	0.008434	0.017639	0.026961	0.035721
6				0.000937	0.006350	0.013755	0.021609
7					0.000706	0.004952	0.011025
8						0.000550	0.003969
9							0.000441

199. Moments due a Total Dead Load of Uniform Panel Weight, $W + L$. — Although the total load is here uniform, the separate or single systems are in no case uniformly loaded throughout the girder's length; and we may find, by equations (40) and (43), the effect of each single weight, $W + L$, at all required points in each single system, or we may sum the values of a' in these two equations for the several cases, as follows: —

1st, When $\frac{n-1}{4}$ is an integer; that is, when $n = 9, 13, 17, 21$, etc.

FIRST SYSTEM. — NINE PANELS.

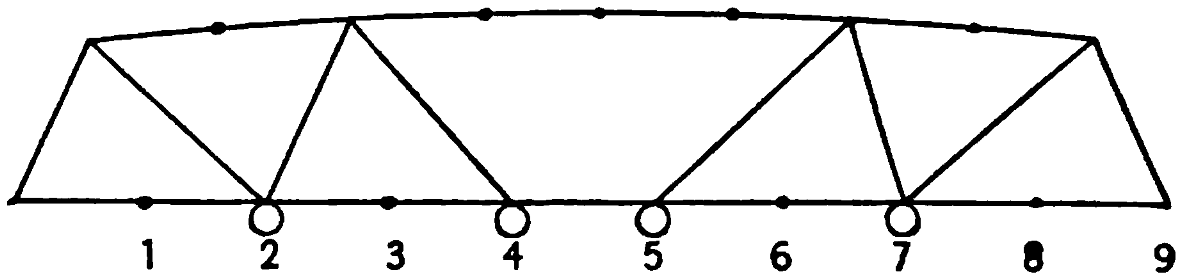


FIG. 121.

Let r denote the number of intervals, each $= 2c = \frac{2l}{n}$, between any weight in the right half-span and the left end of the girder.

Then, when $x \overline{>} a'$, equation (43) applies, giving

$$M = \frac{W + L}{l} a'(l - x).$$

Beginning at the left weight, and summing the values of a', we have

$$\begin{aligned}\Sigma a' = 2c\Big\{&\Big(1 + 2 + 3 + \ldots \frac{n-1}{4}\Big)\\ &+ \Big(\frac{n+1}{4} + \frac{n+5}{4} + \frac{n+9}{4} + \ldots r\Big)\Big\}\\ = c\Big\{&\tfrac{1}{16}(n+3)(n-1) + \Big(r + \frac{n+1}{4}\Big)\Big(r - \frac{n-3}{4}\Big)\Big\};\end{aligned}$$

$$\therefore\ M = \frac{W+L}{n}\Big\{\frac{n}{4} + r(r+1)\Big\}(l - x), \qquad (676)$$

which is the moment due all the weights on the length, $2cr$, measured from the left end, at any point not distant less than $2cr$ from the left end of girder.

$$r \overline{>} \frac{n-3}{4}.$$

For the moments due the weights on the remaining part, $l - 2cr$, we sum equation (40), where now $M = \frac{W+L}{l}(l - a')x$, and $x \overline{<} a'$.

Beginning at the point $2c(r+1)$, we thus sum:

$$\begin{aligned}\Sigma a' &= 2c[(r+1)+(r+2)+(r+3)+\ldots r_1]\\ &= c(r_1+r+1)(r_1-r),\\ \Sigma a'^0 &= r_1 - r = \text{number of terms};\end{aligned}$$

$$\begin{aligned}\therefore\ M &= \frac{W+L}{n}[n(r_1-r)-(r_1+r+1)(r_1-r)]x\\ &= \frac{W+L}{4n}(n-2r-2)(n-2r)x, \qquad (677)\end{aligned}$$

since here $r_1 = \frac{n-2}{2}$.

Adding (676) to (677) gives

$$M = \frac{W+L}{n}\left\{\left[\frac{n}{4}+r(r+1)\right](l-x)\right.$$
$$\left.+\tfrac{1}{4}(n-2r-2)(n-2r)x\right\}, \qquad (678)$$

which is the moment due all weights in the first system, at any point in the second half-span, when $\frac{n-1}{4}$ is an integer; and for the panel points in this system, second half-span, r becomes

$$\frac{n+1}{4},\quad \frac{n+5}{4},\quad \frac{n+9}{4},\quad \text{etc.,}$$

and

$$x = \frac{2rl}{n} = 2cr.$$

Therefore, putting this value of x in (678),

$$M = \frac{(W+L)l}{4n}(2r+1)(n-2r), \qquad (679)$$

which is the moment due all the first system weights at loaded points in the second half-span.

If, in (678), $x = 2(r + \frac{3}{4})\frac{l}{n}$, it becomes

$$M = \frac{(W + L)l}{8n}[4(n - 2r - 4)r + 5n - 9], \quad (680)$$

which is the moment due all weights at the unloaded panel points in second half-span, first system.

Similarly we proceed with the second system when $\frac{n - 1}{4}$ is an integer.

SECOND SYSTEM. — NINE PANELS.

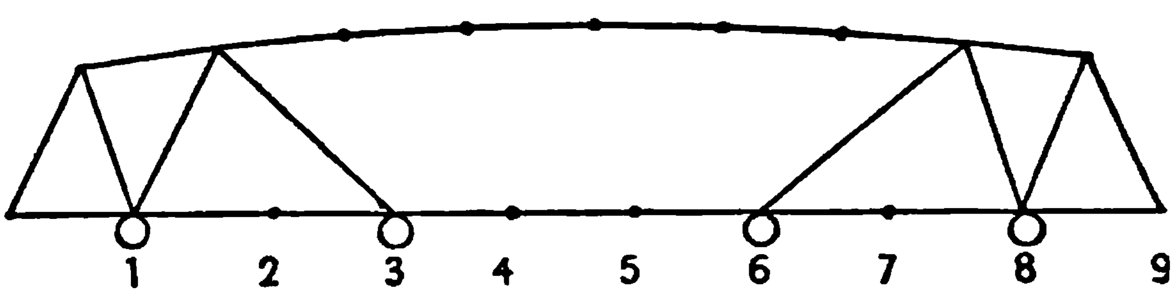


FIG. 122.

Beginning at the left weight, and summing the values of a' in (43), there results

$$\begin{aligned}\Sigma a' &= 2c\left\{\left(\frac{1}{2} + \frac{3}{2} + \frac{5}{2} + \ldots \frac{n - 3}{4}\right)\right. \\ &\qquad\qquad \left. + \left(\frac{n + 3}{4} + \frac{n + 7}{4} + \frac{n + 11}{4} + \ldots r\right)\right\} \\ &= \frac{l}{n}\left\{\left(\frac{n - 1}{4}\right)^2 + \left(r + \frac{n + 3}{4}\right)\left(r - \frac{n - 1}{4}\right)\right\},\end{aligned}$$

$$\begin{aligned}\therefore\ M &= \frac{W + L}{l}(l - x)\Sigma a' \\ &= \frac{W + L}{n}\left\{r(r + 1) - \frac{n - 1}{4}\right\}(l - x), \quad (681)\end{aligned}$$

which is the moment due all the weights on the length, $2rc$, measured from the left end of the girder, at any point not dis-

tant less than $\frac{(n-3)l}{2n}$ from the left end; r being not less than $\frac{(n-1)}{4}$, and increasing by unity for the loaded points in second half-span.

For the remainder, $l - 2cr$, we use (40), and find equation (677), which, since now $r_1 = \frac{n-1}{2}$, becomes

$$M = \frac{W+L}{4n}(n - 2r - 1)^2 x, \qquad (682)$$

where x cannot be greater than $2c(r+1)$, and r not less than $\frac{n-1}{4}$.

Adding (681) to (682), the result is, if, as usual, we call the sum M instead of $2M$,

$$M = \frac{W+L}{n}\left\{\left[r(r+1) - \frac{n-1}{4}\right](l-x) + \tfrac{1}{4}(n-2r-1)^2 x\right\}, \qquad (683)$$

which is the moment due all weights in the second system, at any point in the second half-span, when $\frac{n-1}{4}$ is an integer; the limits of x being $2rc$ and $2(r+1)c$, and the limits of r, $\frac{n-1}{4}$ and $\frac{n-1}{2}$.

If, in (683), we put $x = \frac{2rl}{n}$, we have, for the loaded points in second half-span, second system,

$$M = \frac{(W+L)l}{4n}[(n-2r)(2r-1)+1]; \qquad (684)$$

and, if $x = 2r + \frac{1}{2}\frac{l}{n}$, (68[illegible]) becomes

$$M = \frac{(W + L)l}{n}\left\{ -\left(\frac{n}{2} - r - 1\right) + \frac{1}{2}(n - 1)\right\}, \qquad (68[illegible])$$

which is the moment at all upper or unloaded apices in second half-span, second system, *the sign of the last moment to be changed from* − *to* +.

2d. When $\frac{n - 3}{4}$ is an integer; that is, $n = 11, 15, 19$, etc.

First System.—Eleven Panels.

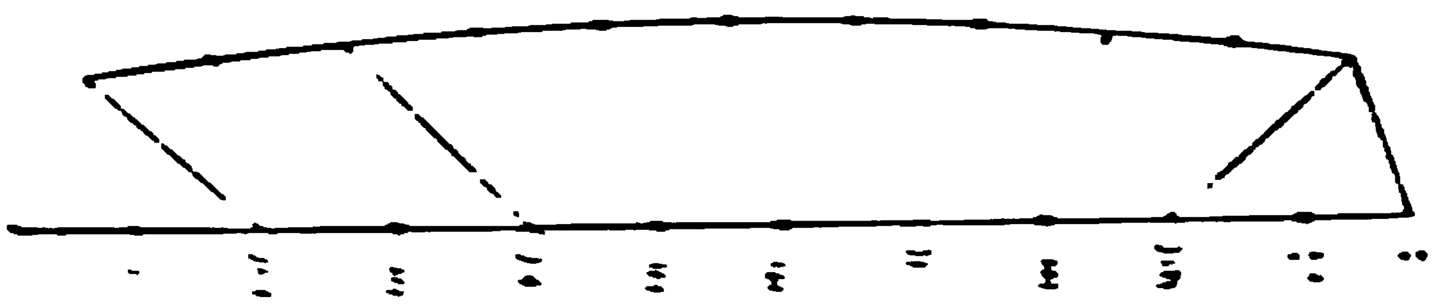


Fig. 2[illegible]

Proceeding as above to sum [illegible] in equations [illegible] and [illegible], in the present case we write

$$M = [illegible]$$

$$= [illegible] \qquad (68[illegible])$$

[illegible]

[illegible]

[illegible]

Again, when $x \overline{<} a'$,

$$M = \frac{W + L}{l}\Sigma(la'^{\circ} - a')x$$

$$= \frac{W + L}{n}(r_1 - r)(n - r_1 - r - 1)x, \qquad (687)$$

since

$$\Sigma a'^{\circ} = r_1 - r = \text{number of terms},$$

and

$$\Sigma a' = 2c[(r + 1) + (r + 2) + (r + 3) + \ldots r_1]$$

$$= \frac{l}{n}(r_1 - r)(r_1 + r + 1).$$

Add (687) to (686), and put $r_1 = \frac{n - 2}{2}$;

$$\therefore \quad M = \frac{W + L}{n}\left\{\left[r(r + 1) - \frac{n}{4}\right](l - x)\right.$$

$$\left. + \tfrac{1}{4}(n - 2r)(n - 2r - 2)x\right\}, \qquad (688)$$

which is the moment due all weights in the first system, at any point in the second half-span, when $\frac{n + 1}{4}$ is an integer; r being $\frac{n - 1}{4}, \frac{n + 3}{4}, \frac{n + 7}{4}$, etc., and x lying between $2rc$ and $2(r + 1)c$.

If $x = 2rc$, equation (688) becomes

$$M = \frac{(W + L)l}{4n}(2r - 1)(n - 2r), \qquad (689)$$

which is the moment due all the first system weights at loaded points in the second half-span.

If, in (688), $x = 2(r + \frac{3}{4})c$, we have

$$M = \frac{(W + L)l}{n}\left\{r\left(\frac{n}{2} - r - 1\right) + \tfrac{1}{8}(n - 3)\right\}, \qquad (690)$$

which is the moment due all weights at the unloaded apices in the second half-span, first system.

Also, for the second system, when $\frac{n + 1}{4}$ is an integer, we write:—

SECOND SYSTEM.—ELEVEN PANELS.

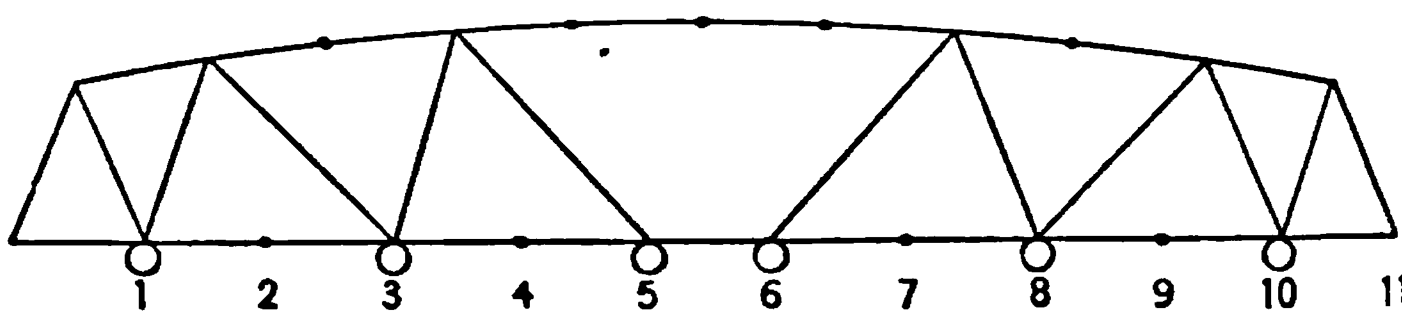


FIG. 124.

From (43), we have

$$x \;\overline{>}\; a',$$

$$\Sigma a' = 2c\left\{\left(\frac{1}{2} + \frac{3}{2} + \frac{5}{2} + \dots \frac{n - 1}{4}\right) + \left(\frac{n + 1}{4} + \frac{n + 5}{4} + \frac{n + 9}{4} + \dots r\right)\right\}$$

$$= c\left\{\left(\frac{n + 1}{4}\right)^2 + \left(r + \frac{n + 1}{4}\right)\left(r - \frac{n - 3}{4}\right)\right\}$$

$$= c[\tfrac{1}{4}(n + 1) + r(r + 1)],$$

$$\therefore\ M = \frac{W + L}{n}[\tfrac{1}{4}(n + 1) + r(r + 1)](l - x). \qquad (691)$$

And, from (40),

$$x \;\overline{<}\; a', \quad (r_1 - r) \text{ terms};$$

$$\Sigma a' = 2c[(r + 1) + (r + 2) + (r + 3) + \dots r_1],$$

$$-\Sigma a' + \Sigma a'^{o} = [-(r_1 + r + 1)(r_1 - r) + n(r_1 - r)]c$$
$$= \tfrac{1}{4}c(n - 2r - 1)^2$$

if $r_1 = \dfrac{n-1}{2}$.

$$\therefore \quad M = \frac{W + L}{4n}(n - 2r - 1)^2 x. \qquad (692)$$

The sum of (691) and (692) is

$$M = \frac{W + L}{4n}\{[n + 1 + 4r(r + 1)](l - x)$$
$$+ (n - 2r - 1)^2 x\}, \qquad (693)$$

which is the moment due all weights in the second system, at any point in the second half-span, when $\dfrac{n+1}{4}$ is an integer; the limits of x being $2rc$ and $2(r + 1)c$, and the limits of r being $\dfrac{n-3}{4}$ and $\dfrac{n-1}{2}$.

Substituting $2rc$ for x in (693), we get

$$M = \frac{(W + L)l}{4n}[n + 1 + r(2n - 4r - 2)], \qquad (694)$$

which is the moment due all weights at the loaded apices, second half-span, second system, $\dfrac{n+1}{4}$ being an integer.

And, if $x = 2(r + \tfrac{3}{4})c$ in (693), we have finally

$$M = \frac{(W + L)l}{4n}[\tfrac{1}{2}(5n - 7) + 2r(n - 2r - 4)], \qquad (695)$$

which is the moment due all weights at all upper or unloaded apices in second half-span, second system; the sign of the last moment to be changed from $-$ to $+$, as in equation (685).

Of course, the total moments for each and both systems will be equal at corresponding points in the two half-spans.

200. Weights of Top and Bottom Chords due a Total Dead Load of Uniform Panel Weight, $W + L$. — Dividing (679) by (674) gives

$$H = \frac{(W + L)l}{h} \times \frac{e_3}{e_1}, \tag{696}$$

which is the horizontal component of strain in top chord over loaded points in first system if

$$e_3 = \frac{(2r + 1)(n - 2r)}{4n},$$

and $\frac{n - 1}{4}$ is an integer.

Also, dividing (684) by (674), calling

$$e_4 = \frac{(n - 2r)(2r - 1) + 1}{4n},$$

we get

$$H = \frac{(W + L)l}{h} \times \frac{e_4}{e_1}, \tag{697}$$

which is horizontal component of strain in top chord over loaded points in second system, for this case. Then, adding the strains due to each panel length of top chord from the two systems, we have total horizontal component of strain due $n(W + L)$ in each panel of top chord,

$$H = \frac{(W + L)l}{h} \times e_5, \tag{698}$$

if $e_5 =$ the sum of the proper values of $\frac{e_3}{e_1}$ and $\frac{e_4}{e_1}$.

Similarly, in case $\frac{n + 1}{4}$ is an integer, making

$$e_3 = \frac{(n - 2r)(2r - 1)}{4n}$$

in equation (689), and

$$\varepsilon_4 = \frac{(n - 2r)(2r + 1) + 1}{4n}$$

in (694).

Computing in this manner, we have, for each half-span,

VALUES OF ε_5 IN (698).

$n =$	9	11	13	15	17	19	21
Panel Pt.							
1	0.704320	0.679863	0.741537	0.718544	0.762139	0.741653	0.775200
2	0.914764	1.036126	1.047237	1.122328	1.121732	1.174723	1.169531
3	1.120256	1.204315	1.341205	1.379895	1.468452	1.487771	1.551306
4	1.120256	1.370786	1.480524	1.629998	1.690715	1.791639	1.828413
5		1.370786	1.621512	1.749982	1.909621	1.988183	2.100001
6			1.621512	1.871858	2.014554	2.182537	2.275878
7				1.871858	2.122266	2.276187	2.451159
8					2.122266	2.372470	2.535517
9						2.372470	2.622746
10							2.622746
$\Sigma\varepsilon_5$	7.719192	11.323752	15.707054	20.688926	26.423490	32.775266	39.864994

$$\left.\begin{aligned} \text{Strain on top chords} &= H \sec\alpha \\ \text{Section of top chords} &= \frac{H \sec\alpha}{Q} \end{aligned}\right\} . \qquad (699)$$

Weight of top chords, in pounds, due $(W + L)n$

$$= \frac{5}{18} \times \frac{12l}{nQ} \Sigma H \sec^2\alpha$$

$$= \frac{(W + L)l^2}{h} \times \frac{10}{3nQ} \Sigma\varepsilon_5 \sec^2\alpha$$

$$= \frac{W + L}{h} \times \frac{10}{3nQ}(l^2\Sigma\varepsilon_5 + h^2\Sigma\varepsilon_2\varepsilon_5), \qquad (700)$$

since, by (675), $\sec^2\alpha = 1 + \frac{h^2}{l^2}\varepsilon_2$.

Using the proper values of e_2 and e_5, we find

VALUES OF $e_2 e_5$ IN (700).

$n =$	9	11	13	15	17	19	21
Panel Pt.							
1	0.109197	0.106613	0.117003	0.113797	0.120991	0.117933	0.123413
2	0.072376	0.098291	0.110614	0.127261	0.133756	0.145430	0.149056
3	0.031901	0.058289	0.085697	0.104742	0.125368	0.138343	0.153928
4	0.003545	0.023632	0.048265	0.074846	0.096626	0.119282	0.136270
5		0.002654	0.019030	0.040999	0.066021	0.088609	0.112058
6			0.002114	0.015787	0.035535	0.058843	0.081297
7				0.001754	0.013476	0.031309	0.052967
8					0.001498	0.011748	0.027954
9						0.001305	0.010410
10							0.001157
$\Sigma e_2 e_5$	0.434038	0.578958	0.765446	0.958372	1.186542	1.425604	1.697020

Hence, if we take, as in article 186, equation (634), $Q = 3.7647$ tons, we have finally

Weight of top chords due $n(W + L)$, in pounds,

$$= \frac{W + L}{h} \times \frac{10}{3 \times 3.7647 n}(l^2 \Sigma e_5 + h^2 \Sigma e_2 e_5) \qquad (701)$$

$$= \frac{W + L}{h} \left| \begin{array}{l|c} 0.759412 l^2 + 0.042701 h^2 & n = 9 \\ 0.906448 l^2 + 0.046602 h^2 & 11 \\ 1.069793 l^2 + 0.052134 h^2 & 13 \\ 1.221223 l^2 + 0.056571 h^2 & 15 \\ 1.376226 l^2 + 0.061799 h^2 & 17 \\ 1.527358 l^2 + 0.066434 h^2 & 19 \\ 1.680811 l^2 + 0.071551 h^2 & 21 \end{array} \right.$$

Again, dividing (680) by (673) gives

$$H = \frac{(W + L)l}{h} \times \frac{\varepsilon_6}{\varepsilon} \qquad (702)$$

for the bottom chord strain under an upper apex in the first system, $\frac{n - 1}{4}$ an integer, and $\varepsilon_6 = \frac{4r(n - 2r - 4) + 5n - 9}{8n}$.

Also, dividing (685) by (673), and putting

$$\varepsilon_7 = \frac{r}{n}\left(\frac{n}{2} - r - 1\right) + \frac{n - 1}{8n},$$

we have

$$H = \frac{(W + L)l}{h} \times \frac{\varepsilon_7}{\varepsilon} \qquad (703)$$

as the bottom chord strain under an upper apex in the second system; $\frac{n - 1}{4}$ being an integer.

Then, adding the two strains thus found for each panel length of bottom chord, we find, for this case,

$$H = \frac{(W + L)l}{h} \times \varepsilon_8, \qquad (704)$$

which is the total strain on bottom chord due $n(W + L)$.

Proceed in like manner in case $\frac{n + 1}{4}$ is an integer, making

$$\varepsilon_6 = \frac{r}{n}\left(\frac{n}{2} - r - 1\right) + \frac{n - 3}{8n}$$

in (690), and

$$\varepsilon_7 = \frac{r}{2n}(n - 2r - 4) + \frac{5n - 7}{8n}$$

in (695).

Computing thus, we find for each half-span, including middle panel, —

VALUES OF e_8 IN (704).

$n =$	9	11	13	15	17	19	21
Panel.							
1	0.246913	0.252525	0.256410	0.259259	0.261438	0.263158	0.264550
2	0.417791	0.489510	0.458860	0.506825	0.481072	0.516987	0.494988
3	0.807154	0.812868	0.894161	0.887470	0.941122	0.932223	0.973214
4	0.957264	1.117273	1.155222	1.249844	1.264131	1.330849	1.336554
5	1.111111	1.238360	1.408483	1.471186	1.578340	1.613270	1.689301
6		1.363636	1.509075	1.687720	1.770084	1.888460	1.940049
7			1.615385	1.774622	1.960186	2.058469	2.186634
8				1.866667	2.036256	2.227605	2.339043
9					2.117647	2.295612	2.491804
10						2.368422	2.553070
11							2.619047
Σe_8	5.969355	9.184709	12.979807	17.540519	22.702905	28.621688	33.966985

All except the middle panel taken twice for Σe_8.

Taking $T = 5$ tons, the allowed inch strain in tension, as in (634), we find, from (704),

$$\text{Cross-section of bottom chords, } S = \frac{(W+L)l}{hT} \times e_8. \qquad (705)$$

Weight of bottom chords due $(W + L)n$, in pounds,

$$= \frac{W+L}{h} \times \frac{5}{18} \times \frac{12l^2}{.5n}\Sigma e_8 \qquad (706)$$

$$= \frac{W+L}{h}\left|\begin{array}{lcr} 0.442174l^2 & n = & 9 \\ 0.556649l^2 & & 11 \\ 0.665631l^2 & & 13 \\ 0.779579l^2 & & 15 \\ 0.890310l^2 & & 17 \\ 1.004269l^2 & & 19 \\ 1.078317l^2 & & 21 \end{array}\right.$$

201. To find the Greatest Strains in the Girder Diagonals, and their Weights. — Equation (148) gives the strain on the counter diagonals in this case in terms of the simultaneous moments in the vertical planes, $AA_{\prime\prime}$, $BB_{\prime\prime}$, etc., Fig. 38. These moments let us compute for the uniform panel load, L tons, advancing over the panel points O, B, D, etc., of the horizontal bottom chord. The difference of the horizontal strains due to these moments at consecutive panel points will be greatest when the foremost panel weight of load is at the foot of a Y diagonal or counter. We use, therefore, the ordinary formulæ (64) and (68), giving simultaneous moments at O and B, B and D, etc.; then, for the moment at $AA_{\prime\prime}$, $CC_{\prime\prime}$, etc., we take one-half the sum of (64) and (68, thus:

$$M_{r+\frac{1}{2}} = \frac{Ll}{2n^2}r(r+1)(n-r-\tfrac{1}{2})$$

$$= Lle_9 \text{ (say)}, \qquad (707)$$

which is the value of M_1 in (148), and would also be obtained by putting $r_2 = 0$, $x = (r+\frac{1}{2})c = (r+\frac{1}{2})\frac{l}{n}$, and L for W, in (60) for the half-intervals OA, BC, etc., Fig. 38.

M_{r+1} in equation (68) takes the place of M_2 in (148).

$$h_1 \text{ in } (148) = AA_{\prime\prime}, \text{ Fig. } 38, \quad = y \text{ in } (673);$$

and

$$h_2 \text{ in } (148) = BB_{\prime\prime}, \text{ Fig. } 38, \quad = y \text{ in } (674).$$

$$a_1 = \tfrac{1}{3}CC_1 \quad = \tfrac{1}{3}y_{r+1}, (673).$$

$$b_1 = 2a_1.$$

With these values of M_1, M_2, h_1, h_2, a_1, b_1, we compute $Y\cos\phi$ of (148), which, with sign changed, becomes

$$Y\cos\phi = \frac{Ll}{h}e_{10}. \qquad (708)$$

VALUES OF e_{10} IN (708).

$n =$	9	11	18	15	17	19	21
Panel.							
2	0.024108	0.016761	0.012321	0.009436	0.007455	0.006038	0.004992
3	0.063502	0.044923	0.033488	0.025928	0.020671	0.016869	0.014025
4	0.115315	0.082009	0.061579	0.048030	0.038533	0.031615	0.026415
5	0.180545	0.127461	0.095784	0.074941	0.060364	0.049719	0.041690
6		0.182225	0.136113	0.106409	0.086241	0.070841	0.059543
7			0.183280	0.142579	0.114847	0.094826	0.079795
8				0.183999	0.147601	0.121698	0.102391
9					0.184512	0.151622	0.127379
10						0.184900	0.154894
11							0.185203
Σe_{10}	0.766940	0.906758	1.045130	1.182644	1.320448	1.456256	1.592654

$$\text{Strain on counter} \quad = Y = \frac{Ll}{h} \times e_{10} \sec \phi. \qquad (709)$$

$$\text{Cross-section of counter} = S = \frac{Lle_{10}}{Th} \sec \phi. \qquad (710)$$

Weight of $(n - 1)$ counters, pounds,

$$= \frac{5}{18} \times \frac{12 \times 3l}{2n} \times \frac{Ll}{5h} \Sigma e_{10} \sec^2 \phi$$

$$= \frac{Ll^2}{nh} \Sigma e_{10} \sec^2 \phi$$

$$= \frac{L}{h}\left(\frac{l^2}{n} \Sigma e_{10} + \frac{4nh^2}{9} \Sigma e_{10} e^2\right) \qquad (711)$$

$$= \frac{L}{h} \left| \begin{array}{l|c} 0.085216 l^2 + 3.033912 h^2 & n = 9 \\ 0.082433 l^2 + 4.386878 h^2 & 11 \\ 0.080395 l^2 + 5.973413 h^2 & 13 \\ 0.078843 l^2 + 7.794240 h^2 & 15 \\ 0.077673 l^2 + 9.855875 h^2 & 17 \\ 0.076645 l^2 + 12.140274 h^2 & 19 \\ 0.075841 l^2 + 14.665989 h^2 & 21 \end{array} \right.$$

since we take $T = 5$ tons per square inch in tension, and

$$\sec^2\phi = 1 + \frac{4n^2}{9l^2}y_{r+1}^2 = 1 + \frac{4n^2h^2}{9l^2}\epsilon^2. \qquad (712)$$

Manifestly ϵ^2 is to be taken from (673), always beginning with $r = 2$.

Main Diagonals. — To find the strains, sections, and weights of the main diagonals of the Post truss with parabolic top chord, we proceed as follows: —

When $\frac{n-1}{4}$ is an integer, use equation (676) for the live load, nL, making $W = 0$; and for moment in second half-span, first-system apices,

At foremost end of live load, put $x = \frac{2rl}{n}$, giving M_0;

At point $1\frac{1}{2}$ panels ahead of foremost end, put

$$x = \frac{2(r + \frac{3}{4})l}{n}, \text{ giving } M_{\frac{3}{4}};$$

At point 2 panels ahead of foremost end, put

$$x = \frac{2(r + 1)l}{n}, \text{ giving } M_1;$$

these three moments being simultaneous. Then

$$\left.\begin{aligned} M_0 &= \frac{Ll}{n^2}\left\{\frac{n}{4} + r(r+1)\right\}(n - 2r) \\ M_{\frac{3}{4}} &= \frac{Ll}{n^2}\left\{\frac{n}{4} + r(r+1)\right\}\left(n - 2r - \frac{3}{2}\right) \\ M_1 &= \frac{Ll}{n^2}\left\{\frac{n}{4} + r(r+1)\right\}(n - 2r - 2) \end{aligned}\right\} . \qquad (713)$$

In a similar manner, for the second half-span, second-system

apices, we find, from (681), simultaneous moments due live load, nL,

$$\left.\begin{aligned} M_0 &= \frac{Ll}{n^2}\left\{r(r+1) - \frac{n-1}{4}\right\}(n-2r) \\ M_{\frac{1}{2}} &= \frac{Ll}{n^2}\left\{r(r+1) - \frac{n-1}{4}\right\}\left(n - 2r - \frac{3}{2}\right) \\ M_1 &= \frac{Ll}{n^2}\left\{r(r+1) - \frac{n-1}{4}\right\}(n-2r-2) \end{aligned}\right\}. \quad (714)$$

Dividing each of these moments, (713), (714), by the height, y, of truss at the section where the moment is taken, we find the horizontal strains at the panel points in second half-span,

At loaded points, $H_0 = \dfrac{M_0}{y_0}$ from (713), (714), (674);

At unloaded points, $H_{\frac{1}{2}} = \dfrac{M_{\frac{1}{2}}}{y_{\frac{1}{2}}}$ from (713), (714), (673);

At unloaded points, $H_1 = \dfrac{M_1}{y_1}$ from (713), (714), (674).

The difference of the two simultaneous horizontal strains at vertical sections through the ends of a diagonal at and next ahead of foremost end of live uniform load is the horizontal component of maximum strain on that diagonal due live load, and is tension on the diagonal whose foot is at the foremost end, but compression on the next.

$$\therefore \quad \Delta H = H_0 - H_{\frac{1}{2}} = \frac{Ll}{n^2h} \times e_{11} \text{ (tension)}, \qquad (715)$$

$$\Delta H = H_{\frac{1}{2}} - H_1 = \frac{Ll}{n^2h} \times e_{12} \text{ (compression)}; \qquad (716)$$

e_{11} and e_{12} being functions of n and r in (673), (674), (713), (714).

For the moments due the dead load, nW, at the same points where the simultaneous moments due live load have been found,

$\frac{n-1}{4}$ being an integer, we use equations (679) and (680), and (679) with $r + 1$ for r, $L = 0$, thus:—

First system,

$$\left.\begin{aligned} M_0 &= \frac{Wl}{4n}(2r+1)(n-2r) \\ M_{\frac{1}{2}} &= \frac{Wl}{4n}\left\{2(n-2r-4)r + \frac{5n-9}{2}\right\} \\ M_1 &= \frac{Wl}{4n}(2r+3)(n-2r-2) \end{aligned}\right\} . \quad (717)$$

Second system, use (684) and (685),

$$\left.\begin{aligned} M_0 &= \frac{Wl}{4n}[(n-2r)(2r-1)+1) \\ M_{\frac{1}{2}} &= \frac{Wl}{4n}\left\{4r\left(\frac{n}{2}-r-1\right) + \tfrac{1}{2}(n-1)\right\} \\ M_1 &= \frac{Wl}{4n}[(n-2r-2)(2r+1)+1] \end{aligned}\right\} \quad (718)$$

Dividing each moment by the proper value of y from (673) and (674), we obtain horizontal strains at all required apices in each system, from the differences of which consecutive horizontal strains comes the horizontal component of maximum diagonal strain due dead load, thus:

$$\Delta H = H_0 - H_{\frac{1}{2}} = \frac{Wl}{4nh} \times e_{13} \text{ (tension)}, \quad (719)$$

$$\Delta H = H_{\frac{1}{2}} - H_{10} = \frac{Wl}{4nh} \times e_{14} \text{ (compression)}; \quad (720)$$

e_{13} and e_{14} being functions of n and r in (673), (674), (717), (718).

INDEX.